ON A DEPTH FORMULA FOR MODULES OVER LOCAL RINGS

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Abstract. We prove that for modules M and N over a local ring R, the depth formula: $\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R = \operatorname{depth}_R \operatorname{Tor}_s^R(M,N) - s$, where $s = \sup\{i \mid \operatorname{Tor}_i^R(M,N) \neq 0\}$, holds under certain conditions. This adds to the list cases where the depth formula, which extends the classical Auslander-Buchsbaum equality, is satisfied.

Introduction

The celebrated Auslander-Buchsbaum equality asserts that if a finitely generated module M over a noetherian local ring R has finite projective dimension, then $\operatorname{depth}_R M + \operatorname{pd}_R M = \operatorname{depth} R$. There have been a number of generalizations of this formula, one of which is due to Auslander [1, 1.2] himself, who proved that with M as before and N a finitely generated R-module, if for $s = \sup\{n \mid \operatorname{Tor}_n^R(M,N) \neq 0\}$ either $s = 0$ or $\operatorname{depth}_R \operatorname{Tor}_s^R(M,N) \leq 1$, then

$$(*) \qquad \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R = \operatorname{depth}_R \operatorname{Tor}_s^R(M,N) - s\,.$$

Observe that with $N = k$, the residue field of R, we recover the Auslander-Buchsbaum equality. As a matter of fact, Auslander established that $\operatorname{depth}_R N = \operatorname{depth}_R \operatorname{Tor}_s^R(M,N) + \operatorname{pd} M - s$, which, in view of the Auslander-Buchsbaum equality, translates to the formula above.

Let us note that all the terms in $(*)$ are defined and finite for *any* pair of R-modules M and N as long as $\sup\{n \mid \operatorname{Tor}_n^R(M,N) \neq 0\} < \infty$. This leads us to the following

Problem. Let M and N be finitely generated modules over a noetherian local ring R such that $s = \sup\{n \mid \operatorname{Tor}_n^R(M,N) \neq 0\}$ is finite. Discover conditions under which (*) holds.

When this occurs, we say that the *depth formula holds for M and N*. Over the last few years it has emerged that if M has *finite CI-dimension* and either one of the following conditions holds:

(1) $s = 0$;
(2) $\operatorname{depth}_R \operatorname{Tor}_s^R(M,N) \leq 1$,

1991 *Mathematics Subject Classification.* 13C15, 18G15.
Key words and phrases. Auslander-Buschsbaum, depth, complete intersection dimension.
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then the depth formula (*) holds for M and N.

The description of modules of finite CI-dimension, first identified by Avramov, Gasharov, and Peeva, [4], is defered to Definition 2; for now, we note only that if either M has finite projective dimension or R is a complete intersection, then M has finite CI-dimension. In particular, these results extend that of Auslander.

The result above was established by Huneke and R. Wiegand [7, 2.5] in the particular case when the ring R is a complete intersection and $s = 0$, also confer [8, 4.3]; the general case is due to Araya and Yoshino [3, 2.5].

We add to this list of cases when the depth formula holds. More precisely, we prove that (*) holds also when M has finite CI-dimension and for some prime ideal $\mathfrak{p} \in \operatorname{Ass}\operatorname{Tor}_s^R(M,N)$, one has

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \geq \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R\,.$$

This is the content of Corollary 4. This result is deduced from the more general, and easier to digest, Theorem 3: If M has finite CI-dimension and $\sup\{n \mid \operatorname{Tor}_n^R(M,N) \neq 0\} = s < \infty$, then

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R \geq -s\,,$$

with equality if and only if $\operatorname{depth}_R \operatorname{Tor}_s^R(M,N) = 0$. This fact is well known in the special case when M has finite projective dimension, and contains a recent result of Jorgensen [9, 2.2].

Now, in [1] Auslander had asked if the depth formula (*) holds for any pair of modules M and N, given that M has finite projective dimension, but Murthy [12, Pg. 565] gave examples to the contrary. Nevertheless, as is explained in Remark 7, for such an M one has always an inequality

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R \geq \inf\{\operatorname{depth}_R \operatorname{Tor}_i^R(M,N) - i\}\,.$$

Thus, it is not unreasonable to ask if there is *some* integer i such that

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R = \operatorname{depth}_R \operatorname{Tor}_i^R(M,N) - i\,.$$

Note that this is a weaker version of the depth formula. Unfortunately, one cannot expect this formula to hold in general, even over regular local rings where all modules have finite projective dimension; this is the content of Proposition 11.

Results and proofs

The reader is referred to Matsumura [11] for basic definitions and notation. In this note, it seems expedient to introduce the following

Notation 1. For modules M and N over a ring R, set

$$\operatorname{fd}_R(M,N) = \sup\{n \mid \operatorname{Tor}_n^R(M,N) \neq 0\}\,.$$

In particular, if $\operatorname{Tor}_n^R(M,N) = 0$ for all n, then $\operatorname{fd}_R(M,N) = -\infty$, else $0 \leq \operatorname{fd}_R(M,N) \leq \infty$. When R is local, by which we understand that it is also noetherian, and k is its residue field, $\operatorname{fd}_R(M,k)$ is the flat dimension of M, which, when M is finitely generated, equals its projective dimension $\operatorname{pd}_R M$. Moreover, for such an M, the number $\operatorname{fd}_R(M,N)$ is finite for each finitely generated N.

Next, we recall the following definition, which surfaced in [4]:

Definition 2. A module M over a local ring R is said to be of *finite CI-dimension* if there is a diagram of local homomorphisms $R \to R' \leftarrow Q$ such that $R \to R'$ is flat, $R' = Q/(x_1, \dots, x_c)$ for a regular sequence $x_1, \dots, x_c$ in Q and $\mathrm{pd}_Q(M \otimes_R R') < \infty$.

There are two important classes of examples: Modules with finite projective dimension, when $R' = Q = R$, and modules over complete intersections, when R' is the completion of R at its maximal ideal and Q is any Cohen presentation of R'.

We are now ready to state our main result; its proof is deferred to 9.

Theorem 3. *Let R be a local ring, and let M and N be finitely generated modules such that $\mathrm{fd}_R(M,N) = s < \infty$. If M has finite* CI-dimension*, then*

$$\mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R \geq -s$$

with equality if and only if $\mathrm{depth}_R \operatorname{Tor}_s^R(M,N) = 0$.

Here is the depth formula advertised in the introduction:

Corollary 4. *Let R, M, and N be as above. If for a prime $\mathfrak{p} \in \operatorname{Ass}\operatorname{Tor}_s^R(M,N)$* $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \mathrm{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \mathrm{depth}\, R_\mathfrak{p} \geq \mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R$, *then*

$$\mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R = -s \quad \textit{and} \quad \mathrm{depth}_R \operatorname{Tor}_s^R(M,N) = 0\,.$$

Proof. Since $\operatorname{Tor}^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{Tor}^R(M,N)_\mathfrak{p}$, one has $\mathrm{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \mathrm{fd}_R(M,N)$ and $\mathrm{depth}_{R_\mathfrak{p}} \operatorname{Tor}_s^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$. This yields the last of the following series of (in)equalities; the first and the third are by Theorem 3, and the second is the hypothesis.

$$\begin{aligned} -\mathrm{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) &= \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \mathrm{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \mathrm{depth}\, R_\mathfrak{p} \\ &\geq \mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R \\ &\geq -\mathrm{fd}_R(M,N) \\ &= -\mathrm{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \end{aligned}$$

Therefore, $\mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R = -\mathrm{fd}_R(M,N)$, and this implies, thanks to Theorem 3, that $\mathrm{depth}_R \operatorname{Tor}_s^R(M,N) = 0$ □

Remark 5. It may be worth pointing out that inequality hypothesised in Corollary 4 is satisfied under either one of the following conditions:

(a) $\mathrm{depth}\, N_\mathfrak{p} \geq \mathrm{depth}\, N$ for a prime $\mathfrak{p} \in \operatorname{Ass}\operatorname{Tor}_s^R(M,N)$; this holds, for example, when $\mathrm{depth}\, N = 0$.

(b) $\text{G-dim}_R\, N$, the Gorenstein dimension of N, cf. [2], is finite and $\mathrm{depth}\, R_\mathfrak{p} \geq \mathrm{depth}\, R$ for a prime $\mathfrak{p} \in \operatorname{Ass}\operatorname{Tor}_s^R(M,N)$.

Indeed, since M has finite CI-dimension, $\mathrm{depth}\, M_\mathfrak{p} - \mathrm{depth}\, R_\mathfrak{p} \geq \mathrm{depth}_R\, M - \mathrm{depth}\, R$, confer [4, 1.4] and [4, 1.6]. Given this fact, it immediate that Condition (a) above implies the desired inequality. In the case of (b), note that

$$\begin{aligned} \mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}} N_\mathfrak{p} &= \text{G-dim}_{R_\mathfrak{p}} N_\mathfrak{p} && \text{by [2, 4.13b]} \\ &\leq \text{G-dim}_R\, N && \text{by [2, 4.15]} \\ &= \mathrm{depth}\, R - \mathrm{depth}_R\, N && \text{by [2, 4.13b]} \end{aligned}$$

Rearranging terms yields $\mathrm{depth}\, N_\mathfrak{p} - \mathrm{depth}_R\, N \geq \mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}\, R$; in particular $\mathrm{depth}\, N_\mathfrak{p} \geq \mathrm{depth}_R\, N$, so that (a) holds.

Here is the local version of Theorem 3; it implies that

$$\inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = -\operatorname{fd}_R(M,N)\,,$$

which has been observed by Jorgensen [9, 2.2].

Corollary 6. *Let M and N be finitely generated modules over a local ring R. If $\operatorname{fd}_R(M,N) < \infty$ and M has finite CI-dimension, then*

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \geq -\operatorname{fd}_R(M,N) \qquad \textit{for} \qquad \mathfrak{p} \in \operatorname{Spec} R\,,$$

with equality if and only if $\mathfrak{p} \in \operatorname{Ass} \operatorname{Tor}_s^R(M,N)$ *.*

Proof. Since M has finite CI-dimension, so does $M_\mathfrak{p}$ for any prime $\mathfrak{p} \in \operatorname{Spec} R$, cf. [4, 1.6], and hence

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \geq -\operatorname{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geq -\operatorname{fd}_R(M,N)\,,$$

where the inequality on the left is the theorem above, and the one on the right is a consequence of the isomorphism $\operatorname{Tor}^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{Tor}^R(M,N)_\mathfrak{p}$. Furthermore, equality holds if and only if $\operatorname{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{fd}_R(M,N)$ and $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} = -\operatorname{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$, which translates to, by Theorem 3, the condition that $\mathfrak{p} \in \operatorname{Ass} \operatorname{Tor}_s^R(M,N)$ for $s = \operatorname{fd}_R(M,N)$. ☐

The proof of Theorem 3 is based on the following observations.

Remark 7. Let M and N be finitely generated modules over a local ring R. If $\operatorname{pd}_R M < \infty$, then

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R \geq \inf\{\operatorname{depth}_R \operatorname{Tor}_i^R(M,N) - i\}\,,$$

with equality if the infimum is achieved at $i = \operatorname{fd}_R(M,N)$. (Note: The depth of the zero module is infinity.)

Indeed, Foxby [6, 12.ai] and Iyengar [8, 2.2] prove that $\operatorname{depth}_R(M \otimes_R^{\mathbf{L}} N) = \operatorname{depth}_R N - \operatorname{pd}_R M$, where $M \otimes_R^{\mathbf{L}} N$ denotes $F \otimes_R N$, for any free resolution F of M; confer [8] for further details. Thanks to the Auslander-Buchsbaum equality this translates to:

$$\operatorname{depth}_R(M \otimes_R^{\mathbf{L}} N) = \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R\,.$$

On the other hand, [8, 2.7.2] applied to the complex $F \otimes_R N$ yields an estimate

$$\operatorname{depth}_R(M \otimes_R^{\mathbf{L}} N) \geq \inf\{\operatorname{depth}_R \operatorname{Tor}_i^R(M,N) - i\}\,,$$

and [8, 2.3] asserts that equality holds if the infimum is achieved at $\operatorname{fd}_R(M,N)$. Combining the preceding equations yields the desired result.

It is useful to note the following special case of the previous remark; it has certainly been observed before, confer [6, 12.ag].

Remark 8. If $\operatorname{pd}_R M < \infty$, then for $s = \operatorname{fd}_R(M,N)$, one has

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R \geq -s$$

with equality if and only if $\operatorname{depth}_R \operatorname{Tor}_s^R(M,N) = 0$.

9. *Proof of* Theorem 3. Let $R \to R' \leftarrow Q$ be the diagram of local homomorphisms provided by the finiteness of the CI-dimension of M, cf. 2. Localizing Q and R' at a minimal prime of $\operatorname{Spec}(R'/\mathfrak{m}R')$, where $\mathfrak{m}$ is the maximal ideal of R, we may assume that $\operatorname{depth}(R'/\mathfrak{m}R') = 0$. This allows us to reduce, by passing to Q, to the case where the projective dimension of M is finite.

Indeed, since $R \to R'$ is flat $\operatorname{Tor}^{R'}(M', N') = \operatorname{Tor}^{R}(M, N) \otimes_R R'$, where $M' = M \otimes_R R'$ and $N' = N \otimes_R R'$, so that $\operatorname{fd}_{R'}(M', N') = \operatorname{fd}_R(M, N)$, Moreover, the depth of $R'/\mathfrak{m}R'$ is 0, hence [11, 23.3] yields

$$\begin{aligned} \operatorname{depth} R' &= \operatorname{depth} R \\ \operatorname{depth}_{R'} M' &= \operatorname{depth}_R M \\ \operatorname{depth}_{R'} N' &= \operatorname{depth}_R N \\ \operatorname{depth}_{R'} \operatorname{Tor}_s^{R'}(M', N') &= \operatorname{depth}_R \operatorname{Tor}_s^R(M, N) \, . \end{aligned}$$

Thus, it suffices to establish the desired result for modules M' and N' over the ring R'. Therefore, by passing to R' we may assume that $R = Q/(x_1, \dots, x_c)$ with $x_1, \dots, x_c$ a Q-regular sequence, and that $\operatorname{pd}_Q M < \infty$.

In this case, as is well known, $\operatorname{Tor}_{s+c}^{Q}(M, N) = \operatorname{Tor}_s^R(M, N)$ and $\operatorname{fd}_Q(M, N) = s + c$; for example, confer [3, 2.6]. Furthermore, $\operatorname{depth} Q = \operatorname{depth} R + c$ and $\operatorname{depth}_Q L = \operatorname{depth}_R L$ for any R-module L, so it suffices to prove that

$$\operatorname{depth}_Q M + \operatorname{depth}_Q N - \operatorname{depth} Q \geq -s - c$$

with equality if and only if $\operatorname{depth}_Q \operatorname{Tor}_{s+c}^{Q}(M, N) = 0$. Since $\operatorname{pd}_Q M < \infty$, it remains to invoke Remark 8. ☐

Remark 10. The reduction technique employed in the argument above also yields another proof of the result of Arraya and Yoshino [3, 2.5] mentioned in the introduction. Indeed, the only change required is that in the last paragraph of the proof one invokes Remark 7 instead of Remark 8.

As has been explained in the introduction, confer also Remark 7, it is reasonable to seek conditions under which there is some integer i such that

$$\operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R = \operatorname{depth}_R \operatorname{Tor}_i^R(M, N) - i \, .$$

One cannot expect this to hold in general, for the next result identifies a class of examples for which the formula above fails for all i, even when both modules have finite projective dimension.

Proposition 11. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and let $x_1, \dots, x_n, y$ be elements in $\mathfrak{m}$ such that the following conditions are satisfied:*

(a) *$x_1, \dots, x_n$ is a regular sequence;*
(b) *the Cohen-Macaulay defect d of the local ring $R/(x_1, \dots, x_n, y)$ is at least 3.*

Then for the modules $M = R/(x_1, \dots, x_n)$ and $N = R/(y)$, one has

$$\begin{aligned} \operatorname{depth}_R \operatorname{Tor}_0^R(M, N) &= \dim R - n - d \\ \operatorname{depth}_R \operatorname{Tor}_1^R(M, N) &= \dim R - n - d + 2 \\ \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R &= \dim R - n - 1 \, . \end{aligned}$$

Proof. It is immediate that $\operatorname{depth}_R M = \dim R - n$ and $\operatorname{depth}_R N = \dim R - 1$; this yields the last equality that we seek. The complex $0 \to R \xrightarrow{y} R \to 0$ is a free resolution of N, so $\operatorname{Tor}^R(M, N)$ is the homology of the complex

$$0 \to M \xrightarrow{y} M \to 0 \, .$$

Thus, $\operatorname{Tor}_0^R(M,N) = R/(x_1,\dots,x_n,y)$, $\operatorname{Tor}_1^R(M,N) = \{m \in M \mid y.m = 0\}$, and these fit into exact sequences

$$0 \to K \to M \to \operatorname{Tor}_0^R(M,N) \to 0$$
$$0 \to \operatorname{Tor}_1^R(M,N) \to M \to K \to 0\,.$$

Since $(x_1,\dots,x_n,y)$ is an ideal of grade n, one has $\dim \operatorname{Tor}_0^R(M,N) = \dim R - n$; since the Cohen-Macaulay defect d of $\operatorname{Tor}_0^R(M,N)$ is at least 3, we obtain

$$\begin{aligned}\operatorname{depth}_R \operatorname{Tor}_0^R(M,N) &= \dim \operatorname{Tor}_0^R(M,N) - d\\ &= \dim R - n - d\\ &\leq \dim R - n - 3\,.\end{aligned}$$

Now $\operatorname{depth}_R M = \dim R - n$, so a routine 'depth chase' on the exact sequences above establishes that $\operatorname{depth}_R K = \operatorname{depth}_R \operatorname{Tor}_0^R(M,N) + 1$ and $\operatorname{depth}_R \operatorname{Tor}_1^R(M,N) = \operatorname{depth}_R \operatorname{Tor}_0^R(M,N) + 2$. □

Here is a concrete example which ensures that the class identified in the previous proposition is not empty. It was cooked up using the recipe outlined in Burch [5], also confer Kohn [10].

Example 12. Let k be a field of characteristic 0 and let R be the polynomial ring $k[a,b,c,d,e]$ localized at the maximal ideal (a,b,c,d,e). The elements $x_1 = abc, x_2 = de(bd+ce)$ and $y = ade + bc(bd+ce)$ are such that

(a) x_1, x_2 is a regular sequence;
(b) $\operatorname{pd}_R(R/(x_1,x_2,y)) = 5$, as is verified by MACAULAY.

Therefore, $\dim R/(x_1,x_2,y) = 3$ and $\operatorname{depth} R/(x_1,x_2,y) = 0$, so that the Cohen-Macaulay defect of $R/(x_1,x_2,y)$ is 3.

Acknowledgements. This collaboration owes its existence to Luchezar Avramov and Craig Huneke, who made each of us aware of the other's work. It is a pleasure to thank them for the same and for their encouragement. Our thanks also to David Jorgensen for his valuable remarks on an earlier version of this paper.

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