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FREE RESOLUTIONS AND CHANGE OF RINGS

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Abstract. Projective resolutions of modules over a ring R are constructed starting from appropriate projective resolutions over a ring Q mapping to R. It is shown that such resolutions may be chosen to be minimal in codimension ≤ 2, but not in codimension 3. This is used to obtain minimal resolutions for essentially all modules over local (or graded) rings R with codimension ≤ 2. Explicit resolutions are given for cyclic modules over multigraded rings, and necessary and sufficient conditions are obtained for their minimality.

Introduction

We construct a projective resolution $\mathrm{F}(X,Y)$ of a module M over a ring R, starting from appropriate projective resolutions, X of R and Y of M, over a ring Q mapping to R.

The interest in such a procedure comes from the fact that homological properties of modules over Q are often better understood. A typical case is when R is a quotient of a polynomial ring Q: resolutions over Q are then finite, but those over R are usually infinite. Thus, one can build, *in a finite number of steps*, suitable Q-resolutions of R and M, and then use our construction to get a resolution over R.

When Q is a field, the classical bar construction of R and M over Q yields an projective resolution of M over R. In Section 1 we generalize this by using differential graded (henceforth abbreviated to DG) bar constructions to obtain a resolution $\mathrm{F}(X,Y)$ of M over R. As an application, we give a short construction of a spectral sequence of Avramov.

Our construction can be used to obtain a free R-resolution of M *only if* both Q-resolutions admit appropriate DG structures, so we consider the following question: *If X is a Q-free resolution of R with a structure of a DG algebra, then does a given Q-free resolution Y of M admit a structure of a DG module over X?* Work of Avramov, Buschsbaum, Eisenbud, Kustin, Miller, Srinivasan, and others, provides answers in many situations. When X is a Koszul complex, the answer is positive for $M = Q/I$ with $\mathrm{pd}_Q(Q/I) \leq 3$, or $\mathrm{pd}_Q(Q/I) = 4$ and Q/I Gorenstein, and may be negative when $\mathrm{pd}_Q(Q/I) = 4$, or $\mathrm{pd}_Q(Q/I) = 5$ with Q/I Gorenstein.

In view of these 'positive' and 'negative' results, the only unresolved case is that of a module M having a resolution of length at most 3. In Section 2, we prove that if Y has length ≤ 2, then Y has a DG module structure over *any DG algebra* X resolving R. On the other hand, using Avramov's obstructions, we show that this result cannot be extended to resolutions of length 3, even when X *is a Koszul complex on a single non-zero divisor.*

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When dealing with local (or graded) rings, the objective is to build *minimal* resolutions. In general, $\mathrm{F}(X,Y)$ is non-minimal over R, even if both resolutions X and Y are minimal over Q. Nevertheless, in codimension ≤ 2 the construction can be used to write down the minimal resolution of an *a priori* specified syzygy module of each module M over R. When $\mathrm{pd}_Q R = 1$, we recover the periodic of period 2 resolutions constructed by Shamash and Eisenbud. Moreover, when $\operatorname{edim} R - \operatorname{depth} R \leq 2$ and R is presented as a quotient of a regular local ring, the entire minimal free resolution of an R-module M can be obtained in a finite number of steps. This is the content of Section 3.

For *multigraded quotients of polynomial rings* there is an explicit, though rarely minimal, DG algebra resolution: the Taylor resolution. In Section 4, we use it to construct resolutions of cyclic modules over multigraded rings. Thus, we recover a result of Charalambous, who constructed by a different procedure, a free resolution of the residue field over a multigraded ring. This result provided the starting point for our investigations.

1. Resolutions

(1.1) Differential Graded algebras. Let Q be a commutative ring. For a complex of Q-modules W, we write $W^\natural$ for the underlying graded Q-module, and $|w|$ for the degree of a homogeneous element in W. We denote a quasi-isomorphism from W to Y by $W \xrightarrow{\simeq} Y$.

A DG algebra over Q is a non-negative complex $X = (X, \partial)$ with $X^\natural$ a graded Q-algebra such that the Leibniz formula $\partial(x_1x_2) = (\partial x_1)x_2 + (-1)^{|x_1|}x_1(\partial x_2)$ holds for all $x_1, x_2 \in X$. A homomorphism $\varphi : (X, \partial) \to (X', \partial')$, of DG algebras is a homomorphism of the underlying graded algebras with $\partial'\varphi = \varphi\partial$. For a DG algebra X, we denote the complex of Q modules $\operatorname{Coker}(Q \to X)$ by $\overline{X}$.

A DG module Y over a DG algebra X is a complex (Y, ∂) with $Y^\natural$ a $X^\natural$-module and such that $\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)$ for all $x \in X$ and $y \in Y$.

The relevance of DG structures as above in the construction of resolutions comes from the following result, cf. [4, Proposition 1.2.5], [6].

For a Q-algebra R and an R-module M, there are Q-projective resolutions X and Y, of R and M respectively, such that X has a structure of a DG Q-algebra and Y has a structure of a DG module over X. Furthermore, X and Y may be chosen to satisfy the following conditions:

1. $\overline{X}_0$ *is a projective Q-module.*
2. $X_i = 0$ *and* $Y_i = 0$ *for* $i > \max\{\mathrm{pd}_Q R, \mathrm{pd}_Q M\}$.
3. *If R (respectively M) admits a resolution by finitely generated projectives, then for each i, the Q-module X_i (respectively Y_i) is finitely generated.* □

The following theorem was suggested by Avramov.

(1.2) Theorem. *Let R be a Q-algebra, and M an R-module.*

Let X be a Q-projective DG algebra resolution of R with $\overline{X}_0$ a projective Q-module. Let Y be an a Q-projective resolution of M such that Y is a DG module over X.

There is a projective resolution $(\mathrm{F}(X,Y), \partial)$ of M as an R-module with

$$F_n(X,Y) = \bigoplus_{p+i_1+\cdots+i_p+j=n} R \otimes \overline{X}_{i_1} \otimes \cdots \otimes \overline{X}_{i_p} \otimes Y_j,$$

and $\partial = \partial' + \partial''$*, where*

$$\begin{aligned}\partial'(x_1 \otimes \cdots \otimes x_p \otimes y) = & - \sum_{i=1}^{p} (-1)^{e_{i-1}} x_1 \otimes \cdots \otimes \partial_{\overline{X}}(x_i) \otimes \cdots \otimes x_p \otimes y \\ & + (-1)^{e_p} x_1 \otimes \cdots \otimes x_p \otimes \partial_Y(y)\end{aligned}$$

$$\begin{aligned}\partial''(x_1 \otimes \cdots \otimes x_p \otimes y) = & \sum_{i=1}^{p-1} (-1)^{e_i} x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_p \otimes y \\ & - (-1)^{e_{p-1}} x_1 \otimes \cdots \otimes x_{p-1} \otimes x_p y\end{aligned}$$

with $e_i = i + |x_1| + \cdots + |x_i|$.

The proof is given in (1.5). It uses bar constructions, which we recall following Mac Lane, [20, Chapter X, §10].

(1.3) Bar Constructions. Let X be a DG algebra, and let W and Y be DG X-modules. A complex of Q-modules $\beta^Q(W, X, Y)$ is defined as follows.

The underlying graded Q-module has

$$\beta_n^Q(W, X, Y) = \bigoplus_{p+k+i_1+\cdots+i_p+j=n} W_k \otimes \overline{X}_{i_1} \otimes \cdots \otimes \overline{X}_{i_p} \otimes Y_j.$$

It is generated by elements $w \otimes x_1 \otimes \cdots \otimes x_p \otimes y = w[x_1|\ldots|x_p]y$. The differential on $\beta^Q(W, X, Y)$ is given by $\partial = \partial' + \partial''$, where

$$\begin{aligned}\partial'(w[x_1|\ldots|x_p]y) = & \,\partial_W(w)[x_1|\ldots|x_p]y \\ & - \sum_{i=1}^{p} (-1)^{e_{i-1}} w[x_1|\ldots|\partial_X(x_i)|\ldots|x_p]y \\ & + (-1)^{e_p} w[x_1|\ldots|x_p]\partial_Y(y)\end{aligned}$$

with $e_i = i + |w| + \sum_{j=1}^{i} |x_j|$

$$\begin{aligned}\partial''(w[x_1|\ldots|x_p]y) = & \,(-1)^{e_0}(wx_1)[x_2|\ldots|x_p]y \\ & + \sum_{i=1}^{p-1} (-1)^{e_i} w[x_1|\ldots|x_i x_{i+1}|\ldots|x_p]y \\ & - (-1)^{e_{p-1}} w[x_1|\ldots|x_{p-1}](x_p y)\end{aligned}$$

The complex of Q-modules $\beta^Q(X, X, Y)$ is a DG module over X, with operation on the left hand factor. It is called the *bar construction* of Y over the DG Q-algebra X.

(1.4) Theorem. *Let $X = (X, \partial)$ be a Q-projective DG algebra resolving R, and such that $\overline{X}_0$ is a projective Q-module. Let $Y = (Y, \partial)$ be an Q-projective resolution of M such that Y is a DG X-module.*

The complex $\beta^Q(R, X, Y)$ is a projective resolution of M as an R module. If $\overline{X}$ and Y are Q-free, then $\beta^Q(R, X, Y)$ is an R-free resolution of M.

Proof. Define $\epsilon\colon \beta^Q(X, X, Y) \longrightarrow Y$ by $\epsilon(x[\,]y) = x \cdot y$ and $\epsilon(x[x_1|\dots|x_p]y) = 0$ for $p > 0$, and note that ϵ is a morphism of complexes.

Let $\pi\colon Y \longrightarrow M$ be the augmentation map, and let $\theta\colon \beta^Q(X, X, Y) \longrightarrow \beta^Q(R, X, Y)$ be the map induced by the augmentation $X \longrightarrow R$. Set $\mu = \pi\epsilon$. In the commutative diagram

$$\begin{array}{ccccc} \beta^Q(R, X, Y) & \xleftarrow{\ \theta\ } & \beta^Q(X, X, Y) & \xrightarrow{\ \epsilon\ } & Y \\ {\scriptstyle R\otimes_X \mu}\downarrow & & {\scriptstyle \mu}\downarrow & & {\scriptstyle \pi}\downarrow \\ M & = & M & = & M \end{array}$$

π is a quasi-isomorphism. Thus, to prove that $R \otimes_X \mu$ is a quasi-isomorphism, it is enough to show that ϵ and θ are quasi-isomorphisms.

The Q-linear homomorphisms

$$\begin{array}{ll} \iota\colon Y \to \beta^Q(X, X, Y) & \sigma : \beta^Q(X, X, Y) \to \beta^Q(X, X, Y) \\ \iota(y) = 1[\,]y & \sigma(x[x_1|\dots|x_p]y) = 1[x|x_1|\dots|x_p]y \quad \text{for } p \geq 0 \end{array}$$

satisfy $\epsilon\iota = 1_Y$ and $\partial\sigma + \sigma\partial = 1_B - \iota\epsilon$. It follows that $\mathrm{H}(\epsilon)$ is an isomorphism.

Filter the complex $C = \beta^Q(X, X, Y)$ by the number of bars; thus F_pC is the submodule generated by elements $x[x_1|\dots|x_i]y$ with $i \leq p$. Let $\{F_pD\}_{p\geq 0}$ be the corresponding filtration on $D = \beta^Q(R, X, Y)$. In the spectral sequences associated to the filtered complexes C and D, we have $\mathrm{E}^0_{p,*}(C) = X \otimes_Q L_p$ and $\mathrm{E}^0_{p,*}(D) = R \otimes_Q L_p$, where $L_p = \big(\overline{X}^{\otimes p} \otimes_Q Y\big)$.

Note that $\theta\colon C \longrightarrow D$ is filtration preserving, and hence yields a homomorphism

$$\mathrm{H}_q(X \otimes_Q L_p) = \mathrm{E}^1_{p,q}(C) \longrightarrow \mathrm{E}^1_{p,q}(D) = \mathrm{H}_q(R \otimes_Q L_p).$$

As $\overline{X}^{\natural}$ and $Y^{\natural}$ are Q-projective, L_p is a bounded below complex of projective modules. Thus the quasi-isomorphism $X \longrightarrow R$ induces a quasi-isomorphism $X \otimes_Q L_p \longrightarrow R \otimes_Q L_p$. So $\mathrm{E}^1_{p,*}(C) \cong \mathrm{E}^1_{p,*}(D)$, and hence $\mathrm{H}(\theta)\colon \mathrm{H}(C) \cong \mathrm{H}(D)$ is an isomorphism.

To complete the proof, note that $\beta^Q(R, X, Y)$ is a complex of projective (respectively, free) R-modules, and hence a projective (respectively, free) resolution of M. □

(1.5) *Proof of Theorem* (1.2). The formulas in (1.3) for the differentials of the bar construction show that the complex of R-modules underlying $\beta^Q(R, X, Y)$ is exactly the complex in (1.2). □

As an application of (1.5) we obtain a spectral sequence and a sufficient condition for its collapse, given in [2, (3.1) and (4.1)]. The sequence below starts on the first page, rather than the second: this is useful in characterizing Golod modules, cf. (1.7)

(1.6) Theorem. *If $Q \to R$ is a finite homomorphism of local rings, inducing the identity on the common residue field k, and M is a finite R-module, then there exists a first quadrant spectral sequence converging to $\operatorname{Tor}^R(M,k)$. The first page of the spectral sequence is the standard reduced bar construction, with*

$$\mathrm{E}^1_{p,q} = \left(\overline{\operatorname{Tor}}^Q(R,k)^{\otimes p} \otimes_k \operatorname{Tor}^Q(M,k)\right)_q$$

$$d^1_{p,*}([x_1|\dots|x_p]y) = \sum_{i=1}^{p-1}(-1)^i[x_1|\dots|x_ix_{i+1}|\dots|x_p]y + (-1)^p[x_1|\dots|x_{p-1}](x_py)$$

where $\overline{\operatorname{Tor}}^Q(R,k) = \operatorname{Coker}(k \longrightarrow \operatorname{Tor}^Q(R,k))$. The second page of the spectral sequence is

$$\mathrm{E}^2_{p,q} = \operatorname{Tor}_p^{\operatorname{Tor}^Q(R,k)}(k, \operatorname{Tor}^Q(M,k))_q.$$

If the minimal Q-free resolution of R has a structure of a DG algebra, and the minimal Q-free resolution of M has a DG module structure over it, then the spectral sequence collapses with $\mathrm{E}^2 = \mathrm{E}^\infty$.

Proof. Choose a DG Q-algebra X and a DG X-module Y satisfying the hypothesis of (1.4). The complex $\beta^Q(R,X,Y)$ is then an R-projective resolution of M, and $\operatorname{Tor}^R(M,k) = \mathrm{H}(C)$, where

$$C = k \otimes_R \beta^Q(R,X,Y) = \beta^k\big(k, (X\otimes k), (Y \otimes k)\big).$$

Consider the spectral sequence associated to the filtration of C defined by the number of bars. Noting that $\mathrm{H}(X\otimes k) = \operatorname{Tor}^Q(R,k)$ and $\mathrm{H}(Y \otimes k) = \operatorname{Tor}^Q(M,k)$, the Künneth formula gives the E^1 page. As k is a field, the expression for $\mathrm{E}^2_{p,q} = [\mathrm{H}_p(\mathrm{E}^1)]_q$ results from [20, Chapter X, (8.2)].

To complete the proof: if X and Y are minimal, then $X \otimes k$ and $Y \otimes k$ have zero differential, and hence $\mathrm{E}^2 = \mathrm{E}^\infty$. □

(1.7) Golod modules. Let $\pi\colon Q \to R$ be as in (1.6). For a finitely generated R-module M, the Poincaré series of M is the formal power series $P^R_M(t) = \sum_{i\geqslant 0}\beta_i t^i$, where the i'th Betti number β_i is the rank of the i'th free module in a minimal free resolution of M.

Let X be a DG algebra and Y a DG module over X satisfying the hypothesis of Theorem (1.4). The convergence of the spectral sequence in (1.6) gives

$$\sum_{p+q=n} \dim_k \mathrm{E}^\infty_{p,q} \le \sum_{p+q=n} \dim_k \mathrm{E}^1_{p,q} \le \sum_{p+q=n} \dim_k \mathrm{E}^0_{p,q}\,.$$

These numerical inequalities yield coefficientwise inequalities of formal power series

$$P^R_M(t) \preccurlyeq \frac{P^Q_M(t)}{1 - t(P^Q_R(t) - 1)} \preccurlyeq \frac{\sum_{i\ge 0}(\operatorname{rank}_Q Y_i)t^i}{1 - t(\sum_{i\ge 0}(\operatorname{rank}_Q X_i)t^i - 1)}.$$

A module M for which equality holds on the left is called π-Golod by Levin [19]. This is equivalent to the condition that the spectral sequence in (1.6) collapses on the first page. If the R-module k is π-Golod, then π is said to be a Golod homomorphism.

Note that equality holds on the right if and only if X and Y are minimal over Q. We are now in a position to prove the following

(1.8) Proposition. *Let X and Y be minimal Q-free resolutions of R and M respectively, and set $X_+ = \operatorname{Ker}(X \to k)$. The following conditions are equivalent:*

(i) *M has a minimal resolution of the form $\mathrm{F}(X,Y)$.*
(ii) *X has a DG algebra structure with $X_+ \cdot X_+ \subset \mathfrak{m}X_+$, and Y has a structure of a DG X-module with $X_+ \cdot Y \subset \mathfrak{m}Y$.*
(iii) *X has a structure of a DG algebra; Y has a structure of a DG X-module, and M is π-Golod.*

Proof. The equivalence of (i) and (ii) follows from the expression for the differential on $\mathrm{F}(X,Y)$, given in (1.2).

Assume that X and Y have the appropriate DG structures. The resolution $\mathrm{F}(X,Y)$ is minimal over R if and only if both inequalities of formal power series above become equalities. By (1.7), as X and Y are minimal, this is equivalent to the condition that M is π-Golod. Thus, (i) $\iff$ (iii). □

2. Existence of multiplicative structures

Let Q be a noetherian local ring and I an ideal in Q. When I is an generated by a regular sequence, the Koszul complex resolving Q/I has a DG algebra structure. This raises the question, first asked by Buchsbaum and Eisenbud [8], as to whether *the minimal resolution of Q/I admits a DG algebra structure* in general. They give a positive answer when $\operatorname{pd}_Q Q/I \leq 3$. When $\operatorname{pd}_Q Q/I = 4$ and Q/I is Gorenstein, such an answer was obtained by Kustin and Miller, in [17], [16].

A related question, also raised in [8], is the following: Let J be an ideal generated by a regular sequence contained in I. Suppose that X is the Koszul complex resolving Q/J. *Does the minimal resolution Y of Q/I over Q have a structure of a DG X-module?* This is a weaker question: Indeed, if Y has a DG algebra structure, then by the universal property of Koszul complexes, it has a structure of a DG X-module. Thus, in view of the discussion above, Y has a DG module structure over X when $\operatorname{pd}_Q Q/I \leq 3$ or $\operatorname{pd}_Q Q/I = 4$ and Q/I is Gorenstein.

On the other hand Srinivasan [25] gives an example of an ideal I such that the minimal resolution Y does not admit a structure of a DG algebra, but for every ideal $J \subseteq I$ generated by a Q-regular sequence, Y has a DG module structure over the Koszul complex resolving Q/J.

In the light of these results, we begin with the following proposition, the proof of which uses an idea from [8, (1.3)]:

(2.1) Proposition. *Let $\pi\colon Q \to R$ be a surjective homorphism of rings, and let X be a Q-free DG algebra resolution of R, with $X_0 = Q$. If Y is a Q-free resolution of M such that $Y_i = 0$ for $i > 2$, then Y has a structure of a DG module over X.*

Proof. Let $\theta\colon R \otimes M \to M$ be the structure map given by $\theta(r \otimes m) = rm$. By the classical comparison theorem for resolutions, there is a morphism of complexes $\mu\colon X \otimes Y \to Y$ such that the following diagram, where the vertical maps are the canonical augumentations, commutes:

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{\ \mu\ } & Y \\ \big\downarrow & & \big\downarrow{\simeq} \\ R\otimes M & \xrightarrow{\ \theta\ } & M \end{array}$$

Furthermore, we can choose μ to extend the canonical isomorphism $Y = X_0 \otimes Y \xrightarrow{\cong} Y$.

Define the action of X on Y by $x \cdot y = \mu(x \otimes y)$ for $x \in X$ and $y \in Y$. The only thing that needs to be proved is that the action is associative, that is, $x_1 \cdot (x_2 \cdot y) = (x_1 x_2) \cdot y$ for $x_1, x_2 \in X$ and $y \in Y$. By construction $X_0 \otimes Y \longrightarrow Y$ is the identity map on Y, and hence, by degree considerations, we need only deal with $x_1, x_2 \in X_1$ and $y \in Y_0$.

In this case, $x_1 \cdot (x_2 \cdot y)$ and $(x_1 x_2) \cdot y$ are in Y_2. As the differential of Y is injective on Y_2, it is enough to prove that $\partial(x_1 \cdot (x_2 \cdot y)) = \partial((x_1 x_2) \cdot y)$. A direct computation shows that

$$\partial(x_1 \cdot (x_2 \cdot y)) = \partial(x_1)\mu(x_2 \otimes y) - \partial(x_2)\mu(x_1 \otimes y) = \partial((x_1 x_2) \cdot y),$$

thus proving associativity. □

In attempting to extend (2.1), one has to contend with the following result proved by Avramov, cf. [2]

Let Q be an arbitrary local ring and m and n be integers subject to the conditions $\operatorname{depth} Q \geq n \geq m \geq 2$. *If $n \geq 4$, then there exists a perfect ideal I in Q of grade n, and an ideal J in I generated by Q-regular sequence of length m, for which the minimal resolution of $M = Q/I$ does not admit a DG module structure over the Koszul complex resolving $R = Q/J$. If $n \geq 6$, then the ideal I can be chosen to be Gorenstein.*

In addition, a recent paper of Srinivasan [26] contains an example of a *grade 5 Gorenstein ideal I*, and an ideal J generated by a regular sequence in I, such that *the minimal resolution of Q/I does not admit a structure of a DG module* over the Koszul complex resolving Q/J.

The following theorem shows that (2.1) cannot be extended to length 3 resolutions:

(2.2) Theorem. *Let Q be an arbitrary local ring and let n be an integer such that* $\operatorname{depth} Q \geq n \geq 3$. *There exists a perfect Q-module L of grade n, and a non-zero divisor $f \in Q$ in the annihilator of L, for which the minimal resolution of L does not admit a DG module structure over any DG algebra resolving $R = Q/(f)$.*

The proof of the theorem is given in (2.6). It uses some facts about obstructions to DG structure on minimal resolutions, which we now recall.

(2.3) Obstructions. Let $\pi\colon Q \longrightarrow R$ be a finite local homomorphism, inducing the identity map on their common residue field k, and let M be an R module. Denote by $\operatorname{Tor}^Q_+(R,k)$ the kernel of the composition $\operatorname{Tor}^Q(R,k) \longrightarrow \operatorname{Tor}^Q_0(R,k) = R \otimes k \longrightarrow k$. In [2] Avramov defines a graded k-module $o^\pi(M)$, called the obstruction to the existence of multiplicative structure, as the kernel of the canonical map of graded k-spaces

$$\frac{\operatorname{Tor}^Q(M,k)}{\operatorname{Tor}^Q_+(R,k) \cdot \operatorname{Tor}^Q(M,k)} \longrightarrow \operatorname{Tor}^R(M,k).$$

Supposing that the minimal Q-free resolution X of R has a structure of a DG algebra, he proves the following results:

If $o^\pi(M) \neq 0$, then the minimal Q-free resolution of M does not admit a structure of a DG module over X, cf. [2, (1.2)].

When $Q = k[x_1, x_2, x_3, x_4]$ and $R = Q/(x_1^2, x_4^2)$, Avramov shows that $o^\pi(M) \neq 0$ for $M = Q/(x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4^2)$. Using this example, and the functorial properties of $o^\pi(-)$ established in [2], he constructs a grade 4 perfect module with a non-trivial obstruction.

We construct a grade 3 example by a similar procedure.

(2.4) Construction. Over the polynomial ring $Q' = k[a, b, c]$ the complex

$$G\colon \quad 0 \to Q' \xrightarrow{\begin{pmatrix} -b \\ a \\ -c \end{pmatrix}} Q'^3 \xrightarrow{\begin{pmatrix} 0 & c^2 & ac \\ 0 & 0 & 0 \\ c^2 & 0 & -bc \\ 0 & -c & -a \\ -a & -b & 0 \end{pmatrix}} Q'^5 \xrightarrow{\begin{pmatrix} b & 0 & a & 0 & c^2 \\ 0 & c & 0 & 0 & 0 \\ c & a & 0 & c^2 & 0 \end{pmatrix}} Q'^3$$

is exact and minimal. Consider the multigraded module $N' = \operatorname{Coker}(Q'^5 \to Q'^3)$, which is generated by elements e_1, e_2, e_3 with multidegrees

$$||e_1|| = (0, -1, 1) \quad ||e_2|| = (1, 0, -1) \quad ||e_3|| = (0, 0, 0).$$

Note that the element ce_1 is non-zero and generates the socle of N'. The element c^2 annihilates N' and so N' is a multigraded $R' = Q'/(c^2)$-module. Let K be the Koszul complex on $\{a, b, c\}$ with generators A, B, C. For a Q'-module P we identify $\operatorname{Tor}^{Q'}(P, k)$ with $\mathrm{H}(P \otimes K)$, and denote homology classes by $\operatorname{cls}(-)$.

The k-vector space $\operatorname{Tor}_1^{Q'}(R', k)$ is generated by $w = \operatorname{cls}(cC)$.

On the other hand, $\operatorname{Tor}_2^{Q'}(N', k)$ is generated by

$$z_1 = \operatorname{cls}(ce_1 \otimes (A \wedge C)) \quad z_2 = \operatorname{cls}(ce_1 \otimes (B \wedge C)) \quad z_3 = \operatorname{cls}(ce_3 \otimes (A \wedge C))$$

Indeed, a direct computation shows that these elements are cycles. On the other hand, the boundaries in $(N' \otimes K)_2$ are elements of the form $m \otimes (aB \wedge C - bA \wedge C + cA \wedge B)$, with $m \in N'$. Comparing multidegrees, one easily sees that z_1, z_2, z_3 are linearly independent. Furthermore $\operatorname{Tor}_2^{Q'}(N', k) = G_3 \otimes k \cong k^3$; hence $\{z_1, z_2, z_3\}$ is a basis for $\operatorname{Tor}_2^{Q'}(N', k)$.

The action of $\operatorname{Tor}^{Q'}(R', k)$ on $\operatorname{Tor}^{Q'}(N', k)$ can be computed from the natural DG module structure of $(N' \otimes K)$ over the DG algebra $(R' \otimes K)$. Using the bases above, we immediately get that

$$\operatorname{Tor}_1^{Q'}(R', k) \cdot \operatorname{Tor}_2^{Q'}(N', k) = (wz_1, wz_2, wz_3) = (0)\ .$$

Thus, to prove that $o^\pi(N') \neq 0$, where $\pi\colon Q' \to R'$ is the canonical surjection, it is enough to exhibit a non-zero element in $\operatorname{Ker}(\operatorname{Tor}_3^{Q'}(N', k) \to \operatorname{Tor}_3^{R'}(N', k))$.

Consider the cycle $u = ce_1 \otimes (A \wedge B \wedge C) \in N' \otimes K$. Let $K' = R' \otimes K$ be the Koszul complex over R', and let $R'\langle S\rangle$ denote the divided powers algebra, with $|S| = 2$. Recall that $R'\langle S\rangle$ is a free module on basis elements $\{S^{(i)}\}_{i\geq 0}$, with $|S^{(i)}| = 2i$, and has a multiplication given by $S^{(i)} \cdot S^{(j)} = S^{(i+j)}$. As R' is a hypersurface, the complex $F = K' \otimes R'\langle S\rangle$ with differential defined by

$$\partial(x \otimes S^{(n)}) = \partial(e) \otimes S^{(n)} + (-1)^{|x|}(x \wedge cC) \otimes S^{(n-1)}$$

is an R'-free resolution of k, cf.[27, Theorem 4]. The map $\operatorname{Tor}^{Q'}(N', k) \longrightarrow \operatorname{Tor}^{R'}(N', k)$ is induced by the natural inclusion $N' \otimes_{Q'} K = N' \otimes_{R'} K' \longrightarrow N' \otimes_{R'} F$. A routine computation shows that

$$u = \partial\big(e_1 \otimes (A \wedge B) \otimes S + e_2 \otimes S^{(2)} + e_3 \otimes (A \wedge C) \otimes S\big)\ .$$

Hence u is a boundary in $N' \otimes_{R'} F$, and $\operatorname{cls}(u) \in \operatorname{Ker}(\operatorname{Tor}_3^{Q'}(N', k) \longrightarrow \operatorname{Tor}_3^{R'}(N', k))$.

Note that the maximal degree of a generator in the minimal resolution of N' is 4. Thus, by [2, (1.5g)], we get that $o^\pi(M') \neq 0$, where $M' = N'/\mathfrak{m}^5 N'$.

The Q'-module M' has finite length, and it is perfect with $\operatorname{pd}_{Q'} M = 3$. For completeness, we note that the computer program MACAULAY gives for M' the following Betti numbers: $3, 18, 27, 12$.

Following a method of Avramov [2, (2.5)], we use (2.4) above to prove

(2.5) Proposition. *Let (Q, m, k) be a local ring and x, y, z be a Q-regular sequence. The Q-module M presented by the 3×18 matrix*

$$\begin{pmatrix} y & 0 & x & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & x^5 & 0 & 0 & x^4y & x^3y^2 & x^2y^3 & xy^4 & y^5 & x^4z & x^3yz & x^2y^2z & xy^3z & y^4z \\ z & x & 0 & 0 & z^2 & 0 & y^5 & y^4z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is perfect with $\operatorname{pd}_Q M = 3$. Furthermore, $z^2M = 0$ and $o^\pi(M) \neq 0$, where $\pi\colon Q \longrightarrow R = Q/(z^2)$ is the natural surjection.

Proof. Denote by $\widetilde{Q}$ the localization of the polynomial ring $Q[a, b, c]$ at the maximal ideal $\mathfrak{m} + (a, b, c)$, and by Q' the localization of $k[a, b, c]$ at (a, b, c). Consider the commutative diagram:

$$\begin{array}{ccccc} Q & \xleftarrow{\ \phi\ } & \widetilde{Q} & \xrightarrow{\ \theta\ } & Q' \\ \downarrow & & \downarrow & & \downarrow \\ R & \longleftarrow & \widetilde{Q}/(c^2) & \longrightarrow & Q'/(c^2) \end{array}$$

where the squares are tensor product diagrams, θ is factorization by $\mathfrak{m}Q$ and ϕ is the unique map of Q-algebras with $\phi(a) = x, \phi(b) = y$, and $\phi(c) = z$. Let $\widetilde{M}$ be the module over $\widetilde{Q}$ presented by 3×18 matrix above, with x, y, z substituted by a, b, c. Set $M = \widetilde{M} \otimes Q$ and $M' = \widetilde{M} \otimes Q'$, and note that $\{Q', Q'/(c^2), M'\}$ is the triple constructed in (2.4).

As the entries in the presentation for $\widetilde{M}$ are monomials in the variables, and x, y, z is a Q-regular sequence, it is easily seen that, for $i > 0$,

$$\begin{aligned} \operatorname{Tor}_i^{\widetilde{Q}}(\widetilde{M}, Q) = 0 \quad &\text{and} \quad \operatorname{Tor}_i^{\widetilde{Q}}(\widetilde{Q}/(c^2), Q) = 0 \\ \operatorname{Tor}_i^{\widetilde{Q}}(\widetilde{M}, Q') = 0 \quad &\text{and} \quad \operatorname{Tor}_i^{\widetilde{Q}}(\widetilde{Q}/(c^2), Q') = 0. \end{aligned} \tag{2.5.1}$$

For the map $\overline{\pi}\colon Q' \to Q'/(c^2)$, we have $o^{\overline{\pi}}(M') \neq 0$ by (2.4). Applying the flat base change property of the obstruction, cf. [2, (1.4)], first to the homomorphism θ and then to the homomorphism ϕ, we deduce that $o^{\pi}(M) \neq 0$.

It remains to be seen that M is perfect of grade 3. From (2.5.1) we get $\operatorname{pd}_Q M = \operatorname{pd}_{\widetilde{Q}} \widetilde{M} = \operatorname{pd}_{Q'} M' = 3$. From the presentation for $\widetilde{M}$, it is clear that (a^5, b^5, c^5) annihilates $\widetilde{M}$, and hence that (x^5, y^5, z^5) annihilates M. As x^5, y^5, z^5 is a Q-regular sequence, we see that $\operatorname{grade} M \geq 3 = \operatorname{pd}_Q M$. Thus, M is perfect. □

(2.6) *Proof of Theorem* (2.2). Suppose that x, y, z is a Q-regular sequence. Let M and π be as in (2.5) and set $f = z^2$. As $\operatorname{depth} Q \geq n \geq 3$ and $\operatorname{depth} M = \operatorname{depth} Q - 3$, there exists a Q-regular sequence $a_1, \ldots, a_{n-3}$ which is also M-regular.

Set $L = M \otimes Q/(a_1, \ldots, a_{n-3})$. Note that L is perfect and also that:

$$\begin{aligned} \operatorname{Tor}^Q(L, k) &= \operatorname{Tor}^Q(M, k) \otimes_k \Lambda \\ \operatorname{Tor}^R(L, k) &= \operatorname{Tor}^R(M, k) \otimes_k \Lambda \end{aligned}$$

where $\Lambda = \operatorname{Tor}^Q(R/(a_1, \ldots, a_{n-3}), k)$ is the exterior algebra on the vector space Λ_1. By naturality we have $o^{\pi}(L) = o^{\pi}(M) \otimes_k \Lambda$, and hence (2.5) shows that $o^{\pi}(L) \neq 0$.

As recalled in (2.3) above, the non-triviality of the obstruction implies that the minimal Q-resolution Y of L has no DG module structure over the Koszul complex K resolving R.

If X is a DG algebra resolving R, then, by the universal property of Koszul complexes, there is a homomorphism of DG algebras $K \to X$. Thus, if Y is a DG module over X, then Y has a structure of a DG module over K, which is a contradiction. □

3. Minimal resolutions

In this section $(Q, \mathfrak{m}, k)$ is a commutative noetherian local ring, $\pi\colon Q \twoheadrightarrow R$ is a surjective homomorphism, and M is a finite R-module. We use (1.2) to construct the minimal R-free resolution of a d'th syzygy of M, where d is known *a priori.*

(3.1) The module M has a minimal R-free resolution

$$\ldots \to G_d \xrightarrow{\partial_d} G_{d-1} \xrightarrow{\partial_{d-1}} \ldots \xrightarrow{\partial_1} G_0$$

which is unique up to an isomorphism. Thus M_d, the image of ∂_d is well defined and is called the d'th syzygy module of M over R.

For $d > \max\{\operatorname{depth} R - \operatorname{depth} M, 0\} + 1$ the R-module M_d has no free direct summands, cf. [3, (4.7)]. Furthermore, $\operatorname{depth} M_d = \operatorname{depth} R$, by [22, Proposition 10], and hence $\operatorname{pd}_Q M_d = \operatorname{pd}_Q R$ if $\operatorname{pd}_Q R$ and $\operatorname{pd}_Q M$ are finite.

(3.2) Theorem. *Assume that* $\operatorname{pd}_Q R = 1$ *and that* $\operatorname{pd}_Q M$ *is finite.*

For $d \geq \operatorname{depth} R - \operatorname{depth} M + 2$*, the dth syzygy module* M_d*, of* M *over* R*, has a minimal free resolution* Y *with a DG module structure over the Koszul complex* X *resolving* R*.*

The free resolution of $\mathrm{F}(X,Y)$ *of* M_d *given by Theorem* (1.2) *is minimal.*

Remark. The resolution described above is the *periodic of period 2* minimal resolution constructed by Shamash [24] and Eisenbud [10].

Proof. The Koszul complex $X\colon 0 \longrightarrow Qe \longrightarrow Q \longrightarrow 0$, where $\partial(e) = x$ is a non-zero divisor, is a free resolution of R. It has a DG algebra structure with $e \cdot e = 0$.

If $x \notin \mathfrak{m}^2$, then by a theorem of Nagata [21, (27.4)] we have $\operatorname{pd}_R M = \operatorname{pd}_Q M - 1$; hence $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth} M$ and $M_d = 0$.

Assume $x \in \mathfrak{m}^2$. By (3.1), we have $\operatorname{pd}_Q M_d = \operatorname{pd}_Q R$ and so M_d has a minimal free resolution

$$Y : 0 \longrightarrow Y_1 \xrightarrow{\xi} Y_0 \longrightarrow 0 \quad \text{with} \quad Y_0 \cong Q^n \quad \text{and} \quad Y_1 \cong Q^n.$$

As M_d is an R-module, multiplication by x on Y is homotopic to zero. Choose one such homotopy σ, and note that for degree reasons $\sigma^2 = 0$. Setting $e \cdot y = \sigma(y)$, for $y \in Y$, gives Y the structure of a DG X-module. With $e^{(n)} = e \otimes \cdots \otimes e$ (n copies), the free resolution $F = \mathrm{F}(X,Y)$ given by Theorem (1.2) is:

$$\cdots \longrightarrow Re^{(n)} \otimes Y_1 \longrightarrow Re^{(n)} \otimes Y_0 \longrightarrow Re^{(n-1)} \otimes Y_1 \longrightarrow \cdots$$

with maps $e^{(n)} \otimes y_0 \mapsto e^{(n-1)} \otimes \sigma(y_0)$ and $e^{(n)} \otimes y_1 \mapsto e^{(n-1)} \otimes \xi(y_1)$ for $y_i \in Y_i$. Setting $\overline{Y}_i = Re^{(n)} \otimes Y_i$, the R-free resolution of M_d takes the form

$$\cdots \longrightarrow \overline{Y}_1 \xrightarrow{\overline{\xi}} \overline{Y}_0 \xrightarrow{\overline{\sigma}} Y_1 \longrightarrow \cdots \longrightarrow \overline{Y}_1 \xrightarrow{\overline{\xi}} \overline{Y}_0$$

where $\overline{\xi} = R \otimes \xi$, and $\overline{\sigma} = R \otimes \sigma$.

As Y is minimal $\overline{\xi}(\overline{Y}_1) \subset \mathfrak{m}\overline{Y}_0$. To show that the periodic resolution is minimal, we need to check that $\overline{\sigma}(\overline{Y}_0) \subset \mathfrak{m}\overline{Y}_1$. This follows from the fact that M_d has no free direct summands by an arugument of Eisenbud [10,(6.1)], which we reproduce for completion. As F is exact, $\overline{\sigma}(\overline{Y}_0) \cong \operatorname{Coker}(\overline{\xi}) \cong M_d$. If $\overline{\sigma}(\overline{Y}_0)$ were not contained in $\mathfrak{m}\overline{Y}_1$, then $\overline{\sigma}(\overline{Y}_0)$ would contain a basis element of $\overline{Y}_1$, so $\overline{\sigma}(\overline{Y}_0)$ would have a R-free direct summand. Since M_d has not free direct summands, this concludes the proof. □

(3.3) If $\operatorname{pd}_Q R = 2$, then by the Hilbert-Birch theorem R has a minimal free resolution

$$X : 0 \longrightarrow \bigoplus_{j=1}^{n} Qf_j \xrightarrow{\phi} \bigoplus_{i=1}^{n+1} Qe_i \xrightarrow{\theta} Q \xrightarrow{\pi} R \longrightarrow 0$$

with $\theta(e_i) = (-1)^i a\phi_i$, where a is a non-zero divisor and for $I, J \subset \mathbb{N}$, the element ϕ_I^J is the minor obtained by deleting the rows indexed by I and columns indexed by J.

Herzog [14] has shown that the formulas

$$e_i \cdot e_j = \begin{cases} 0 & \text{if } i = j; \\ -a \sum_{k=1}^{n} (-1)^{i+j+k} \phi_{i,j}^{k} f_k & \text{if } i < j; \\ -e_j \cdot e_i & \text{if } i > j. \end{cases}$$

define on X a structure of a commutative DG algebra.

(3.4) Theorem. *Assume that* $\operatorname{pd}_Q R = 2$ *and that* $\operatorname{pd}_Q M$ *if finite.*

Let X *be the free resolution of* R *with the DG algebra structure as above. For* $d \geq \operatorname{depth} R - \operatorname{depth} M + 2$*, the d'th syzygy module* M_d*, of* M *over* R*, has a free resolution* Y *with a DG module structure over* X*.*

The free resolution $\mathrm{F}(X, Y)$ *of* M_d *given by Theorem* (1.2) *is minimal if and only if the kernel of* $\pi\colon Q \longrightarrow R$ *is a not generated by a regular sequence.*

Remark. For the case not covered by the theorem, that is when $\ker(\pi)$ is generated by a regular sequence, Avramov and Buchweitz [5] build tail ends of minimal resolutions using different techniques.

Proof. By the assumption on d and (3.1), we have $\operatorname{depth} M_d = \operatorname{depth} R$. Thus the minimal free resolution of M_d is

$$Y : 0 \longrightarrow Y_2 \xrightarrow{\zeta} Y_1 \xrightarrow{\xi} Y_0 \longrightarrow 0,$$

and has a structure of a DG module over X by (2.1).

If $\ker(\pi)$ is generated by a regular sequence, then X is a Koszul complex and hence $X_+ \cdot X_+ \not\subset \mathfrak{m}X_+$. Thus, condition (ii) of (1.9) fails to hold and $\mathrm{F}(X, Y)$ cannot be minimal.

If $\ker(\pi)$ is not generated by a regular sequence, then π is a Golod homorphism. cf. [1, (7.1)]. As the minimal resolutions X and Y have the appropriate DG structures, if M_d is π-Golod, then $\mathrm{F}(X, Y)$ is minimal by (1.9).

To prove that M_d is π-Golod, it is enough, by [19, (1.1)], to show that $\operatorname{Tor}_i^Q(M_d, k) \longrightarrow \operatorname{Tor}_i^R(M_d, k)$ is injective for $i \geq 1$. Lescot [18, Lemma, p. 43] proves that if π is a Golod homomorphism and $\operatorname{Tor}_i^Q(M, k) \longrightarrow \operatorname{Tor}_i^R(M, k)$ is injective for all $i \geq p$, then $\operatorname{Tor}_i^Q(M_{p-1}, k) \longrightarrow \operatorname{Tor}_i^R(M_{p-1}, k)$ is injective for all $i \geq 1$. As $\operatorname{pd}_Q M$ is finite, this applies to $p = \operatorname{pd}_Q M + 1 = d + 1$. □

(3.5) Example. Let $Q = k[x, y]$, for a field k. Set $R = Q/(x^2y, xy^2)$ and $M = R/(x^2, y^2)$. We have $\operatorname{depth} M = \operatorname{depth} R = 0$. Thus $d = 2$, and the second syzygy module is the direct sum of $M_2' = k[x, y]/(x^2, xy)$ and $M_2'' = k[x, y]/(y^2, xy)$. The Q-free resolution of M_2' is

$$Y : 0 \longrightarrow Qc \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} Qb_1 \oplus Qb_2 \xrightarrow{(\,x^2 \quad xy\,)} Qa \longrightarrow 0$$

By (3.3), the multiplication on X is defined by $e_1 \cdot e_2 = -xyf = -e_2 \cdot e_1$. Following through the proof of (2.1) one sees that the action of X on Y is given by:

$$e_1 \cdot a = yb_1 \qquad e_1 \cdot b_2 = xyc \qquad e_2 \cdot a = yb_2 \qquad e_2 \cdot b_1 = -xyc \qquad f \cdot a = yc$$

Now, Theorem (1.2) gives the beginning of the resolution of $k[x, y]/(x^2, xy)$ over R:

$$\cdots \longrightarrow R^5 \xrightarrow{\begin{pmatrix} -y & 0 & xy & -xy & 0 \\ y & x^2 & xy & 0 & 0 \\ -x & 0 & 0 & x^2 & xy \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & y & 0 \\ x & 0 & y \end{pmatrix}} R^2 \xrightarrow{(\,x^2 \quad xy\,)} R\ .$$

The resolution of $k[x, y]/(y^2, xy)$ is obtained by interchanging x and y. The Poincaré series of M is $(1 + t - t^2)(1 - t - t^2)^{-1}$.

4. Monomial quotients of monomial rings.

Let k be a field and $Q = k[x_1, \dots, x_n]$ the polynomial ring with the natural n-grading, and unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. If $\mathfrak{a}$ is an ideal generated by monomials $(a_1, \dots, a_s)$, then the quotient ring $R = Q/\mathfrak{a}$ is n-graded with the multigrading induced from Q.

(4.1) Taylor resolutions. Define X to be the graded Q-module $\Lambda \oplus_{i=1}^{s} Qf_i$, with X_p free of rank $\binom{s}{p}$. For $I = \{i_1, \dots, i_p\}$ set $f_I = f_{i_1} \wedge \cdots \wedge f_{i_p}$ and a_I to be the least common multiple of the monomials indexed by I. The Taylor resolution, cf. [28], of R over Q is the graded module X with differential

$$\partial(f_I) = \sum_{k=1}^{p} (-1)^k \frac{a_I}{a_{\{I \setminus i_k\}}} f_{\{I \setminus i_k\}}.$$

As proved by Gemeda [13], cf. also [12], the formula

$$f_{I_1} \cdot f_{I_2} = \gcd(a_{I_1}, a_{I_2}) f_{I_1} \wedge f_{I_2}.$$

defines a structure of a DG algebra on the Taylor resolution.

(4.2) Multi-homogeneous quotients. Let $\mathfrak{b}$ be an ideal generated by monomials $b_1, \dots, b_t$, and such that $\mathfrak{a} \subseteq \mathfrak{b}$. Let Y be the Taylor resolution of $Q/\mathfrak{b}$ and $h_J = h_{j_1} \wedge \cdots \wedge h_{j_q}$, where $J = \{j_1, \dots, j_q\}$.

For each i there is an integer $\nu(i)$ such that $b_{\nu(i)}$ divides a_i. For $I = \{i_1, \dots, i_p\}$, set $\nu(I) = \{\nu(i_1), \dots, \nu(i_p)\}$ and $h_{\nu(I)} = h_{\nu(i_1)} \wedge \cdots \wedge h_{\nu(i_p)}$ in Y.

The consideration of such "monomials" is suggested by the formulas for the differential in the resolution of the residue field given in [9]. A direct computation shows that

Lemma. *The canonical surjection $R = Q/\mathfrak{a} \twoheadrightarrow Q/\mathfrak{b}$ extends to a homomorphism, $\phi : X \to Y$, of DG Q-algebras defined by*

$$\phi(f_J) = \frac{a_J}{b_{\nu(J)}} h_{\nu(J)}$$

In particular, Y is a DG X-module. □

As an application of (1.2) and (4.2), we get an explicit free resolution of the R-module $M = Q/\mathfrak{b}$.

(4.3) Theorem. *With X and Y as above, set $\overline{X} = X/Q$. There is a projective resolution of M over R, which in degree n is the R-module*

$$\bigoplus_{p+i_1+\cdots+i_p+j=n} R \otimes \overline{X}_{i_1} \otimes \cdots \otimes \overline{X}_{i_p} \otimes Y_j$$

and differential $\partial = \partial' + \partial''$, *where*

$$\partial'(f_{I_1} \otimes \cdots \otimes f_{I_p} \otimes h_J) = -\sum_{i=1}^{p} (-1)^{e_{i-1}} f_{I_1} \otimes \cdots \otimes \partial_X(f_{I_i}) \otimes \cdots \otimes f_{I_p} \otimes h_J + (-1)^{e_p} f_{I_1} \otimes \cdots \otimes f_{I_p} \otimes \partial_Y(h_J)$$

$$\partial''(f_{I_1} \otimes \cdots \otimes f_{I_p} \otimes h_J) = \sum_{i=1}^{p-1} (-1)^{e_i} \gcd(a_{I_1}, a_{I_2}) \big(f_{I_1} \otimes \cdots \otimes f_{I_i} \wedge f_{I_{i+1}} \otimes \cdots \otimes f_{I_p} \otimes h_J\big) + (-1)^{e_p} \frac{a_{I_p} b_J}{b_{\{\nu(I_p) \cup J\}}} \big(f_{I_1} \otimes \cdots \otimes f_{I_{p-1}} \otimes h_{\nu(I_p)} \wedge h_J\big)$$

with $e_i = i + \operatorname{card}(I_1) + \cdots + \operatorname{card}(I_i)$. □

We have the following combinatorial criterion for the minimality of the resolution:

(4.4) Proposition. *The resolution of Theorem* (4.3) *is minimal if and only if the following conditions hold:*

(1) $a_I \neq a_{\{I \setminus i\}}$ *for* $1 \leq i \leq s$, *where* $I = \{1, \dots, s\}$.

(2) $b_J \neq b_{\{J \setminus j\}}$ *for* $1 \leq j \leq t$, *where* $J = \{1, \dots, t\}$.

(3) $\gcd(a_i, a_j) \neq 1$ *for* $1 \leq i, j \leq s$.

(4) $a_I \neq b_{\nu(I)}$ *for all* $I \subseteq \{1, \dots, s\}$.

Proof. Let I_1 and I_2 be subsets of $\{1, \dots, s\}$, and J a subset of $\{1, \dots, t\}$. Evidently, the resolution is minimal if and only if X and Y satisfy the following conditions:

(1') X is a minimal resolution.
(2') Y is a minimal resolution.
(3') $\gcd(a_{I_1}, a_{I_2}) \neq 1$.
(4') $a_{I_p} b_J \neq b_{\{\nu(I_p) \cup J\}}$.

A result of Fröberg [12, (5.1)] shows that conditions (1) and (1'), respectively, (2) and (2'), are equivalent.

Assuming that (1) holds, a direct computation shows that (3) and (3′), respectively, (4) and (4'), are equivalent. □

As a corollary of (4.3) and (4.4), we recover the main result of [9].

Corollary. *Let* $Q = k[x_1, \dots, x_n]$ *be a polynomial ring over a field* k, *and let* $\mathfrak{a}$ *be an ideal minimally generated by monomials* $a_1, \dots, a_s$. *With* X *the Taylor resolution of* $R = Q/\mathfrak{a}$, *and* Y *the Koszul complex on* $\{x_1, \dots, x_n\}$, *Theorem* (4.3) *yields a free resolution of the residue field over* R.

Furthermore, if $a_I \neq a_{I \setminus i}$ *and* $\gcd(a_i, a_j) \neq 1$ *for* $i, j \in I = \{1, \dots, s\}$, *then the resolution is minimal.*

Proof. When $M = k$, the Taylor resolution is the Koszul complex on the variables $\{x_1, \dots, x_n\}$. Thus, with X and Y as above, (4.3) is an R-free resolution of the residue field k.

Under the assumptions of the corollary, conditions (1) and (3) of Proposition (4.4) are satisfied. As $\mathfrak{a} \subset (x_1, \dots, x_n)^2$, conditions (2) and (4) are easily seen to hold. Thus, the resolution is minimal. □

Remark. Recently, Backelin [7] and Eisenbud, Reeves and Totaro [11] have given iterative procedures for constructing a resolution of a multi-graded module over a monomial ring. Their purpose is to give upper bounds on the degrees of the generators in the minimal resolution, in terms of the degrees of the generators of the first syzygy module. Our aim in this paper has been to give explicit resolutions, and the price we pay is that the upper bounds we get from Theorem (4.3) are weaker.

Remark. Suppose that I, J are 0-Borel fixed ideals in a polynomial ring Q, and such that $I \subseteq (x_1, \dots, x_n)J$. One can use a recent result of Peeva [23], and Theorem (1.2) to construct the minimal resolution of the Q/I-module Q/J. This is a special case of a result in [15].

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