The Bousfield lattice of the stable module category of a finite group

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Let G be a finite group, k a field whose characteristic divides the order of G, and $\mathsf{StMod}\, kG$ the stable module category of all (and not only the finite dimensional) kG-modules, with its natural structure of a triangulated category. Benson, Krause, and I [3, 4, 6] have been investigating global structural properties of $\mathsf{StMod}\, kG$; to be precise, the classification of its localizing subcategories and its colocalizing subcategories. The aim of my talk was to cast our results in a different light, by using them to discover the structure of certain lattices naturally associated to the stable module category. For a more systematic treatment, in the context of tensor triangulated categories, see [10]. This line of development is inspired by Bousfield's work [7] in stable homotopy theory; see also [9].

For any kG-modules M, N, the k-vectorspace $M \otimes_k N$ has a diagonal kG-action:

$$g(m \otimes n) = gm \otimes gn \quad \text{for } g \in G \text{ and } m \otimes n \text{ in } M \otimes_k N.$$

This induces a tensor product on $\mathsf{StMod}(kG)$ as well.

Definition 1. The *Bousfield class* of a kG-module M is the full subcategory

$$\mathrm{A}(M) = \{X \in \mathsf{StMod}(kG) \mid M \otimes_k X = 0 \text{ in } \mathsf{StMod}(kG)\}$$

Recall that $M \otimes_k X$ is zero in $\mathsf{StMod}(kG)$ precisely when it is projective. Modules in $\mathrm{A}(M)$ are said to be *M-acyclic*, whence the notation. Modules M and N are *Bousfield equivalent* if $A(M) = A(N)$.

A basic problem is to classify kG-modules, up to Bousfield equivalence. To this end we mimic [7], and endow the collection of all Bousfield classes, $\mathrm{A}(\mathsf{StMod}\, kG)$, with the following partial order:

$$\mathrm{A}(M) \leq \mathrm{A}(N) \quad \text{if} \quad \mathrm{A}(M) \supseteq \mathrm{A}(N)\,.$$

A priori, it is not even clear that $\mathrm{A}(\mathsf{StMod}\, kG)$ is a set. That it is so, and much more, is contained in the following:

Theorem 2. *The collection $\mathrm{A}(\mathsf{StMod}\, kG)$ with partial order $\leq$ is a lattice, with supremum and infimum given by*

$$\mathrm{A}(M) \vee \mathrm{A}(N) = \mathrm{A}(M \oplus N) \quad \textit{and} \quad \mathrm{A}(M) \wedge \mathrm{A}(N) = \mathrm{A}(M \otimes_k N)\,.$$

Moreover, the lattice $\mathrm{A}(\mathsf{StMod}\, kG)$ is distributive and complete.

Assume for the moment that $\mathrm{A}(\mathsf{StMod}\, kG)$ is a set. It is clear that it is partially ordered under $\leq$. Moreover, since $- \otimes_k X$ commutes with (arbitrary) direct sums, any set $\{M_i\}$ of kG-modules has a supremum:

$$\bigvee_i \mathrm{A}(M_i) = \mathrm{A}(\bigoplus_i M_i)\,.$$

It then follows from general principles, see [8], that any subset of $\mathrm{A}(\mathsf{StMod}\, kG)$ also has a infimum; that is to say, the lattice $\mathsf{StMod}\, kG$ is complete. The non-trivial part in Theorem 2 is the explicit identification of the infimum; given that, it is clear also that the lattice is distributive.

Localizing subcategories. The tensor product on $\mathsf{StMod}\,kG$ is compatible with its structure as a triangulated category. A subcategory S is *tensor closed* if whenever M is in S so is $M \otimes_k X$ for any kG-module X. A *localizing* subcategory is a triangulated subcategory that is closed under all set-indexed coproducts. We write $\mathrm{L}(M)$ for the smallest (with respect to inclusion) tensor closed localizing subcategory of $\mathsf{StMod}\,kG$ containing M, and $\mathrm{L}(\mathsf{StMod}\,kG)$ for the collection of all such subcategories, with the (natural !) partial order:

$$\mathrm{L}(M) \leq \mathrm{L}(N) \quad \text{if} \quad \mathrm{L}(M) \subseteq \mathrm{L}(N)\,.$$

There is an analogue of Theorem 2 for this collection. There is a map of lattices from $\mathrm{L}(\mathsf{StMod}\,kG)$ and $\mathrm{A}(\mathsf{StMod}\,kG)$, the key point being the following:

Lemma 3. *If* $\mathrm{L}(M) \leq \mathrm{L}(N)$, *then* $\mathrm{A}(M) \leq \mathrm{A}(N)$. □

Corollary 8 contains the converse to the preceding lemma. Its proof uses the theory of support, which we now recall.

Support. Let $H^*(G,k)$ be the cohomology algebra, $\mathrm{Ext}^*_{kG}(k,k)$, of G. This is a k-algebra which is graded-commutative, because kG is a Hopf algebra, and also finitely generated; the last statement is due to Evens and Venkov, and the starting point of the cohomology study of modular representations of finite groups; see, for instance, [1] for details. Set

$$\mathcal{V}_G = \text{homogeneous prime ideals in } H^*(G,k), \text{ except } H^{\geqslant 1}(G,k).$$

For each $\mathfrak{p} \in \mathcal{V}_G$ Benson, Carlson, and Rickard [2] (see also [3]) construct certain idempotent exact functors on $\mathsf{StMod}\,kG$, which we denote $\varGamma_{\mathfrak{p}}$. A crucial property of these functors is that

$$\varGamma_{\mathfrak{p}} M \cong \varGamma_{\mathfrak{p}} k \otimes_k M\,.$$

The *support* of a kG-module is the subset

$$\mathrm{supp}_G\, M = \{\mathfrak{p} \in \mathcal{V}_G \mid \varGamma_{\mathfrak{p}} k \otimes_k M \neq 0\}$$

For finite dimensional modules, this coincides with the usual cohomological support; see [3]. We remark that when M is non-zero $\mathrm{supp}_G\, M$ is non-empty. The relevance of support to us is that there are maps:

$$\begin{array}{ccc} \mathrm{L}(\mathsf{StMod}\,kG) & \xleftarrow{\qquad\qquad \sigma \qquad\qquad} & \{\text{subsets of } \mathcal{V}_G\} \\ & {\scriptstyle \iota}\searrow \qquad\qquad \nearrow{\scriptstyle \tau} & \\ & \left\{ \begin{array}{c} \text{tensor closed localizing} \\ \text{subcategories of } \mathsf{StMod}\,kG \end{array} \right\} & \end{array}$$

where ι is the obvious inclusion, and τ and σ are defined as follows:

$$\tau(\mathsf{S}) = \bigcup_{M \in \mathsf{S}} \mathrm{supp}_G\, M \quad \text{and} \quad \sigma(\mathcal{U}) = \mathrm{L}\,\Big(\bigoplus_{\mathfrak{p} \in \mathcal{U}} \varGamma_{\mathfrak{p}} k\Big)$$

It is not hard to see that [4, Theorem 10.3] is equivalent to the following:

Theorem 4. *The composition of any three consecutive maps in the diagram above is the identity. In particular, the maps are all bijections.* □

From this one can deduce the 'tensor product theorem'; see [4, Theorem 11.1].

Corollary 5. *For any kG-modules M and N one has*

$$\operatorname{supp}_G(M \otimes_k N) = \operatorname{supp}_G M \cap \operatorname{supp}_G N\,.$$

In particular, $\mathrm{A}(M) = \{N \mid \operatorname{supp}_G N \cap \operatorname{supp}_G M = \varnothing\}$. □

Using this result one can prove Theorem 2 without much ado. The next corollary extends Lemma 3 and characterizes Bousfield equivalent modules.

Corollary 6. *One has* $\mathrm{L}(M) \le \mathrm{L}(N)$ *if and only if* $\mathrm{A}(M) \le \mathrm{A}(N)$*, if and only if* $\operatorname{supp}_R M \subseteq \operatorname{supp}_R N$. □

Local objects. In what follows, the set of morphisms in $\mathsf{StMod}\,kG$ between kG-modules M and N is denoted $\underline{\operatorname{Hom}}_G(M,N)$. Once again inspired by the work in [7], we consider the right orthogonal of the M-acyclic modules:

$$\mathrm{A}(M)^{\perp} = \{N \in \mathsf{StMod}\,kG \mid \underline{\operatorname{Hom}}_G(X,N) = 0 \text{ for all } X \in \mathrm{A}(M)\}\,.$$

The modules in this subcategory are said to be *M-local.* Note that the subcategory of M-local objects is equivalent to the Verdier quotient of $\mathsf{StMod}\,kG$ by $\mathrm{A}(M)$. Again, one is faced with the problem of classifying such subcategories. To address it, we consider the right adjoint $\varLambda^{\mathfrak{p}} = \operatorname{Hom}_k(\varGamma_{\mathfrak{p}} k, -)$ to $\varGamma_{\mathfrak{p}}$. In [6] we introduced the *cosupport* of a kG-module M to be the subset

$$\operatorname{cosupp}_R M = \{\mathfrak{p} \in \mathcal{V}_G \mid \varLambda^{\mathfrak{p}} M \neq 0\}\,.$$

The cosupport of M is non-empty when $M \neq 0$; see [6, Theorem 4.5].

In what follows $\operatorname{Hom}_k(M,N)$ is viewed as a kG-module with diagonal action. The theorem below is a consequence of [6, Theorem 9.5] and [4, Theorem 10.3], which are the central results of the corresponding articles. Theorem 4, and the other results described above, can be easily deduced from it.

Theorem 7. *For any kG-modules M and N one has*

$$\operatorname{cosupp}_G \operatorname{Hom}_k(M,N) = \operatorname{supp}_G M \cap \operatorname{cosupp}_G N\,.$$

In particular, $\operatorname{Hom}_k(M,N) = 0$ *if and only if* $\operatorname{supp}_G M \cap \operatorname{cosupp}_G N = \varnothing$. □

This result and Corollary 5 yield

Corollary 8. *One has* $\mathrm{A}(M)^{\perp} = \{N \mid \operatorname{cosupp}_G N \subseteq \operatorname{supp}_G M\}\,.$ □

Using this result and [6, Theorem 11.3], one can prove an analogue of Theorem 2, yielding bijections between subcategories of form $\mathrm{A}(M)^{\perp}$, the Hom closed colocalizing subcategories of $\mathsf{StMod}\,kG$, and the set of subsets of $\mathcal{V}_G$.

In all this the cosupport of modules plays a central role, but we do not yet have a good understanding of its significance. In my lecture, I mentioned some examples from commutative algebra where we have been able to compute the cosupport of all finitely generated modules. These are discussed in detail in [6], where it is also explained that the functor $\varLambda^{\mathfrak{p}}$ is akin to completion at $\mathfrak{p}$, in the sense of commutative algebra.

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