Stratifying the derived category of a complete intersection

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Let A be a commutative noetherian ring and D the bounded derived category of finitely generated A-modules; its objects are complexes M of A-modules such that A-module $\mathrm{H}_i(M)$ is finitely generated for each i and zero when $|i| \gg 0$. There is a natural triangulated category structure on D, with exact triangles arising from mapping cone sequences of morphisms of complexes. A non-empty full subcategory of D is *thick* if it is a triangulated subcategory and closed under retracts; see [15].

An intersection of thick subcategories is again thick so each M in D is contained in a smallest, with respect to inclusion, thick subcategory, which I denote $\operatorname{thick}_A(M)$. The objects of $\operatorname{thick}_A(M)$ are exactly those complexes which can be built out of M using suspensions, finite direct sums, exact triangles, and retracts; in fact, the last two operations suffice. Thus, for example, a complex is in $\operatorname{thick}_A(A)$ if and only if it is *perfect*, i.e. isomorphic in $\mathsf{D}^{\mathsf{f}}(R)$ to a finite complex of finitely generated projective modules.

My talk was concerned with the following problem: *Classify the thick subcategories of* D. I started by trying to explain why thick subcategories of $\mathsf{D}^{\mathsf{f}}(A)$ are interesting from the point of view of homological algebra; this is discussed also in [11]. Such investigations concerning derived categories started with a remarkable result of Hopkins [10] and Neeman [13]:

If M, N are perfect complexes with $\operatorname{supp}_A M \subseteq \operatorname{supp}_A N$, then $M \in \operatorname{thick}_A(N)$.

Here $\operatorname{supp}_A M$ is the set $\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathrm{H}(M)_{\mathfrak{p}} \neq 0\}$, the *support* of M. Various proofs of this theorem are discussed in [12]; for applications, see [8]. Given this theorem, it is easy to prove, see [13], that there is a bijection of sets:

$$\left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \operatorname{thick}_A(A)\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\text{Specialization closed}\\ \text{subsets of } \operatorname{Spec} A\end{matrix}\right\}$$

where a subset of $\operatorname{Spec} A$ is *specialization closed* if it is a (possibly infinite) union of closed subsets. The maps in question are

$$\sigma(\mathsf{C}) = \bigcup_{M \in \mathsf{C}} \operatorname{supp}_R M \quad \text{and} \quad \tau(\mathcal{V}) = \{M \mid \operatorname{supp}_R M \subseteq \mathcal{V}\}$$

This 'thick subcategory' theorem solves the classification problem stated when A is regular, for then $\operatorname{thick}_A(A) = \mathsf{D}$. Similar results have since been established for the derived category of perfect complexes of coherent sheaves on a noetherian scheme, by Thomason [14]; the stable module category of finite dimensional modules over the group algebra of a finite group, by Benson, Carlson, and Rickard [5]; and the category of perfect differential modules over a commutative noetherian ring, by Avramov, Buchweitz, Christensen, Piepmeyer and myself [2].

Let now A be a complete intersection; for simplicity assume $A = k[x_1, \dots, x_n]/I$, where k is a field, $x_1, \dots, x_n$ are indeterminates, and I is generated by a regular sequence. Set $c = n - \dim A$ and let $A[\chi_1, \dots, \chi_c]$ be the ring of cohomology operators constructed by Avramov and Sun [4]. Thus, $\chi_1, \dots, \chi_c$ are indeterminates over A of cohomological degree 2, and for each pair of complexes M, N of A-modules, $\operatorname{Ext}^*_A(M,N)$ is a graded R-module, which is finitely generated when M, N are in D; see [4, §§2,5], and also Gulliksen [9], for details. Set

$$\mathcal{V}_A(M) = \operatorname{supp}_R \operatorname{Ext}^*_A(M,M) \subseteq \operatorname{Spec} A[\chi_1, \dots, \chi_c]\,.$$

This construction is akin to the support variety of M in the sense of Avramov and Buchweitz [1]; only, it takes into account also the support of M as a complex of A-modules; see [7, §11]. A positive answer to the conjecture below takes us a long way towards a classification of thick subcategories of D for complete intersections.

Conjecture: For any M, N in D, if $\mathcal{V}_A(M) \subseteq \mathcal{V}_A(N)$, then $M \in \operatorname{thick}_A(N)$.

There are two points of view concerning homological algebra over complete intersections which lead one to such a statement: it is akin to that over regular rings, once we take into account the action of the cohomology operators; it is akin to that of group algebras of finite groups. Indeed, a result from [5] settles the conjecture above for the case when k is of positive characteristic p and $I = (x_1^p, \dots, x_n^p)$, for then A is the group algebra of $(\mathbb{Z}/p\mathbb{Z})^n$.

The simplest ring not covered by [5] is $A = k[x]/(x^d)$ with $d \geq 3$. The indecomposable A-modules are precisely $M_i = k[x]/(x^i)$, for $1 \leq i \leq d$. It is easy to verify that

$$\mathcal{V}_A(M_i) = \begin{cases} \{(x)\} & \text{for } i \neq d \\ \{(x), (x, \chi)\} & \text{for } i = d \end{cases}$$

Since $M_1 = k$ and $M_d = A$, the conjecture postulates that for $1 \leq i \leq d-1$ the subcategory $\operatorname{thick}_A(M_i)$ contains both A and k. In my talk, I demonstrated that this is indeed the case. This example is atypical for a general complete intersection is not of finite representation type, and one cannot expect to settle the conjecture with such direct computations.

Recently Benson, Krause, and I [6] gave a rather different proof of the result in [2]. It builds on the work in [3], which develops new tools for studying modules and complexes over complete intersections, and in [7], which develops a theory of local cohomology for the action of the ring of cohomology operators $A[\chi_1, \dots, \chi_c]$ on complexes of A-modules. The technique in [6] can be adapted to settle the conjecture above for all Artin complete intersection rings. The general case remains open, but I am optimistic that it will be settled in the near future.

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