Classifying Polynomial and Rational Solution of a Functional Equation

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Abstract

The research aims to investigate solutions of the functional equation

$$U(z)f(z) + zU(z)f(U(z)) = 0$$

For particular linear fractional maps $U(z)$, we checked different types of functions for f (polynomials and rational functions) to see whether they solved the functional equation or not. In some scenarios, we have classified the solutions or have shown a class of functions that are not solutions.

Introduction

Definition. Given an analytic function φ, the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$, where φ is referred to as the symbol of the composition operator.

φ is defined as a rational function of degree two with real coefficients, φ has the form

$$\varphi = \frac{a_1 z^2 + b_1 z + c_1}{a_2 z^2 + b_2 z + c_2} \tag{1}$$

where $c_2 \neq 0$ and there's at least one nonzero quadratic term. Furthermore, f is in the Hardy space.

Definition. The Hardy space, H^2, is the Hilbert space of analytic functions for which $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $z \in \mathbb{D}$ and $a_n \in \mathbb{C}$, satisfies $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

Theorem 1. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant rational map, and let C_φ act on H^2. Set $\sigma(z) = \frac{1}{\varphi^{-1}(\sqrt{z})}$, $\psi(z) = \frac{z\sigma\prime(z)}{\sigma(z)}$, and $\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z)$. Then,

$$(C_\varphi^* f)(z) = \frac{f(0)}{1-\varphi(\infty)z} + \sum \psi(z) f(\sigma(z))$$

where the sum is taken over the branches of σ.

Definition. Define σ_1 and σ_2 are two branches of σ on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \bigcup \infty$, which implies $(C_\varphi^* f)(z) = \frac{f(0)}{1-\varphi(\infty)z} + \sum_{j=1}^{2} \psi_j(z) f(\sigma_j(z))$.

Theorem 2. There exists a function U, such that $U(z) = \frac{1}{\varphi^{-1}(\varphi(\sqrt{z}))}$, and $U(\sigma_1) = \sigma 2$, where $sigma_1$ and σ_2 are branches of σ in Theorem 1. Furthermore,

$$U(z) = -\frac{(a_1 b_2 - b_1 a_2) + (a_1 c_2 - c_1 a_2)z}{(a_1 c_2 - c_1 a_2) + (b_1 c_2 - c_1 b_2)z} \tag{2}$$

Definition. The **kernel** of C_φ^* is a subset of H^2 such that if $f \in ker(C_\varphi^*)$, then $C_\varphi^*(f) = 0$

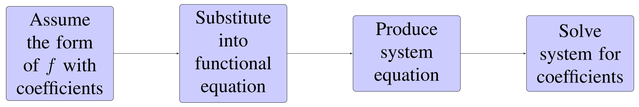
Theorem 3. Let φ be a rational map of degree two mapping $\mathbb{D}$ into $\mathbb{D}$ and let Cφ act on the Hardy space. For U as in Theorem 2, $f \in ker(C_\varphi^*)$ if and only if

$$U(z)f(z) + zU'(z)f(U(z)) = 0 \tag{3}$$

U is always represented by one of following **5 forms**: $-z$, $\frac{m}{z}$, $m - z$, $\frac{-z}{1+mz}$, $\frac{-m+z}{1+tz}$ where $m, t \in \mathbb{R}$ and $m \neq 0, t \neq 0$

The primary goals of the research were to identify the non-zero polynomial and rational solutions of f for the functional equation (3), where $U(z)$ is represented by one of the 5 forms above.

Method



Main Result

Theorem. Let f be a polynomial solution of equation (3) when $U(z) = m - z$. If f has an even degree, then the first coefficient is 0. Furthermore, if f has an odd degree, then the first coefficient is a free variable

When $U(z) = m - z$, functional equation is

$$(m-z)f(z) - zf(m-z) = 0 \tag{4}$$

Substituting a degree n polynomial into Equation 4 gives

$$\begin{aligned}&(m-z)(\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_1 z + \alpha_0)\\ &-z(\alpha_n (m-z)^n + \alpha_{n-1}(m-z)^{n-1} + \ldots + \alpha_1(m-z) + \alpha_0) = 0\end{aligned} \tag{5}$$

The coefficients of each power of z in equation (5) produce the following system of equations. Generally, these equations consist of a general system equation shown as follow:

$$\begin{cases} 0 = -\alpha_n\left((-1)^n + 1\right) z^{n+1} \\ 0 = \alpha_n m\left((-1)^{n-1}\binom{n}{1} + 1\right) z^n - \alpha_{n-1}\left((-1)^{n-1}\binom{n-1}{0} + 1\right) z^n \\ \quad \vdots \\ 0 = -\alpha_n m^{n-1}\binom{n}{n-1} z^2 - \cdots - \alpha_3 m\left(\binom{2}{1} - 1\right) z^2 - \alpha_2\left(\binom{1}{0} + 1\right) z^2 \\ 0 = \alpha_n m^n z + \alpha_{n-1} m^{n-1} z + \cdots + \alpha_2 (m-m) z + \alpha_1 (1+1) z \\ 0 = \alpha_0 m \end{cases}$$

From equation (6), the highest power term of z is $n+1$, the corresponding equation is

$$-\alpha_n((-1)^n + 1)z^{n+1} = 0 \tag{6}$$

when n is odd, α_n is always a free variable, which indicate that there always exists a class of solutions for the coefficients of odd degree polynomial.

As shown in equation (7), when n is even, the coefficient of highest power of z is always 0 ($z \neq 0$). Then, the rest of polynomial becomes to the odd degree polynomial which is already solved. Therefore, when $U(z) = m - z$, even degree polynomials can not be the solutions of equation (3).

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Example

Let $f(z)$ be a forth degree polynomial $f(z) = \alpha_1 z^4 + \alpha_2 z^3 + \alpha_3 z^2 + \alpha_4 z^1 + \alpha 5$.

Substitute $f(z)$ into equation (4). Follow the method stated in the above section, the coefficients of each power of z produce the system equation shown as follow:

$$\begin{cases} 0 = \alpha_1 \\ 0 = 5\alpha_1 m \\ 0 = 3\alpha_1 m^2 + \alpha_2 m + \alpha_3 \\ 0 = 4\alpha_1 m^3 + 3\alpha_2 m^2 + 3\alpha_3 m \\ 0 = \alpha_1 m^4 + \alpha_2 m^3 + \alpha_3 m^2 + 2\alpha_5 \\ 0 = \alpha_5 m \end{cases}$$

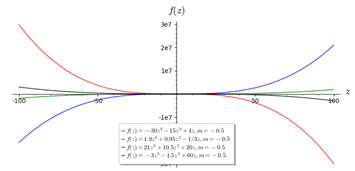

Figure 1: *Possible $f(z)$ when $m = -0.5$*

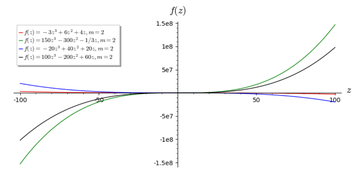

Figure 2: *Possible $f(z)$ when $m = 2$*

Solve the system equation in terms of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, the solution is $\alpha_1 = 0, \alpha_2 = r_1, \alpha_3 = -mr_1, \alpha_4 = r_2, \alpha_5 = 0$, where r_1 and r_2 are free variables.

This matches with the theorem that that coefficient of forth degree is 0. If we consider the solution as a third degree polynomial, the coefficient of highest power of z is a free variable.

Additional result and future work

- For $U(z) = m - z$ we have only explored the polynomial, can 'even degree solution does not exist' also apply to even degree rational function?
- Is there an efficient method to explore the solutions for the functional equation (3)?
- Will the solutions for all 5 forms $U(z)$ follow some routine based on the character of $U(z)$

Reference

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