The Shape of Large Soap Bubbles

California State University, Fullerton

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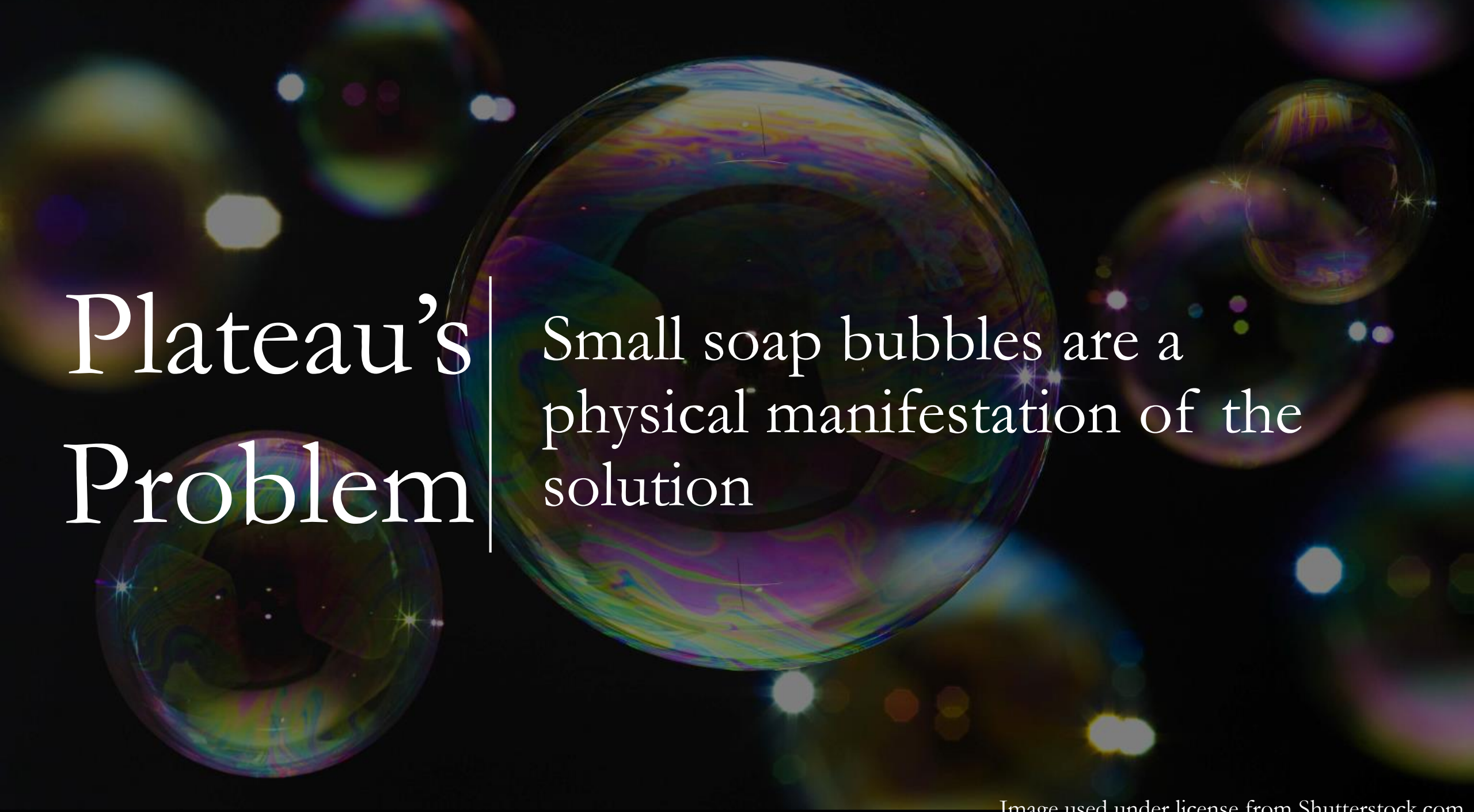
CALIFORNIA STATE UNIVERSITY FULLERTON™

GRAM

Joseph Plateau

- Belgian Physicist, 1801-1883
- Studied capillary action, surface tension, and minimal surfaces
- Observed soap bubbles and films to form Plateau's Laws
- **Plateau's Problem: Find a surface containing a specified volume that minimizes surface area.**

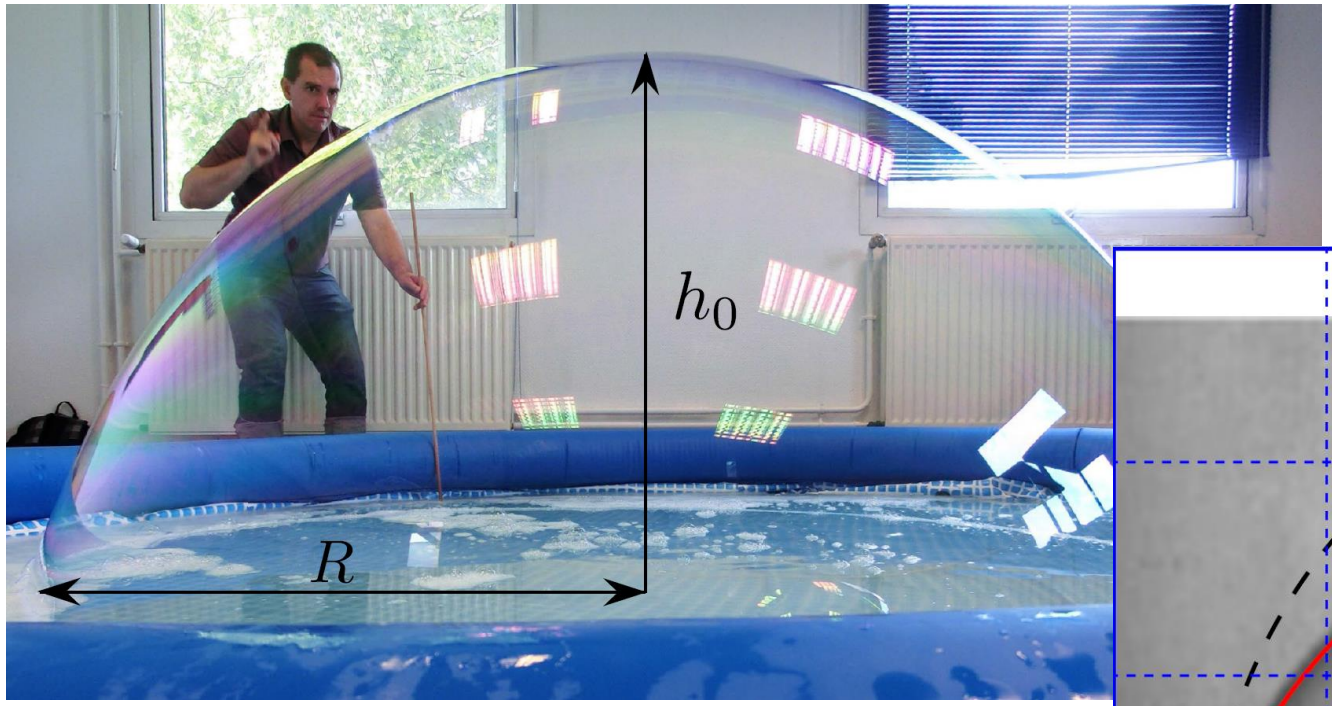


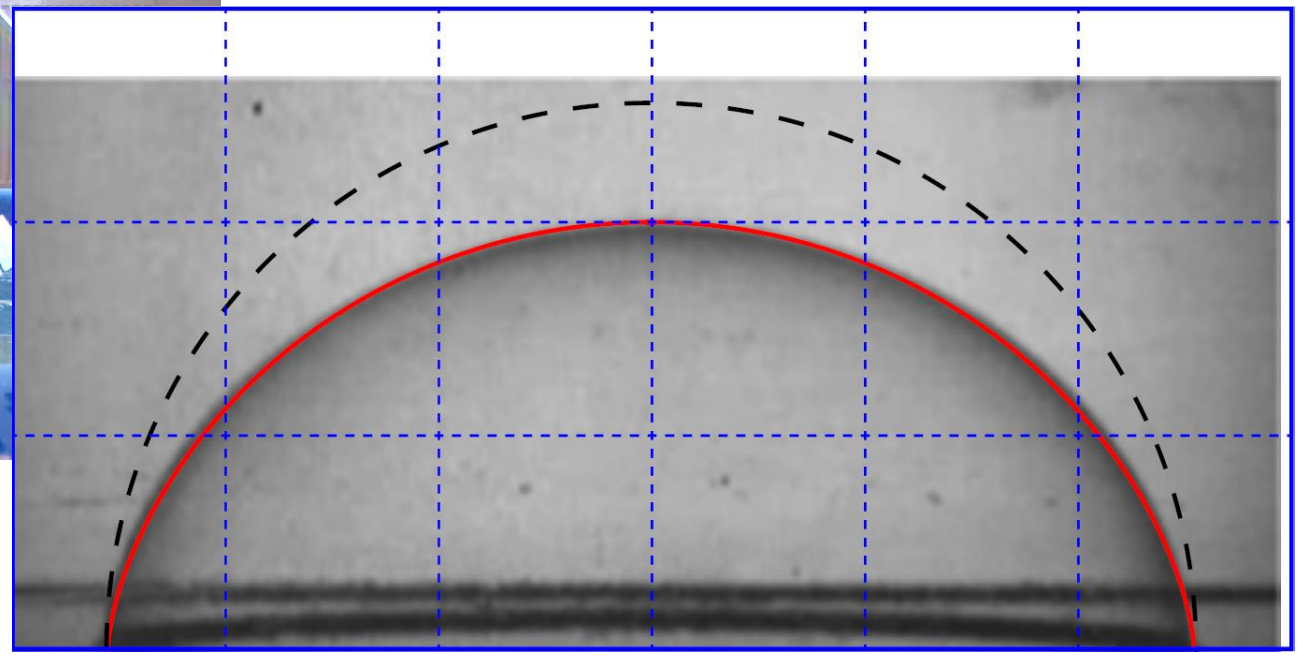
Plateau's Problem

Small soap bubbles are a physical manifestation of the solution

Large Soap Bubbles

- Increased material needed to contain large volume gives significant mass for gravity to act upon, flattening large bubbles





Applications

Large inflatable structures, like Festo's Airquarium!

Formulating the Model

Principle of Virtual Work – Static systems minimize potential energy

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Bubbles take the shape that minimizes potential energy

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Find the equation for the shape of bubbles of a given volume with the **least potential energy!**

How do we do this?

Functionals: Map a function to a real number

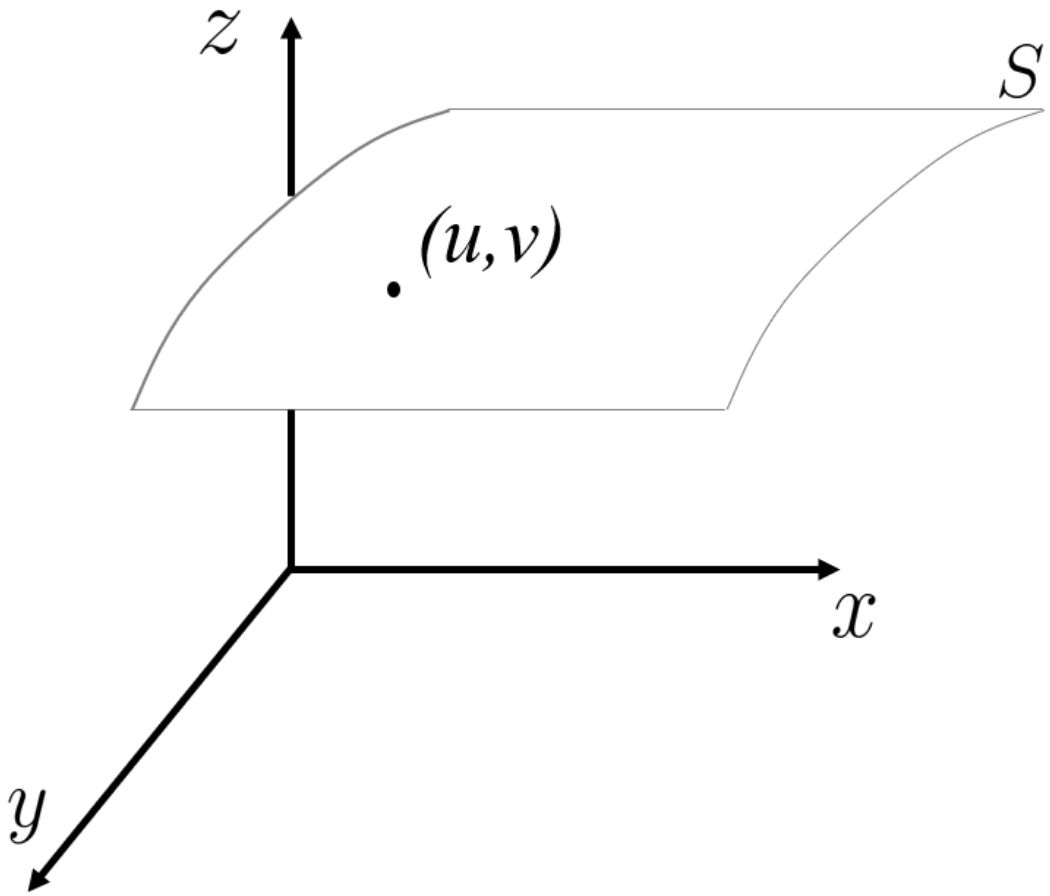
- Create a functional that represents the total potential energy of the bubble to be minimized

Calculus of variations: Find the extrema of functionals

- Minimize our functional and produce equations governing the shape of the surface

Parameterization

$$(u, v, z(u, v))$$



Constructing the functional

- Interfacial Free Energy: Difference between cohesive and adhesive forces at the membrane.

$$\gamma \iint_{\Omega} |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

- Gravitational Energy: Effect of gravity on the membrane.

$$\beta \iint_{\Omega} z(u,v) |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

Result 1

We can remove the effect of interfacial free energy with a vertical translation.

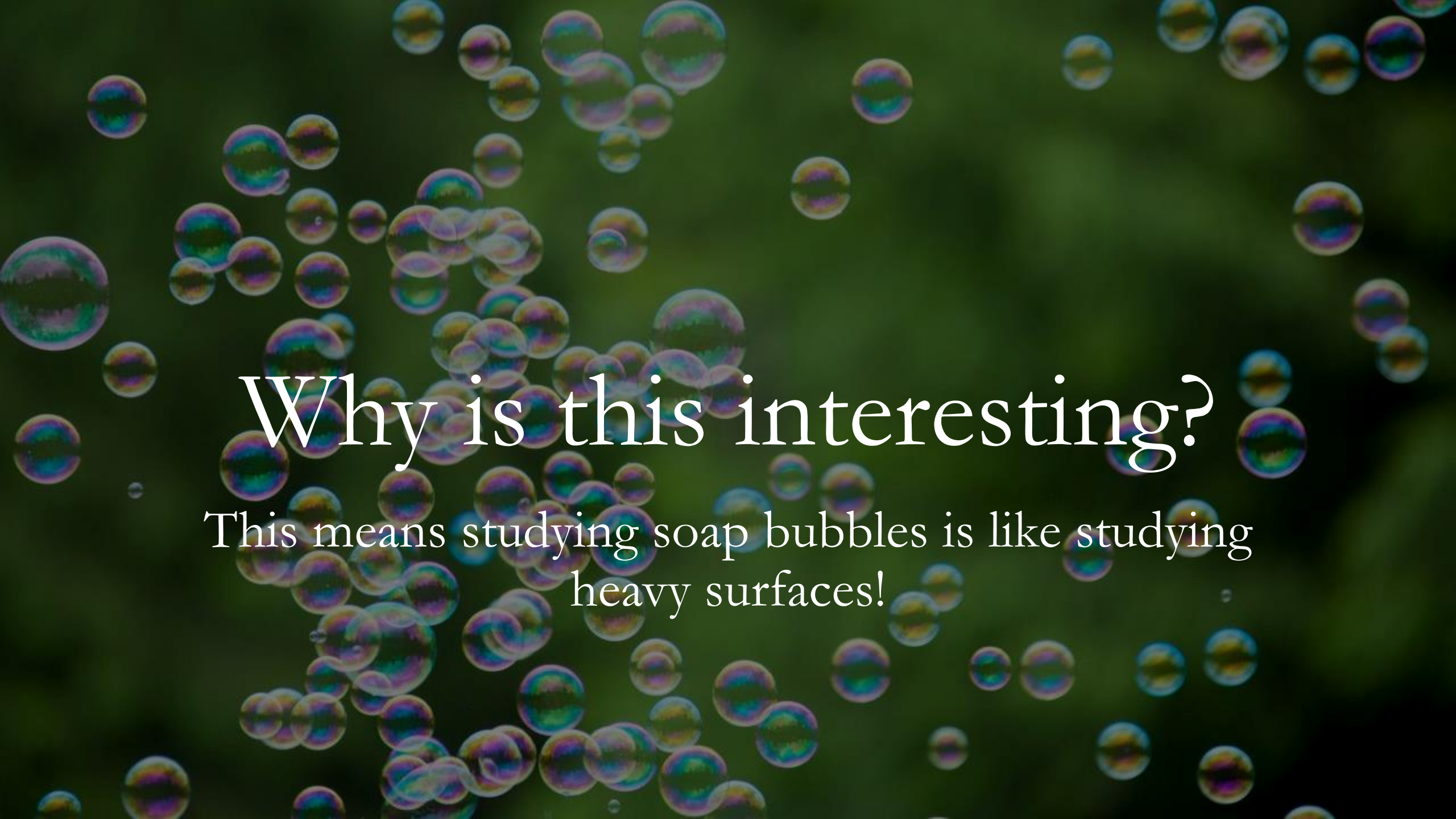
$$E[z(u,v)] = \gamma \iint_\Omega |\mathbf{r}_u \times \mathbf{r}_v| dudv + \beta \iint_\Omega z(u,v)|\mathbf{r}_u \times \mathbf{r}_v| dudv.$$

Combine Terms

$$E[z(u,v)] = \beta \iint_\Omega \left(\frac{\gamma}{\beta} + z(u,v)\right) |\mathbf{r}_u \times \mathbf{r}_v| dudv.$$

Make substitution such that $z(u,v) = \zeta(u,v) - \frac{\gamma}{\beta}$.

$$E[\zeta(u,v)] = \beta \iint_\Omega \zeta(u,v)|\mathbf{r}_u \times \mathbf{r}_v| dudv$$

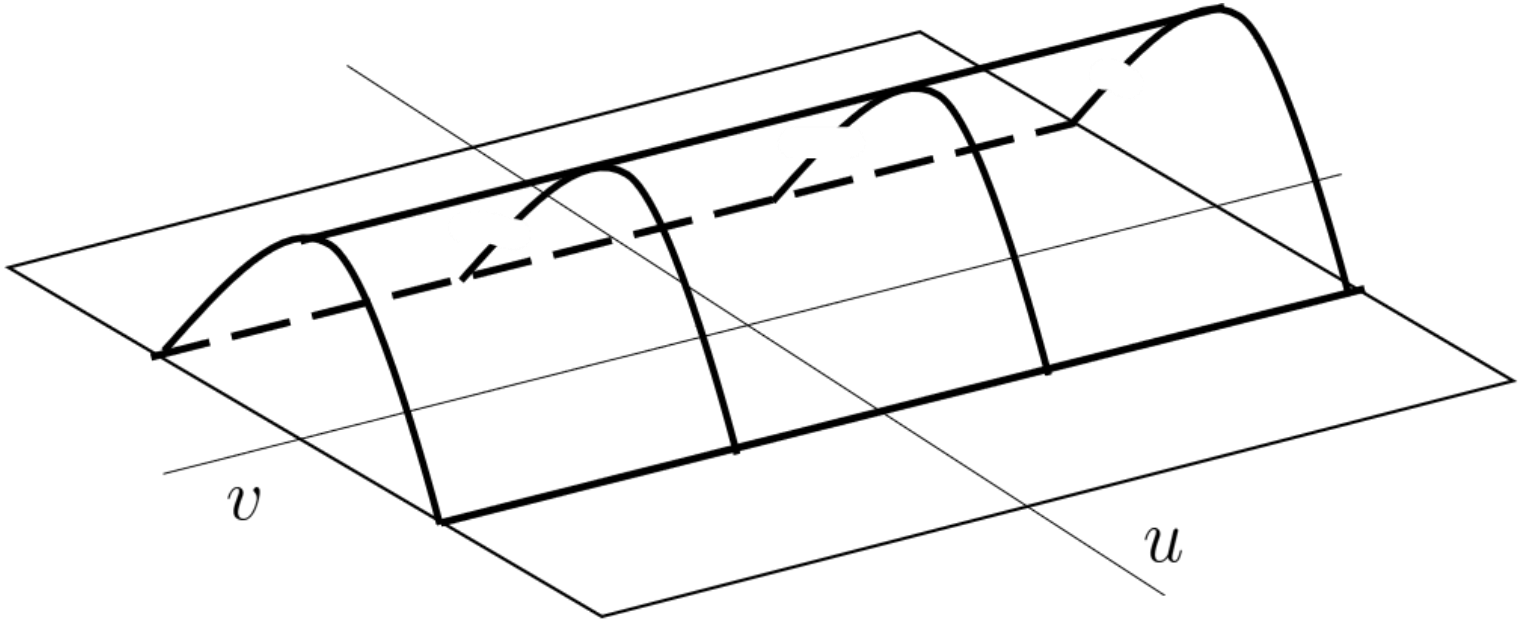


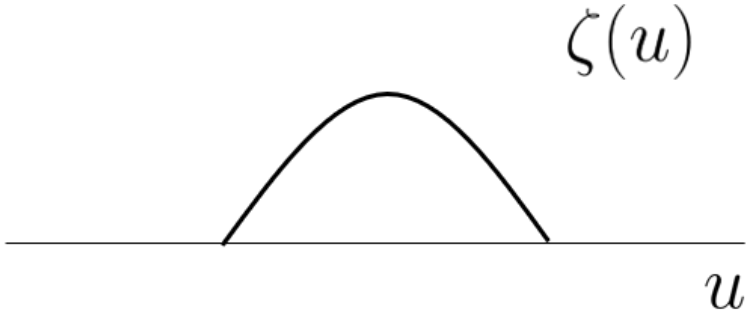
Why is this interesting?

This means studying soap bubbles is like studying heavy surfaces!

Translationally Invariant Surfaces

To simplify analysis, we reparametrize our surface such that $\mathbf{r} = (u, v, \zeta(u))$.





New Equation with Volume Constraint

$$E[\zeta(u)] = \int_{-a}^{a} \zeta(u)\sqrt{1 + (\zeta(u)')^2}du + \Lambda\left(\int_{-a}^{a} \zeta(u)du - A_0\right)$$

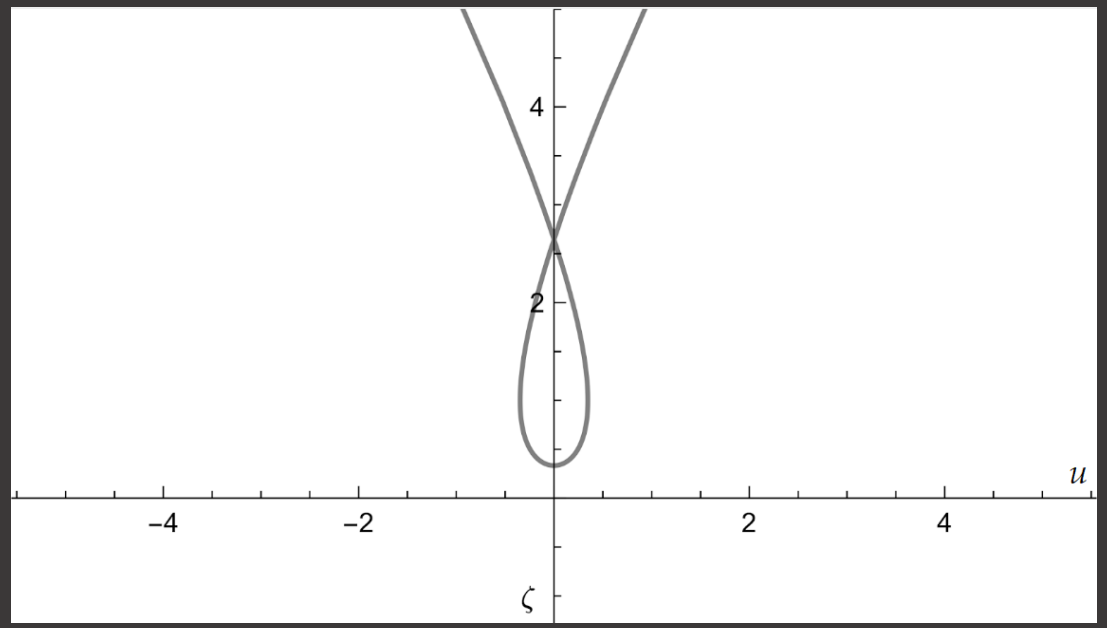
The Model

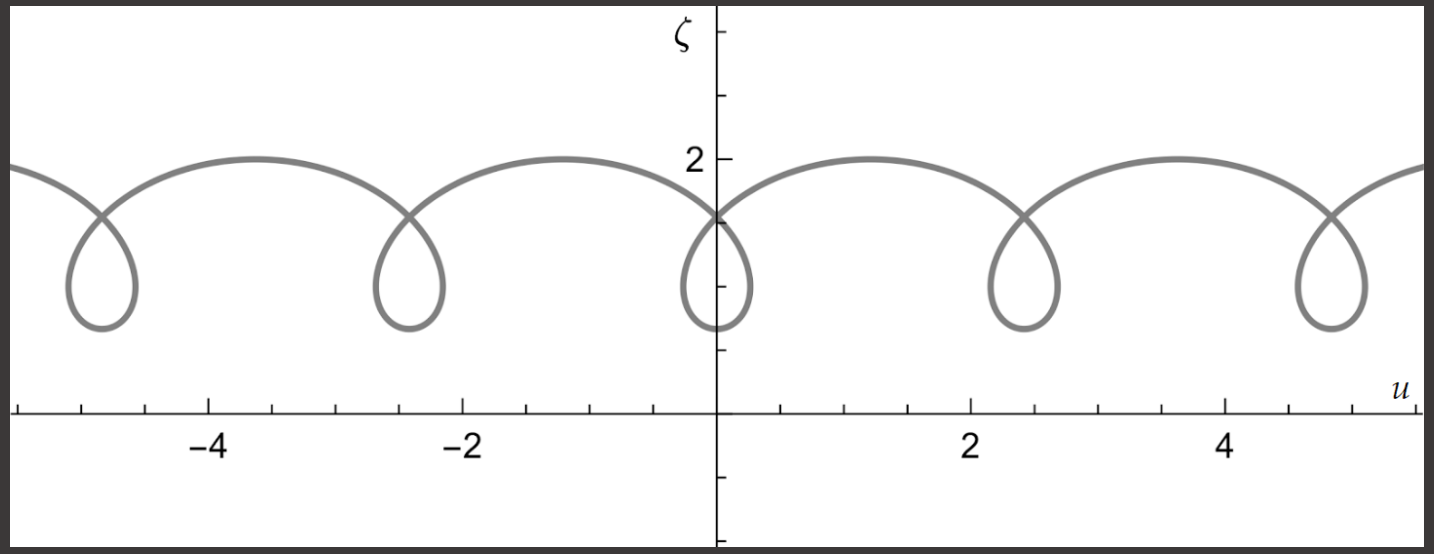
With our chosen parameterization, we obtain the following governing equation from the Euler-Lagrange equation.

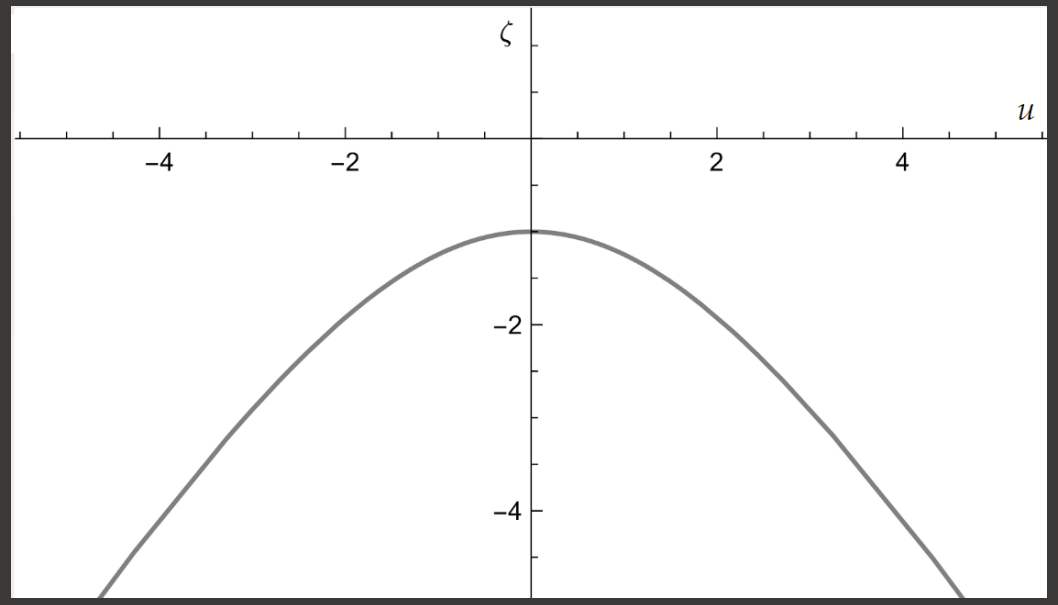
$$\zeta \frac{\zeta''}{\left(1+(\zeta')^2\right)^{\frac{3}{2}}} = \frac{1}{\sqrt{1+(\zeta')^2}} + \Lambda.$$

Result 2

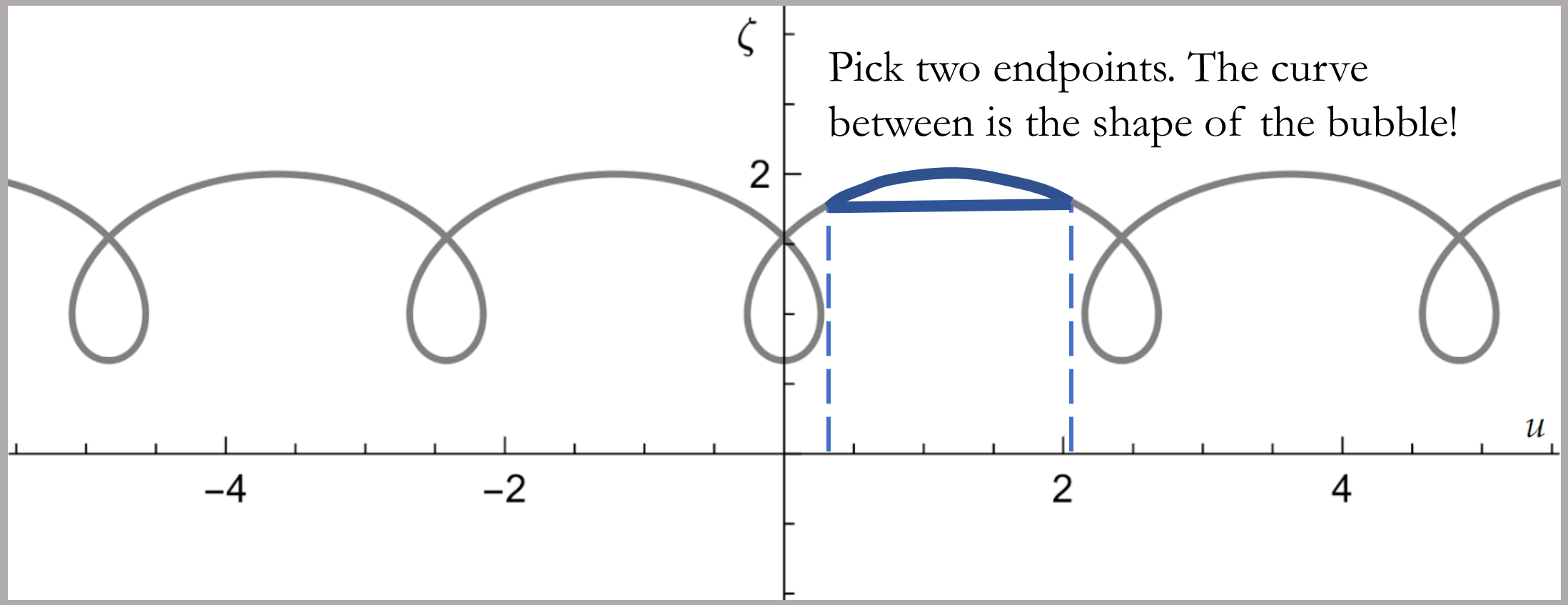
There are four distinct one-parameter families of curves for translationally invariant bubbles.





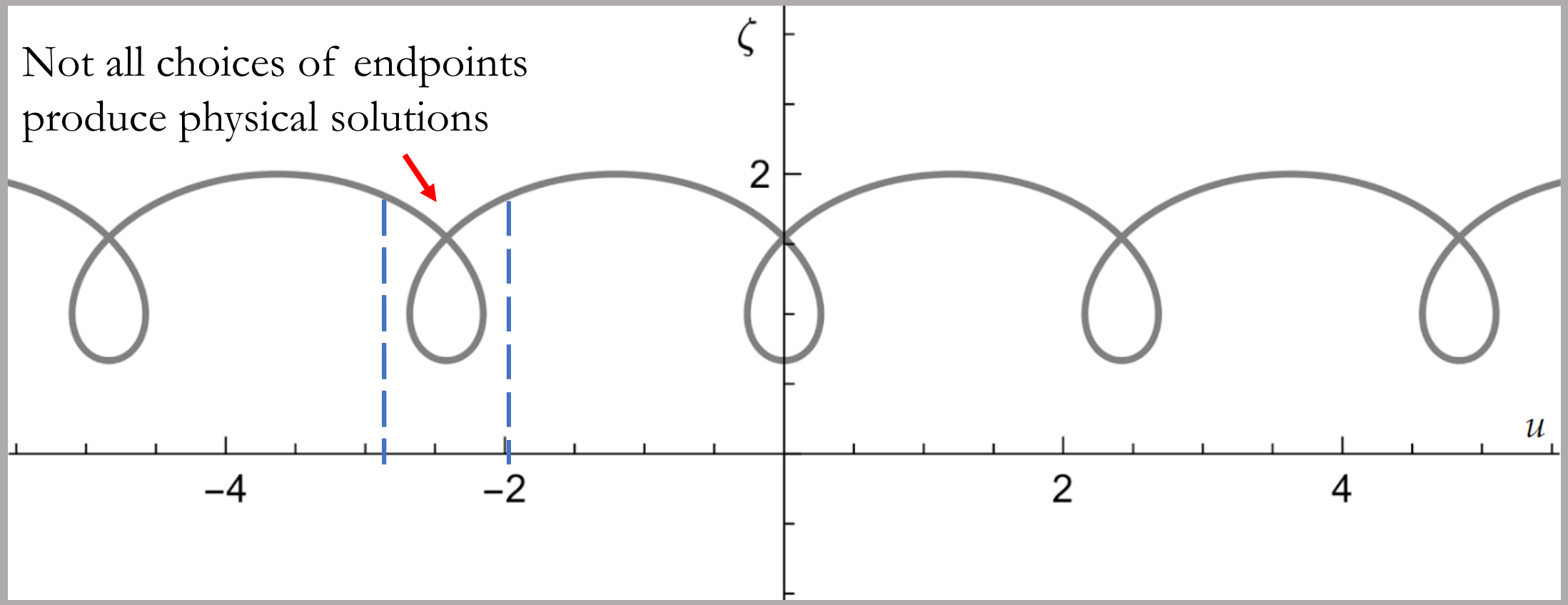


Where are the bubbles?



Pick two endpoints. The curve between is the shape of the bubble!

Where are the bubbles?



Future Work

Classify	Classify critical points to confirm which solutions are minimizing equations
Analyze	Perform stability analysis to determine which forms bubbles will take
Extend	Extend this analysis to rotationally symmetric bubbles

Acknowledgements

- I would like to thank professor Nicholas Brubaker for being my mentor.
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Further Reading

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Thank you for listening!

Any questions?

bmcbean@csu.fullerton.edu