Jacobian Groups of Bipartite Graphs

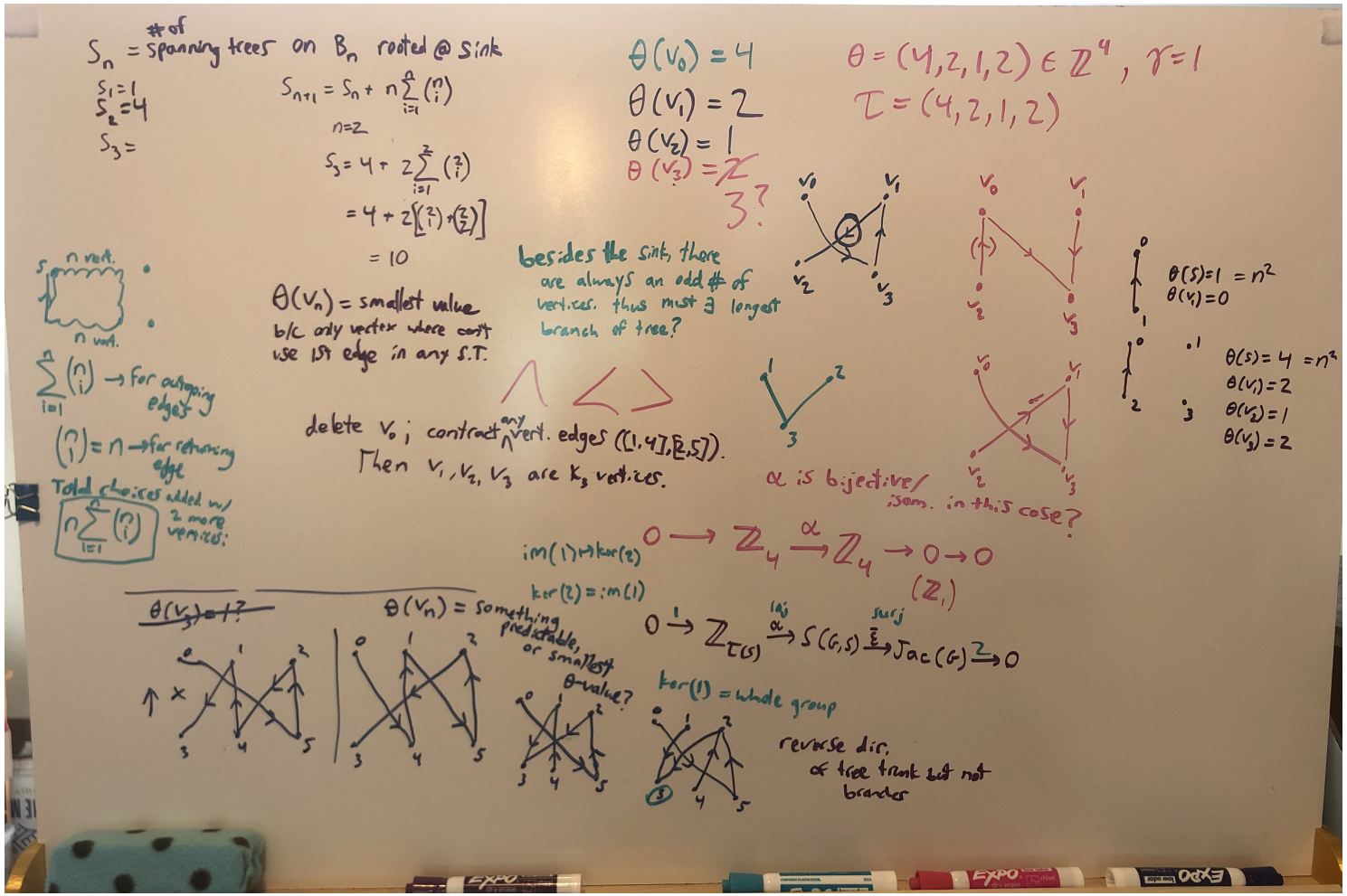
Miriam Syvertsen

Lawrence University

January 23, 2021

A Little Motivation

??



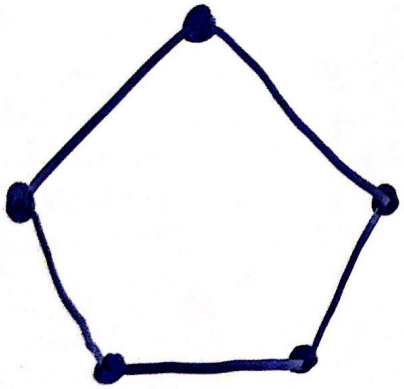
??

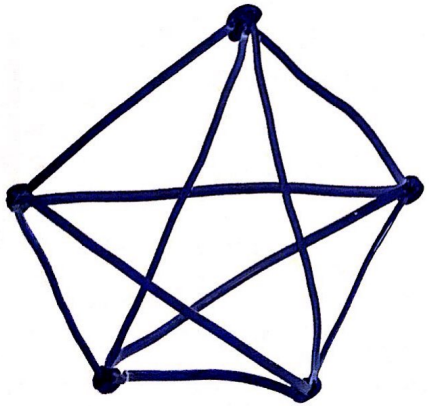
Defining Graphs

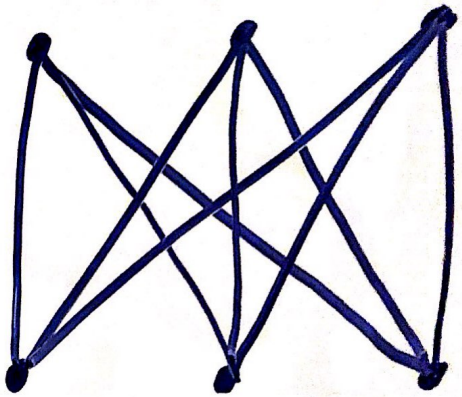
Figure: Three Equivalent Graphs

- What matters is what is connected to what
- No coordinate system, unlike in calculus

Some Common Graphs







The Jacobian Group

What is the Jacobian group?

- The Jacobian group of every finite graph can be written as the Cartesian product of $\mathbb{Z}_i$'s for some (often multiple) values of $i \in \mathbb{N}$.

Example: Jacobian Groups of Some Actual Graphs

Revisiting the graphs from before, with their Jacobian groups:

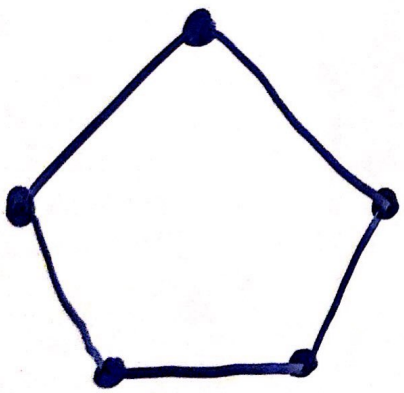
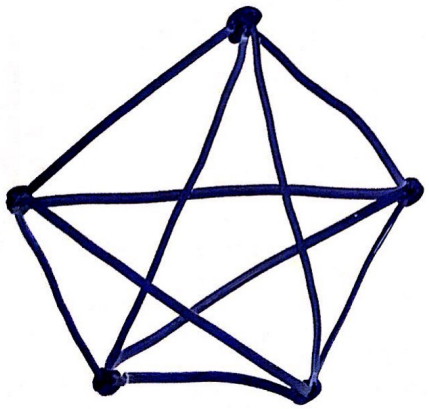
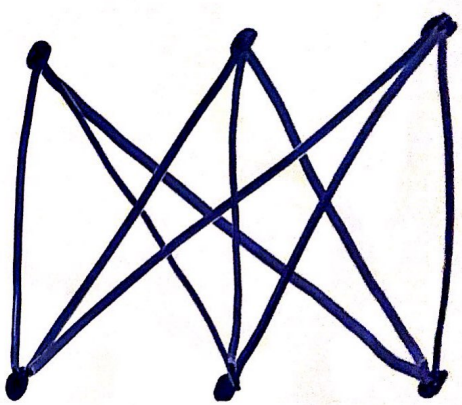


Figure: $\mathbb{Z}_5$



$\mathbb{Z}_5^3$



$\mathbb{Z}_3^2 \times \mathbb{Z}_9$

Example: Jacobian Groups of Some Actual Graphs

Revisiting the graphs from before, with their Jacobian groups:

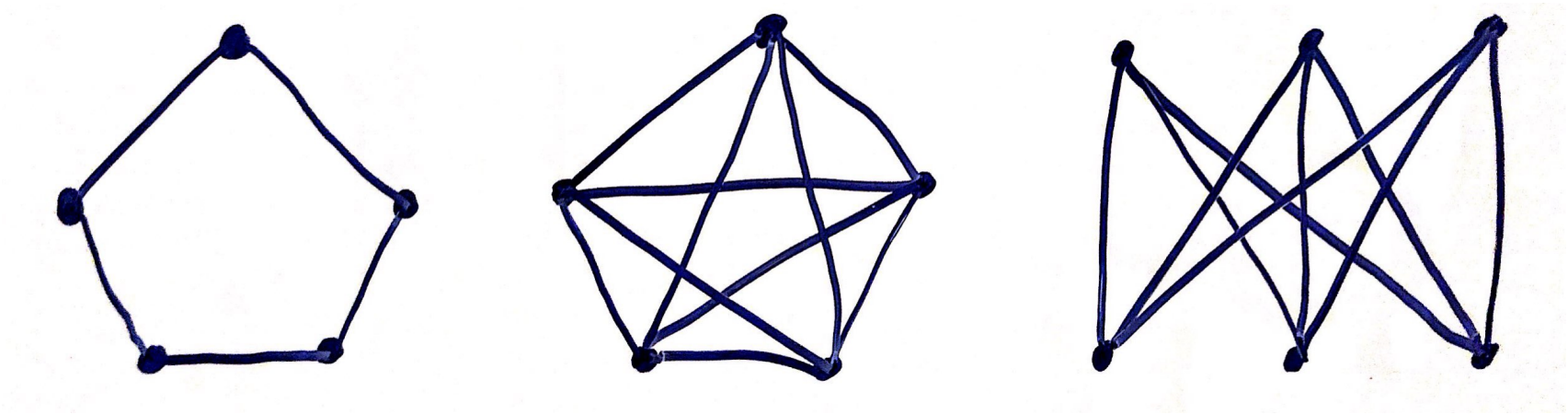


Figure:	$\mathbb{Z}_5$	$\mathbb{Z}_5^3$	$\mathbb{Z}_3^2 \times \mathbb{Z}_9$
Pattern:	$\mathbb{Z}_n$	$\mathbb{Z}_n^{n-2}$	$\mathbb{Z}_n^{2n-4} \times \mathbb{Z}_{n^2}$

Usefulness

Is this useful for anything!?

- The Matrix-Tree Theorem tells us that the size of the Jacobian group of a graph is the number of spanning trees on that graph.

Spanning Trees

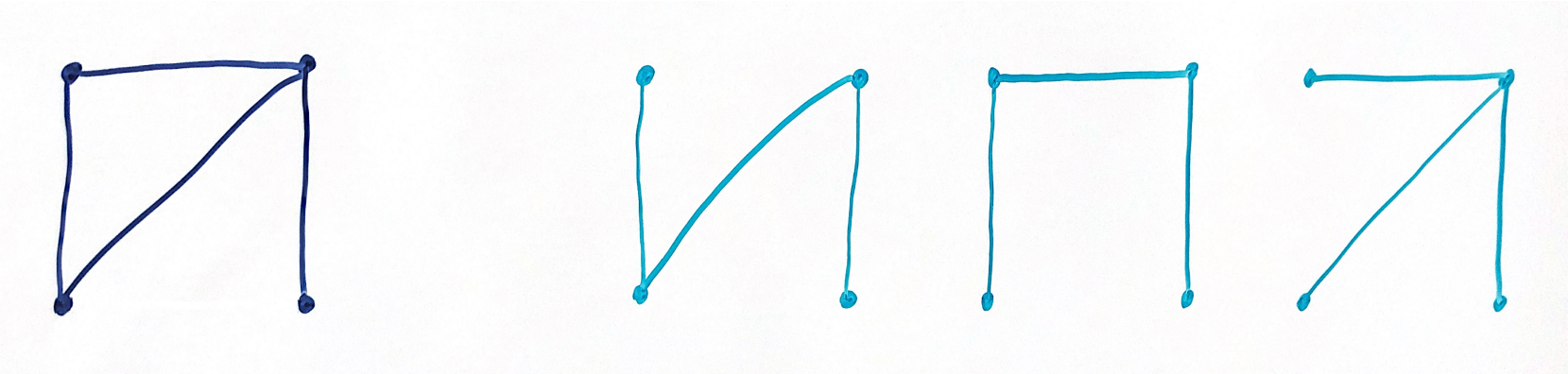


Figure: All 3 spanning trees of a graph on 4 vertices

Spanning Trees: Your Turn!

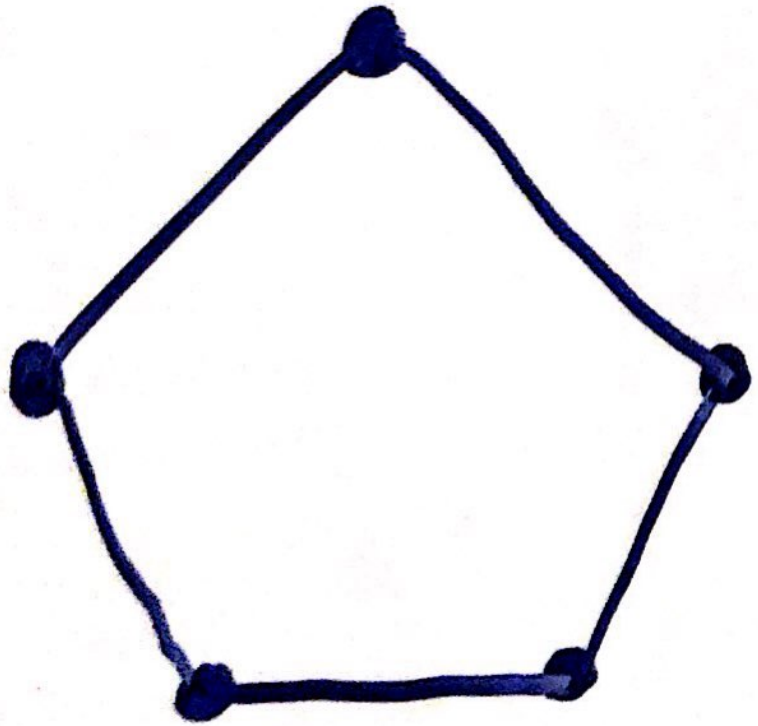


Figure: Can you find any of this graph's spanning trees?

Spanning Trees: Your Turn!

Figure: All 5 spanning trees of C_5

Bipartite Graphs

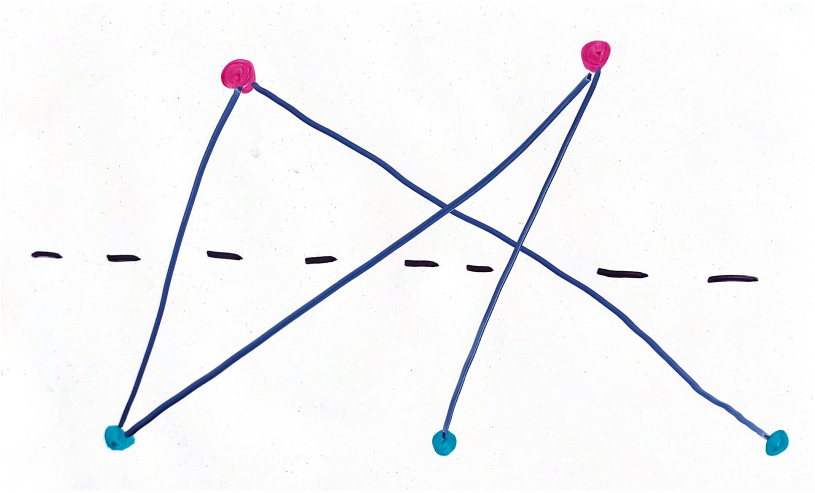


Figure: Bipartite Graph

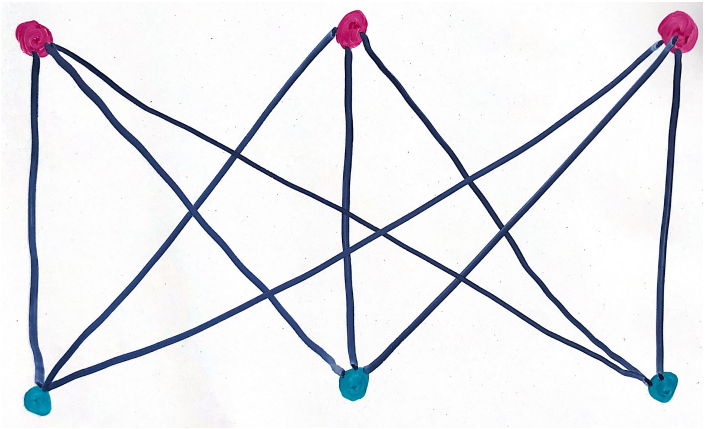


Figure: $K_{3,3}$

- $Jac(K_{n,n}) \cong \mathbb{Z}_n^{2n-4} \times \mathbb{Z}_{n^2}$

Question

What happens to the Jacobian group if we take a graph and specify a single direction for some edge(s)?

The B_n Graph

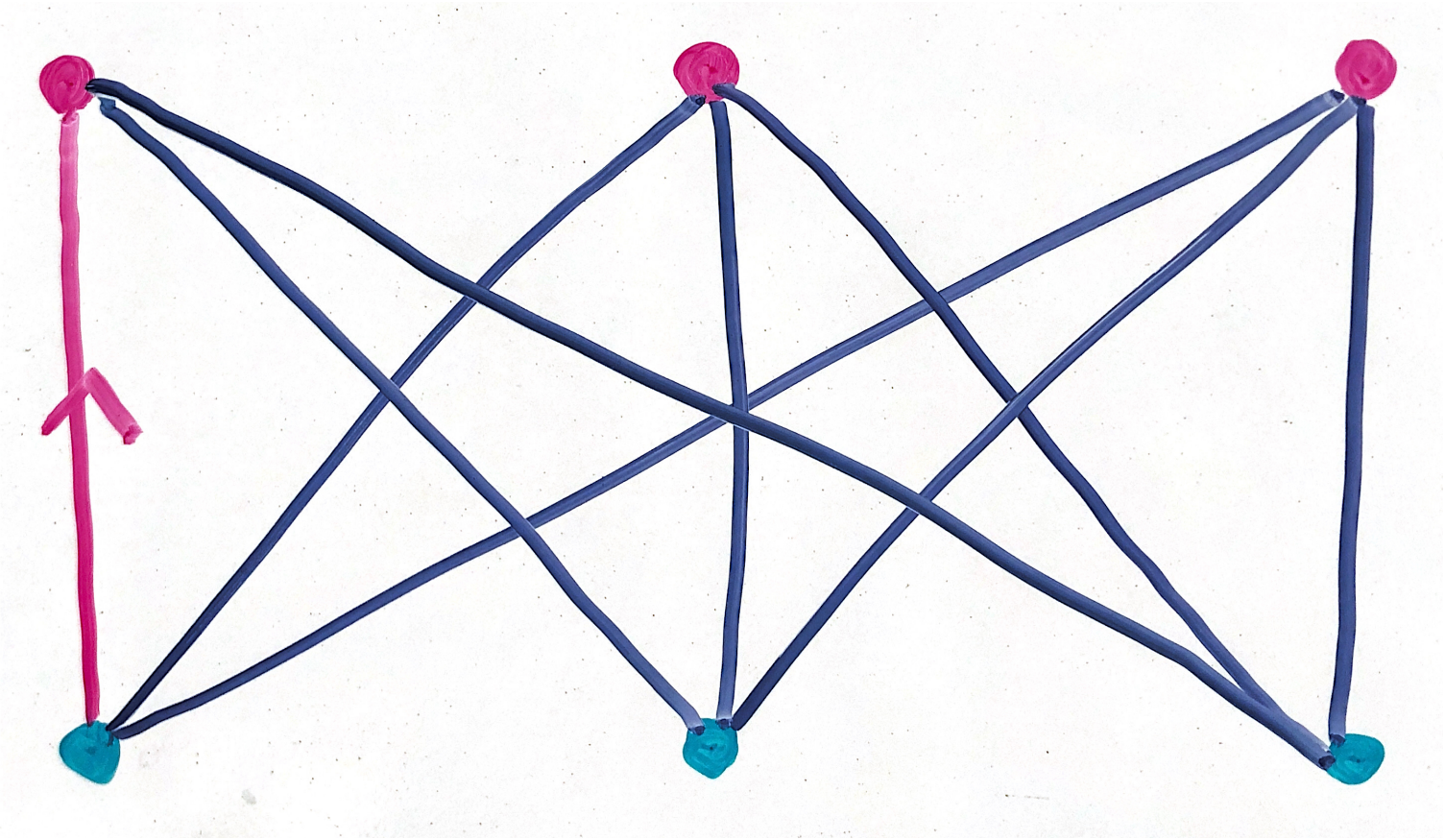


Figure: The B_3 Graph

$\mathrm{Jac}(B_n)$

Theorem

$\mathrm{Jac}(B_n) \cong \mathbb{Z}_n^{2n-4}$ for $n \geq 2$

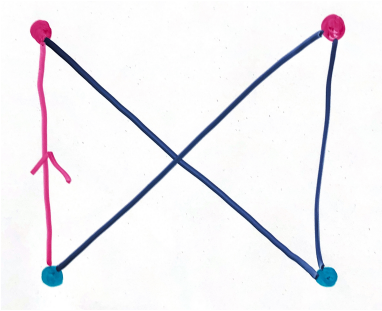


Figure: $\mathrm{Jac}(B_2) \cong \mathbb{Z}_1$

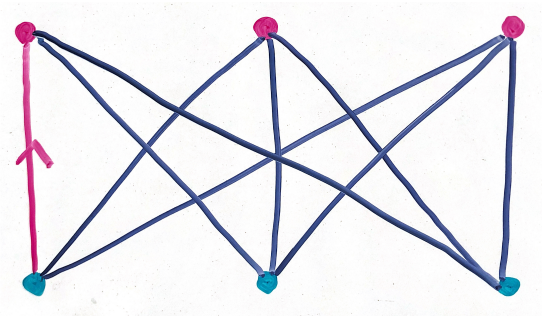


Figure: $\mathrm{Jac}(B_3) \cong \mathbb{Z}_3^2$

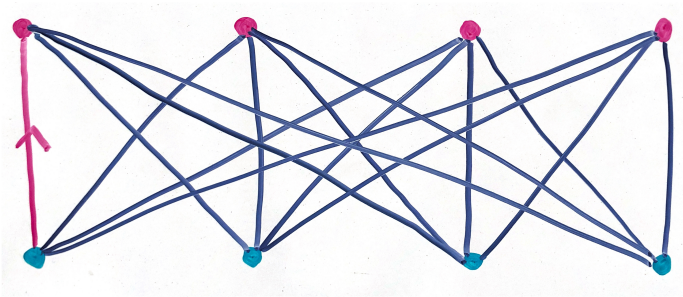


Figure: $\mathrm{Jac}(B_4) \cong \mathbb{Z}_4^4$

Ideas From My Proof

- Counting spanning trees
- Directed spanning trees rooted at a given vertex
- Linear algebra: The Matrix-Tree Theorem
- Group theory: Factor groups; Short exact sequences

Next Steps

A next goal: Generalizing this pattern to take off any number of outgoing edges from one vertex

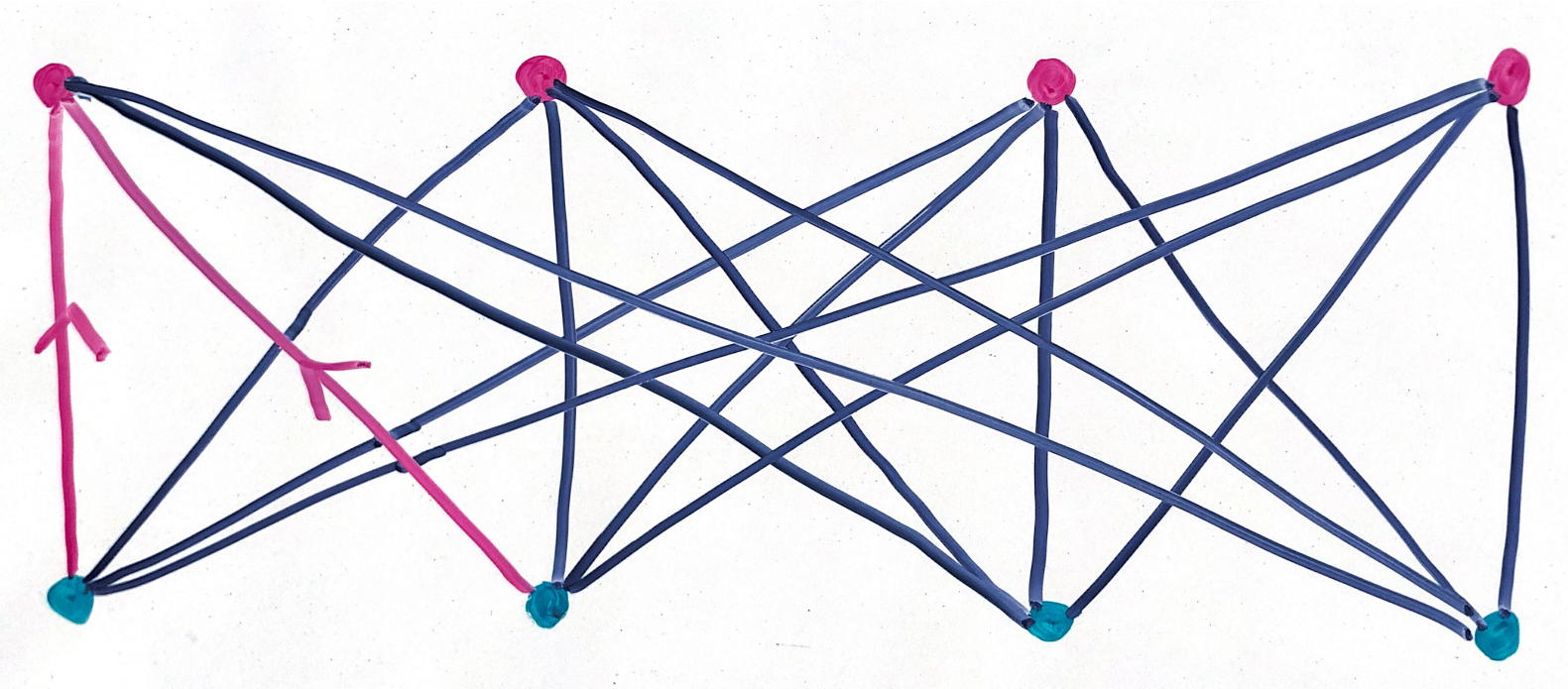
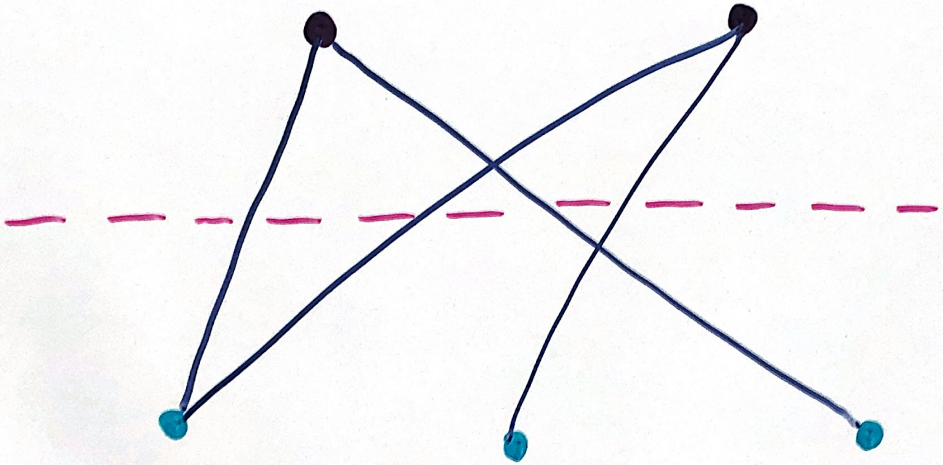


Figure: $\mathrm{Jac}(G) \cong \mathbb{Z}_4^3 \times \mathbb{Z}_8$. What is this pattern?

Thank You!



Thank You to:

- You
- Prof. Corry and Prof. Rana
- Gabi Stokely