CLASSIFYING QUANTUM ADJACENCY MATRICES

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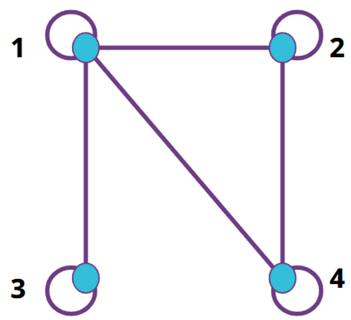
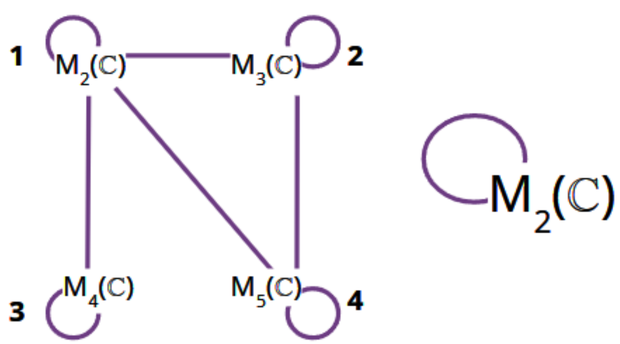
CLASSICAL VS QUANTUM GRAPHS



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{array}$$

This is an example of a **classical graph** with a corresponding matrix that describes the connections between vertices.



This is a representation of **a quantum graph**, where the vertices are vector spaces of matrices rather than points. On the right is what has been studied in this project: one space of matrices mapping back to itself.

WHAT IS A QUANTUM ADJACENCY MATRIX?

A matrix A must be **quantum idempotent** (the quantum analog of being Schur idempotent) in order to be a quantum adjacency matrix. That is, A must satisfy this property:

$$m(A \otimes A)m^*(x) = n^2 A(x) \quad \text{for all } x \in M_n(\mathbb{C})$$

$$\begin{tikzcd} M_n \otimes M_n \arrow[r, "A\otimes A"] & M_n \otimes M_n \arrow[d, "m"] \\ M_n \arrow[u, "m^*"] \arrow[r, "n^2 A"'] & M_n \end{tikzcd}$$

Figure 1: this is a commutative diagram illustrating the quantum idempotent property

KNOWN EXAMPLES OF QUANTUM GRAPHS

We started by investigating types of quantum adjacency matrices and their properties that were already known.
Junichiro Matsuda (J. Math. Phys. 63, no. 9 (2022): 0902201)

- **Reflexive:** A quantum adjacency matrix A is reflexive if:

$$m(A \otimes \mathrm{id})m^* = n^2 \mathrm{id}$$

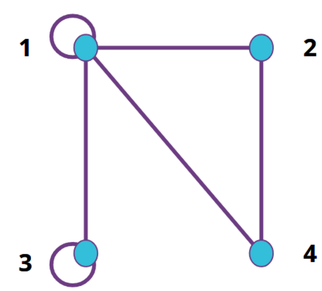
$$\begin{tikzcd} M_n \otimes M_n \arrow[r, "A\otimes \mathrm{id}"] & M_n \otimes M_n \arrow[d, "m"] \\ M_n \arrow[u, "m^*"] \arrow[r, "n^2 \mathrm{id}"'] & M_n \end{tikzcd}$$

Figure 2: this is a commutative diagram illustrating the quantum reflexive property

OUR RESULTS: NEW QUANTUM GRAPHS

Using **figure 1**, we derived 16 coefficient equations which classify quantum adjacency matrices for n=2.

The following classical graph is not reflexive. That is, two vertices do not point back to themselves, resulting in two zeros in the diagonal of the corresponding matrix.



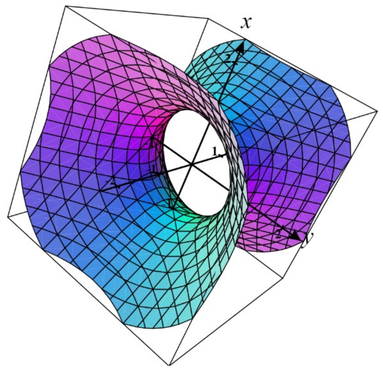
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 \end{array}$$

We wanted to find quantum adjacency matrices that did not satisfy the property illustrated in **figure 2**. We used the 16 derived equations to study the relationships of the entries of the following matrix and found that it is **not reflexive.**

$$\begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix} = \begin{bmatrix} 2-x & 0 & 0 & 2-y \\ 0 & a & b & 0 \\ 0 & \frac{2y-y^2}{b} & \frac{2x-x^2}{a} & 0 \\ y & 0 & 0 & x \end{bmatrix}$$

The matrix on the right is a general form of the one on the left for any x, y, a, b ≠ 0.

$$\begin{bmatrix} 2-x & 0 & 0 & 2-y \\ 0 & a & b & 0 \\ 0 & \frac{2y-y^2}{b} & \frac{2x-x^2}{a} & 0 \\ y & 0 & 0 & x \end{bmatrix}$$



The image on the right is a graphical representation of the relationship of the circled entries in the matrix above.

CODE

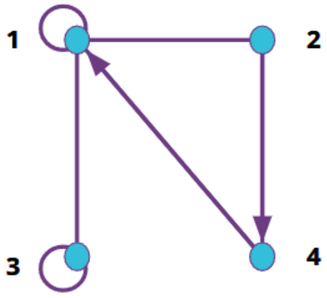
After deriving equations for a quantum adjacency matrix by hand, we created scripts that were able to do much of the computation for us.

Currently, I have scripts that can determine if a given matrix of any size is a quantum adjacency matrix. It also checks if a quantum adjacency matrix is reflexive, regular, and a *-homomorphism.

For the case where n=2, I wrote code to compute the quantum edge checker which generates the quantum analog of an edge set.

Future Work

Another property of quantum adjacency matrices that we are interested in is symmetry. For a classical graph, symmetry means that the edges are undirected, or that the matrix representation of the graph is equal to its conjugate transpose. The classical graph below is **not symmetric**:



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 \end{array}$$

The definition of **symmetry** for quantum adjacency matrices is described by the diagram below:

$$\begin{tikzcd} M_n \otimes M_n \otimes M_n \arrow[r, "\mathrm{id}\otimes A\otimes \mathrm{id}"] & M_n \otimes M_n \otimes M_n \arrow[d, "\psi m\otimes \mathrm{id}"] \\ M_n \arrow[u, "\mathrm{id}\otimes(\psi m)^*"] \arrow[r, "A"'] & M_n \end{tikzcd}$$

In the future, we have several next steps that we would like to take:

- Investigating the property of symmetry and finding forms of quantum graphs that are not symmetric
- Developing code to check if any sized matrix is symmetric
- Investigating more forms of quantum adjacency matrices
- Researching the relationships between quantum adjacency matrices and their corresponding quantum edge checkers