Fantastic points and where to find them

Cristiano Bocci

Department of Information Engineering and Mathematics, Siena

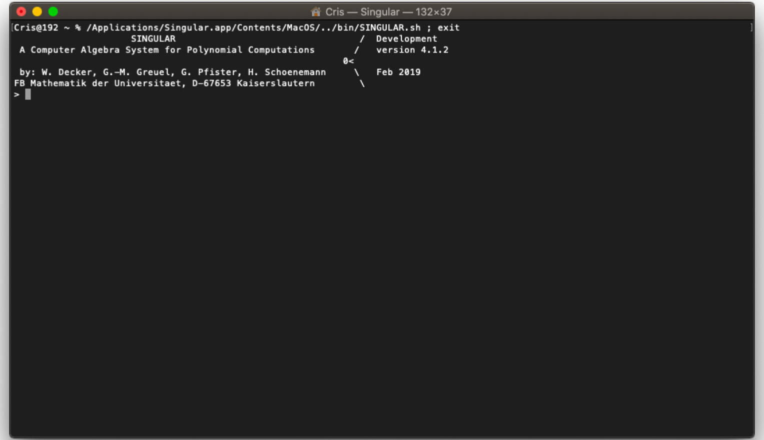
BrianFest - August 12, 2023

Cristiano Bocci

Department of Information Engineering
and
Mathematics

BrianFest

August 12, 2023


Cris — Singular — 132×37
Cris@192 ~ % /Applications/Singular.app/Contents/MacOS/../bin/SINGULAR.sh ; exit
SINGULAR / Development
A Computer Algebra System for Polynomial Computations / version 4.1.2
0<
by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann \ Feb 2019
FB Mathematik der Universitaet, D-67653 Kaiserslautern \
>

Cris — Singular — 132×37

```
. for (i=1;i<=t+1;i++)
. {
. for (j=1;j<=dif[i];j++)
. {
. for (k=dif[i]+1;k<=s-t+2;k++)
. {
.
. G[counter,1]=-((A[1,1]+u[i]*A[1,2])*(v[j]*A[1,1]+A[1,2])*(v[k]*A[1,1]+A[1,2]))/(A[1,1]*A[1,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[1,1
]*A[3,2]-A[3,1]*A[1,2])*(A[1,1]*A[4,2]-A[4,1]*A[1,2]));
. G[counter,2]=((A[2,1]+u[i]*A[2,2])*(v[j]*A[2,1]+A[2,2])*(v[k]*A[2,1]+A[2,2]))/
. (A[2,1]*A[2,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2]));
. G[counter,3]=-((A[3,1]+u[i]*A[3,2])*(v[j]*A[3,1]+A[3,2])*(v[k]*A[3,1]+A[3,2]))/
. (A[3,1]*A[3,2]*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. G[counter,4]=((A[4,1]+u[i]*A[4,2])*(v[j]*A[4,1]+A[4,2])*(v[k]*A[4,1]+A[4,2]))/
. (A[4,1]*A[4,2]*(A[1,1]*A[4,2]-A[4,1]*A[1,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. counter=counter+1;
. }
. }
. }
. for (i=1;i<=t;i++)
. {
. for (k=i+1;k<=t+1;k++)
. {
. down=min(dif[i],dif[k]);
. up=max(dif[i],dif[k])-1;
. for (j=down;j<=up;j++)
. {
. G[counter,1]=-((A[1,1]+u[i]*A[1,2])*(A[1,1]+u[k]*A[1,2])*(v[j+1]*A[1,1]+A[1,2]))/
. (A[1,1]*A[1,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[1,1]*A[4,2]-A[4,1]*A[1,2]));
. G[counter,2]=((A[2,1]+u[i]*A[2,2])*(A[2,1]+u[k]*A[2,2])*(v[j+1]*A[2,1]+A[2,2]))/
. (A[2,1]*A[2,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2]));
. G[counter,3]=-((A[3,1]+u[i]*A[3,2])*(A[3,1]+u[k]*A[3,2])*(v[j+1]*A[3,1]+A[3,2]))/
. (A[3,1]*A[3,2]*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. G[counter,4]=((A[4,1]+u[i]*A[4,2])*(A[4,1]+u[k]*A[4,2])*(v[j+1]*A[4,1]+A[4,2]))/
. (A[4,1]*A[4,2]*(A[1,1]*A[4,2]-A[4,1]*A[1,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. counter=counter+1;
. }
```

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```
. G[counter,2]=((A[2,1]+u[i]*A[2,2])*(v[j]*A[2,1]+A[2,2])*(v[k]*A[2,1]+A[2,2]))/
. (A[2,1]*A[2,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2]));
. G[counter,3]=-((A[3,1]+u[i]*A[3,2])*(v[j]*A[3,1]+A[3,2])*(v[k]*A[3,1]+A[3,2]))/
. (A[3,1]*A[3,2]*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. G[counter,4]=((A[4,1]+u[i]*A[4,2])*(v[j]*A[4,1]+A[4,2])*(v[k]*A[4,1]+A[4,2]))/
. (A[4,1]*A[4,2]*(A[1,1]*A[4,2]-A[4,1]*A[1,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. counter=counter+1;
. }
. }
. }
. for (i=1;i<=t;i++)
. {
. for (k=i+1;k<=t+1;k++)
. {
. down=min(dif[i],dif[k]);
. up=max(dif[i],dif[k])-1;
. for (j=down;j<=up;j++)
. {
. G[counter,1]=-((A[1,1]+u[i]*A[1,2])*(A[1,1]+u[k]*A[1,2])*(v[j+1]*A[1,1]+A[1,2]))/
. (A[1,1]*A[1,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[1,1]*A[4,2]-A[4,1]*A[1,2]));
. G[counter,2]=((A[2,1]+u[i]*A[2,2])*(A[2,1]+u[k]*A[2,2])*(v[j+1]*A[2,1]+A[2,2]))/
. (A[2,1]*A[2,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2]));
. G[counter,3]=-((A[3,1]+u[i]*A[3,2])*(A[3,1]+u[k]*A[3,2])*(v[j+1]*A[3,1]+A[3,2]))/
. (A[3,1]*A[3,2]*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. G[counter,4]=((A[4,1]+u[i]*A[4,2])*(A[4,1]+u[k]*A[4,2])*(v[j+1]*A[4,1]+A[4,2]))/
. (A[4,1]*A[4,2]*(A[1,1]*A[4,2]-A[4,1]*A[1,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. counter=counter+1;
. }
. }
. }
. ideal I=IdProjPoints(G);
. print(G);
. return(I);
. }
> matrix A[4][2]=1,1,1,2,1,3,1,4;
> intvec h=1,3,5,3,1;
> ideal K=Gor(h,A,2,2);
```

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```
. down=min(dif[i],dif[k]);
. up=max(dif[i],dif[k])-1;
. for (j=down;j<=up;j++)
. {
. G[counter,1]=-((A[1,1]+u[i]*A[1,2])*(A[1,1]+u[k]*A[1,2])*(v[j+1]*A[1,1]+A[1,2]))/
. (A[1,1]*A[1,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[1,1]*A[4,2]-A[4,1]*A[1,2]));
. G[counter,2]=((A[2,1]+u[i]*A[2,2])*(A[2,1]+u[k]*A[2,2])*(v[j+1]*A[2,1]+A[2,2]))/
. (A[2,1]*A[2,2]*(A[1,1]*A[2,2]-A[2,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2]));
. G[counter,3]=-((A[3,1]+u[i]*A[3,2])*(A[3,1]+u[k]*A[3,2])*(v[j+1]*A[3,1]+A[3,2]))/
. (A[3,1]*A[3,2]*(A[1,1]*A[3,2]-A[3,1]*A[1,2])*(A[2,1]*A[3,2]-A[3,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. G[counter,4]=((A[4,1]+u[i]*A[4,2])*(A[4,1]+u[k]*A[4,2])*(v[j+1]*A[4,1]+A[4,2]))/
. (A[4,1]*A[4,2]*(A[1,1]*A[4,2]-A[4,1]*A[1,2])*(A[2,1]*A[4,2]-A[4,1]*A[2,2])*(A[3,1]*A[4,2]-A[4,1]*A[3,2]));
. counter=counter+1;
. }
. }
. }
. ideal I=IdProjPoints(G);
. print(G);
. return(I);
. }
> matrix A[4][2]=1,1,1,2,1,3,1,4;
> intvec h=1,3,5,3,1;
> ideal K=Gor(h,A,2,2);
-1/2, 2, -5/2,  1,
-5/6, 3, -7/2,  4/3,
-7/6, 4, -9/2,  5/3,
-5/2, 15,-49/2, 12,
-7/2, 20,-63/2, 15,
-15/2,30,-245/6,18,
-21/2,40,-105/2,45/2,
-25/6,27,-91/2, 68/3,
-35/6,36,-117/2,85/3,
-25/2,54,-455/6,34,
-35/2,72,-195/2,85/2,
-3/2, 5, -35/6, 9/4,
-5/2, 9, -65/6, 17/4
> 
```

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```
. }
. ideal I=IdProjPoints(G);
. print(G);
. return(I);
. }
> matrix A[4][2]=1,1,1,2,1,3,1,4;
> intvec h=1,3,5,3,1;
> ideal K=Gor(h,A,2,2);
-1/2, 2, -5/2,  1,
-5/6, 3, -7/2,  4/3,
-7/6, 4, -9/2,  5/3,
-5/2, 15,-49/2, 12,
-7/2, 20,-63/2, 15,
-15/2,30,-245/6,18,
-21/2,40,-105/2,45/2,
-25/6,27,-91/2, 68/3,
-35/6,36,-117/2,85/3,
-25/2,54,-455/6,34,
-35/2,72,-195/2,85/2,
-3/2, 5, -35/6, 9/4,
-5/2, 9, -65/6, 17/4
> ideal K=Gor(h,A,7,5);
// ** redefining K (ideal K=Gor(h,A,7,5);)
-1,   7/2,   -4,     3/2,
-11/6,6,     -13/2,  7/3,
-8/3, 17/2,  -9,     19/6,
-44/3,90,    -143,   203/3,
-64/3,255/2, -198,   551/6,
-88,  315,   -1144/3,609/4,
-128, 1785/4,-528,   1653/8,
-55/2,174,   -559/2, 133,
-40,  493/2, -387,   361/2,
-165, 609,   -2236/3,1197/4,
-240, 3451/4,-1032,  3249/8,
-8,   105/4, -88/3,  87/8,
-15,  203/4, -172/3, 171/8
>
```

Cris — Singular — 132×37

```
> intvec h=1,3,5,3,1;
> ideal K=Gor(h,A,2,2);
-1/2, 2, -5/2,  1,
-5/6, 3, -7/2,  4/3,
-7/6, 4, -9/2,  5/3,
-5/2, 15,-49/2, 12,
-7/2, 20,-63/2, 15,
-15/2,30,-245/6,18,
-21/2,40,-105/2,45/2,
-25/6,27,-91/2, 68/3,
-35/6,36,-117/2,85/3,
-25/2,54,-455/6,34,
-35/2,72,-195/2,85/2,
-3/2, 5, -35/6, 9/4,
-5/2, 9, -65/6, 17/4
> ideal K=Gor(h,A,7,5);
// ** redefining K (ideal K=Gor(h,A,7,5);)
-1,    7/2,   -4,      3/2,
-11/6,6,      -13/2,  7/3,
-8/3, 17/2,  -9,      19/6,
-44/3,90,     -143,    203/3,
-64/3,255/2, -198,    551/6,
-88,  315,    -1144/3,609/4,
-128, 1785/4,-528,    1653/8,
-55/2,174,    -559/2, 133,
-40,  493/2, -387,    361/2,
-165, 609,    -2236/3,1197/4,
-240, 3451/4,-1032,   3249/8,
-8,   105/4, -88/3,   87/8,
-15,  203/4, -172/3,  171/8
> hvec(K);
1
3
5
3
1
> ideal K=Gor(h,A,7,5);
```

Cris — Singular — 132×37

```
> hvec(K);
1
3
5
3
1
> intvec h=1,3,6,6,6,3,1;
// ** redefining h (intvec h=1,3,6,6,6,3,1;)
> ideal K=Gor(h,A,3,4);
// ** redefining K (ideal K=Gor(h,A,3,4);)
-5/6,  3,    -7/2,   4/3,
-3/2,  5,    -11/2,  2,
-13/6, 7,    -15/2,  8/3,
-17/6, 9,    -19/2,  10/3,
-7/2,  11,   -23/2,  4,
-6,    35,   -55,    26,
-26/3, 49,   -75,    104/3,
-34/3, 63,   -95,    130/3,
-14,   77,   -115,   52,
-30,   105,  -385/3, 52,
-130/3,147,  -175,   208/3,
-170/3,189,  -665/3, 260/3,
-70,   231,  -805/3, 104,
-91/6, 91,   -285/2, 200/3,
-119/6,117,  -361/2, 250/3,
-49/2, 143,  -437/2, 100,
-455/6,273,  -665/2, 400/3,
-595/6,351,  -2527/6,500/3,
-245/2,429,  -3059/6,200,
-273/2,455,  -1045/2,200,
-357/2,585,  -3971/6,250,
-441/2,715,  -4807/6,300,
-10/3, 21/2, -35/3,  13/3,
-35/6, 39/2, -133/6, 25/3,
-21/2, 65/2, -209/6, 25/2,
-42,   455/2,-1045/3,325/2
>
```

Cris — Singular — 132×37

```
> ideal K=Gor(h,A,3,4);
// ** redefining K (ideal K=Gor(h,A,3,4);)
-5/6,  3,    -7/2,   4/3,
-3/2,  5,    -11/2,  2,
-13/6, 7,    -15/2,  8/3,
-17/6, 9,    -19/2,  10/3,
-7/2,  11,   -23/2,  4,
-6,    35,   -55,    26,
-26/3, 49,   -75,    104/3,
-34/3, 63,   -95,    130/3,
-14,   77,   -115,   52,
-30,   105,  -385/3, 52,
-130/3,147,  -175,   208/3,
-170/3,189,  -665/3, 260/3,
-70,   231,  -805/3, 104,
-91/6, 91,   -285/2, 200/3,
-119/6,117,  -361/2, 250/3,
-49/2, 143,  -437/2, 100,
-455/6,273,  -665/2, 400/3,
-595/6,351,  -2527/6,500/3,
-245/2,429,  -3059/6,200,
-273/2,455,  -1045/2,200,
-357/2,585,  -3971/6,250,
-441/2,715,  -4807/6,300,
-10/3, 21/2, -35/3,  13/3,
-35/6, 39/2, -133/6, 25/3,
-21/2, 65/2, -209/6, 25/2,
-42,   455/2,-1045/3,325/2
> hvec(K);
1
3
6
6
6
3
1
>
```

The beginning...

Definition

Let $P, Q \in \mathbb{P}^n$ be two points of coordinates respectively

$$[a_0 : a_1 : \cdots : a_n] \text{ and } [b_0 : b_1 : \cdots : b_n].$$

If $a_i b_i \neq 0$ for some i, the Hadamard product $P \star Q$ of P and Q, is defined as

$$P \star Q = [a_0 b_0 : a_1 b_1 : \cdots : a_n b_n].$$

If $a_i b_i = 0$ for all $i = 0, \ldots, n$ then we say $P \star Q$ is not defined.

The Hadamard product of two varieties $X, Y \in \mathbb{P}^n$ is

$$X \star Y = \overline{\{P \star Q : P \in X, Q \in Y, P \star Q \text{ is defined}\}}.$$

Definition

Let A and B be $m \times n$ matrices. The Hadamard product $A \star B$ of A and B is defined as

$$(A \star B)_{ij} = (A)_{ij}(B)_{ij} \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Definition

Let A and B be $m \times n$ matrices. The Hadamard product $A \star B$ of A and B is defined as

$$(A \star B)_{ij} = (A)_{ij}(B)_{ij} \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Definition

Given varieties $X, Y \subset \mathbb{P}^n$ we consider the usual Segre product $X \times Y \subset \mathbb{P}^N$, where $N = (n+1)^2 - 1$, given by

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \longrightarrow [a_0 b_0 : a_0 b_1 : \cdots : a_n b_n]$$

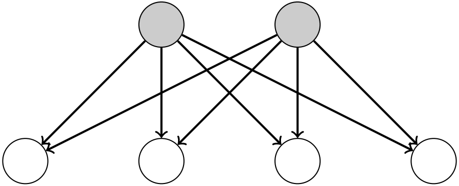
and we denote with z_{ij} the coordinates in $\mathbb{P}^N$.
Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ be the projection map from the linear space Λ defined by equations $z_{ii} = 0, i = 0, \ldots, n$. The Hadamard product of X and Y is

$$X \star Y = \overline{\pi(X \times Y)}.$$

Motivations (from Algebraic Statistics)

"Statistical Models are Algebraic Varieties'

- M.A. Cueto, E.A. Tobis and J. Yu, *An implicitization challenge for binary factor analysis*, J. Symbolic Comput. **45** (2010), no. 12, 1296–1315.
- M.A. Cueto, J. Morton and B. Sturmfels, *Geometry of the restricted Boltzmann machine*, Alg. Methods in Statistics and Probability, AMS, Contemporary Mathematics **516** (2010) 135–153.



where each node represents a binary random variable.

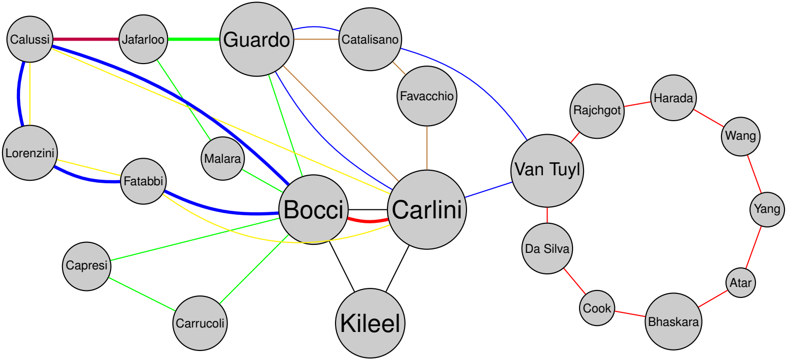
$$V_{\mathcal{M}} = S_2((\mathbb{P}^1)^4) \star S_2((\mathbb{P}^1)^4)$$

What are the properties of $X \star Y$ w.r.t the properties of X and Y ?

- C. Bocci, E. Carlini, J. Kileel, *Hadamard Products of Linear Spaces*, J. of Algebra 448 (2016), 595–617.

What are the properties of $X \star Y$ w.r.t the properties of X and Y ?

- C. Bocci, E. Carlini, J. Kileel, *Hadamard Products of Linear Spaces*, J. of Algebra 448 (2016), 595–617.
  - C. Bocci, G. Calussi, G. Fatabbi and A. Lorenzini, *On Hadamard product of linear varieties*, J. Algebra its Appl. **16**(8) (2017), 155–175.
  - C. Bocci, G. Calussi, G. Fatabbi and A. Lorenzini, *The Hilbert function of some Hadamard products*, Collect. Math. **69**(2) (2018), 205–220.
  - G. Calussi, E. Carlini, G. Fatabbi, and A. Lorenzini, *On the Hadamard product of degenerate subvarieties*, Port. Math. **76**(2) (2019), 123–141.
  - E. Carlini, M. V. Catalisano, E. Guardo, and A. Van Tuyl, *Hadamard star configurations*, Rocky Mt. J. Math. **49**(2) (2019), 419–432.
  - C. Bocci and E. Carlini, *Hadamard products of hypersurfaces*, J. Pure Appl. Algebra **226**(11) (open access) (2022).
  - I. Bahmani Jafarloo and G. Calussi, *Weak Hadamard star configurations and apolarity*, Rocky Mt. J. Math. **50**(3) (2020), 851–862.
  - I. Bahmani Jafarloo, C. Bocci, E. Guardo and G. Malara, *Hadamard products of symbolic powers and Hadamard fat grids*, Mediterranean J. of Maths, https://doi.org/10.1007/s00009-023-02375-5 (open access) (2023).
  - C. Bocci, C. Capresi and D. Carrucoli, *Gorenstein points in $\mathbb{P}^3$ via Hadamard products of projective varieties*, Collect. Math. 10.1007/s13348-022- 00362-9 (open access) (2022).
  - B. Atar, K. Bhaskara, A. Cook, S. Da Silva, M. Harada, J. Rajchgot, A. Van Tuyl, R. Wang and, J. Yang, *Hadamard products of binomial ideals*, arXiv:2211.14210
  - I. Bahmani Jafarloo, *Hadamard*, arXiv:2012.10398.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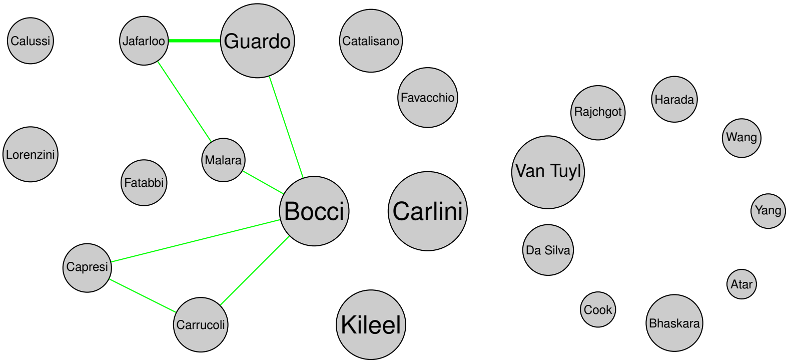
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Basic facts

Definition

Let $H_i \subset \mathbb{P}^n$, $i = 0, \ldots, n$, be the hyperplane $x_i = 0$ and set

$$\Delta_i = \bigcup_{0 \leq j_1 < \ldots < j_{n-i} \leq n} H_{j_1} \cap \ldots \cap H_{j_{n-i}}.$$

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Δ_i is the i–dimensional variety of points having *at most* $i + 1$ non-zero coordinates, or equivalently, with *at least* $n - i$ zero coordinates.

- Δ_0 is the set of coordinates points
- Δ_{n-1} is the union of the coordinate hyperplanes.
- $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{P}^n$.

$P \in \mathbb{P}^n \setminus \Delta_{n-1} \iff P$ has no zero coordinates

Hadamard transformations

Definition

Let $f \in \mathbb{K}[\mathbf{X}]$ be a homogenous polynomial, of degree d, of the form

$$f = \sum_{|I|=d} a_I \mathbf{X}^I$$

where $I = (i_0, \ldots, i_n)$ and $\mathbf{X}^I = x_0^{i_0} \cdots x_n^{i_n}$.
Consider a point $P \in \mathbb{P}^n \setminus \Delta_{n-1}$. The Hadamard transformation of f by P is the polynomial

$$f^{\star P} = \sum_{|I|=d} \frac{a_I}{\mathbf{P}^I} \mathbf{X}^I.$$

where $\mathbf{P}^I$ is the monomial $\mathbf{X}^I$ evaluated in P.

Theorem (B.-Carlini, 2022)

Let $V \subset \mathbb{P}^n$ be a variety and consider a point $P \in \mathbb{P}^n \setminus \Delta_{n-1}$. If $f_1, \ldots, f_s$ is a generating set (resp. a Gröbner bases) for $\mathbb{I}(V)$ with respect to a monomial order $<$, then $f_1^{\star P}, \ldots, f_s^{\star P}$ is a generating set (resp. a Gröbner bases) for $\mathbb{I}(P \star V)$ with respect to the same monomial order $<$.

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Theorem (Atar et Al.,2022)

Let $V \subset \mathbb{P}^n$ be a projective variety and $P = [p_0 : \cdots : p_n]$ with $P \in \mathbb{P}^n \setminus \Delta_{n-1}$. Suppose $\{f_1 \dots, f_s\}$ is a reduced Gröbner basis with respect to a monomial order $<$ for $\mathbb{I}(V)$ with $LT(f_i) = \mathbf{X}^{I_i}$ for $i = 1, \dots, s$. Then

$$\{\mathbf{P}^{I_1} f_1^{\star P}, \dots, \mathbf{P}^{I_s} f_s^{\star P}\}$$

is a reduced Gröbner basis for $\mathbb{I}(P \star V)$ with respect to the same monomial order $<$.

Definition

Let I and J be homogeneous ideals in $\mathbb{K}[x_0, \ldots, x_n]$. The Hadamard product of ideals I and J, denoted $I \star J$, is the ideal constructed via the following algorithm

- Define the polynomial ring $\mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n]$.
- Let $I(\mathbf{y}) := \langle f(y_0, \ldots, y_n) \mid f(x_0, \ldots, x_n) \in I \rangle$ to be the ideal obtained by replacing x_i with y_i for all elements of I, and similarly, let $J(\mathbf{z})$ be the ideal obtained by replacing x_i with z_i for all elements of J.
- Define the ideal

$$K := I(\mathbf{y}) + J(\mathbf{z}) + \langle x_0 - y_0 z_0, x_1 - y_1 z_1, \ldots, x_n - y_n z_n \rangle.$$

- Finally, define

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Lemma

Let $X, Y \subset \mathbb{P}^n$ be projective varieties with defining (radical) ideals $\mathbb{I}(X)$ and $\mathbb{I}(Y)$ in $\mathbb{K}[x_0, \ldots, x_n]$. Then $\mathbb{I}(X) \star \mathbb{I}(Y) = \mathbb{I}(X \star Y)$.

```
proc HadProd(ideal I, ideal J)
{
def @r=basering;
ideal @M=maxideal(1);
int n=nvars(basering);
int @char=char(basering);
int i;
ring RH=@char,(x(1..n),y(1..n),z(1..n)),dp;
map F1=@r,y(1..n);
map F2=@r,z(1..n);
ideal S=F1(I),F2(J);
ideal H=0;
for (i=1; i<=n; i=i+1)
   {
   H=H+ideal(x(i)-y(i)*z(i));
   }
ideal T=H+S;
ideal K=elim(T,n+1..2*n);
setring @r;
map f=RH,@M;
return(std(f(K)));
}
```

Gorenstein points in $\mathbb{P}^3$

In

- C. Bocci, E. Carlini, J. Kileel, *Hadamard Products of Linear Spaces*, J. of Algebra 448 (2016), 595–617.

we show how to build star configurations via Hadamard products...

..and more general results can be found in

- E. Carlini, M. V. Catalisano, E. Guardo, and A. Van Tuyl, *Hadamard star configurations*, Rocky Mt. J. Math. **49**(2) (2019), 419–432.
- I. Bahmani Jafarloo and G. Calussi, *Weak Hadamard star configurations and apolarity*, Rocky Mt. J. Math. **50**(3) (2020), 851–862.

Thus, the question if other interesting geometrical objects can be obtained by Hadamard products naturally arises.

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- C. Bocci and G. Dalzotto, *Gorenstein points in* $\mathbb{P}^3$, Rend. Sem. Mat. Univ. Politec. Torino **59**(1) (2001), 155–164.

Thus, the question if other interesting geometrical objects can be obtained by Hadamard products naturally arises.

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The Gorenstein set of points is obtained, by Liasion Theory, as the intersection of two aCM curves, linked by a complete intersection. The approach here is related to the well-known construction of Migliore and Nagel, where the complete intersection is a stick figure of lines.

Definition

A generalized stick figure is a union of linear subvarieties of $\mathbb{P}^n$, of the same dimension d, such that the intersection of any three components has dimension at most $d - 2$ (the empty set has dimension -1).

Planar complete intersections

We start building a zero-dimensional planar complete intersection $Z \subset \mathbb{P}^3$, as the Hadamard product of two sets of collinear points X and X'.

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Proposition (B.-Calussi-Fatabbi-Lorenzini, 2018)

Let L, L' be two generic distinct lines in $\mathbb{P}^3$. There is a generic choice of a finite set of points $X \subseteq L$ for which it is possible a generic choice of a finite set of points $X' \subseteq L'$ such that:

(1) $X \star X' = (X \star L') \cap (X' \star L)$ *and* $|X \star X'| = |X||X'|$.

(2) $L \star L'$ *is an irreducible and non-degenerate quadric, and* $X \star L'$ *and* $X' \star L$ *are lines of the two different rulings.*

what about coplanar lines in $\mathbb{P}^3$?

what about coplanar lines in $\mathbb{P}^3$?

$$L_1 = \begin{cases} 3x_1 + 4x_2 - 7x_3 = 0 \\ 7x_0 - 4x_1 - 3x_2 = 0 \end{cases} \qquad L_2 = \begin{cases} x_1 + 24x_2 - 25x_3 = 0 \\ 10x_0 - x_1 - 9x_2 = 0; \end{cases}$$

L_1 and L_2 are coplanar but $L_1 \star L_2$ is the quadric of equation

$1120x_0^2 - 68x_0x_1 + x_1^2 + 1056x_0x_2 - 30x_1x_2 + 216x_2^2 - 3500x_0x_3 + 110x_1x_3 - 1530x_2x_3 + 2625x_3^2$

A tricky construction

Let $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ be a collection of four distinct points in $\mathbb{P}^1 \setminus \Delta_0$, where $A_i = [\alpha_i : \beta_i]$, for $i = 0, \ldots, 3$,

We define two families of points in $\mathbb{P}^3$ associated to the set $\mathcal{A}$

$$P_k^{\mathcal{A}} = \left[\frac{\alpha_0 + k\beta_0}{\alpha_0} : \frac{\alpha_1 + k\beta_1}{\alpha_1} : \frac{\alpha_2 + k\beta_2}{\alpha_2} : \frac{\alpha_3 + k\beta_3}{\alpha_3}\right] \quad k \in \mathbb{N}$$

$$Q_k^{\mathcal{A}} = \left[\frac{k\alpha_0 + \beta_0}{\beta_0} : \frac{k\alpha_1 + \beta_1}{\beta_1} : \frac{k\alpha_2 + \beta_2}{\beta_2} : \frac{k\alpha_3 + \beta_3}{\beta_3}\right] \quad k \in \mathbb{N}.$$

$$P_0^{\mathcal{A}} = Q_0^{\mathcal{A}} = [1 : 1 : 1 : 1]$$

It is easy to verify that, for $i \geq 2$,

$$P_i^{\mathcal{A}} = (1-i)P_0^{\mathcal{A}} + iP_1^{\mathcal{A}} \qquad Q_i^{\mathcal{A}} = (1-i)Q_0^{\mathcal{A}} + iQ_1^{\mathcal{A}}.$$

- let ℓ^P be the line spanned by $P_0^{\mathcal{A}}$ and $P_1^{\mathcal{A}}$
  - for any fixed k, the points $P_0^{\mathcal{A}}, \dots, P_k^{\mathcal{A}}$ are collinear.
- let ℓ^Q be the line spanned by $Q_0^{\mathcal{A}}$ and $Q_1^{\mathcal{A}}$
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  - for any fixed k, the points $Q_0^{\mathcal{A}}, \ldots, Q_k^{\mathcal{A}}$ are collinear.

- ℓ^P and ℓ^Q are two distinct coplanar lines.
- One has $P_i^{\mathcal{A}} \neq Q_j^{\mathcal{A}}$, for every $i, j \geq 1$.

Example

Consider $A_0 = [1:1]$, $A_1 = [1:2]$, $A_2 = [1:3]$, *and* $A_3 = [1:4]$.

Hence we get

$$P_1^{\mathcal{A}} = [2:3:4:5]\,, P_2^{\mathcal{A}} = [3:5:7:9]\,, P_3^{\mathcal{A}} = [4:7:10:13]\,, \ldots$$

$$Q_1^{\mathcal{A}} = \left[2:\frac{3}{2}:\frac{4}{3}:\frac{5}{4}\right], Q_2^{\mathcal{A}} = \left[3:2:\frac{5}{3}\,\frac{3}{2}\right], Q_3^{\mathcal{A}} = \left[4:\frac{5}{2}:2:\frac{7}{4}\right], \ldots$$

For our construction we need to avoid the situation in which $P_i^{\mathcal{A}}$ and $Q_j^{\mathcal{A}}$ have zero coordinate (for any i and j).

$$P_k^{\mathcal{A}} = \left[\frac{\alpha_0 + k\beta_0}{\alpha_0} : \frac{\alpha_1 + k\beta_1}{\alpha_1} : \frac{\alpha_2 + k\beta_2}{\alpha_2} : \frac{\alpha_3 + k\beta_3}{\alpha_3}\right] \quad k \in \mathbb{N}$$

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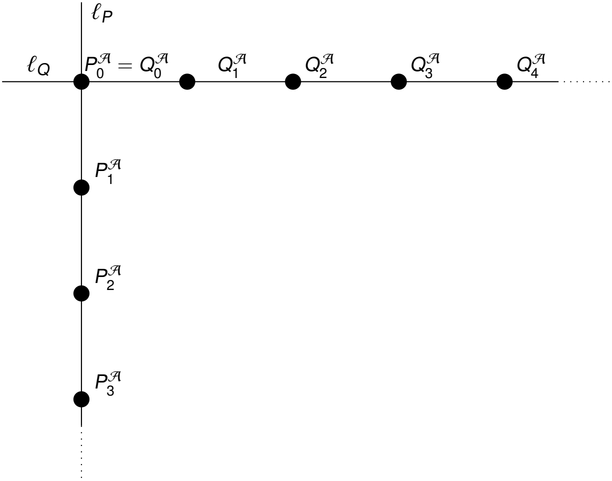
$$\boxed{\mathcal{W} = \bigcup_{i \in \mathbb{N}^*} \left\{[1 : -i], \left[1 : -\frac{1}{i}\right]\right\} \subset \mathbb{P}^1}$$

Lemma

If $A_i \notin \mathcal{W}$, for $i = 0, \ldots, 3$, then $P_i^{\mathcal{A}} \notin \Delta_2$, for any i, and $Q_j^{\mathcal{A}} \notin \Delta_2$, for any j, that is they do not have any zero coordinate.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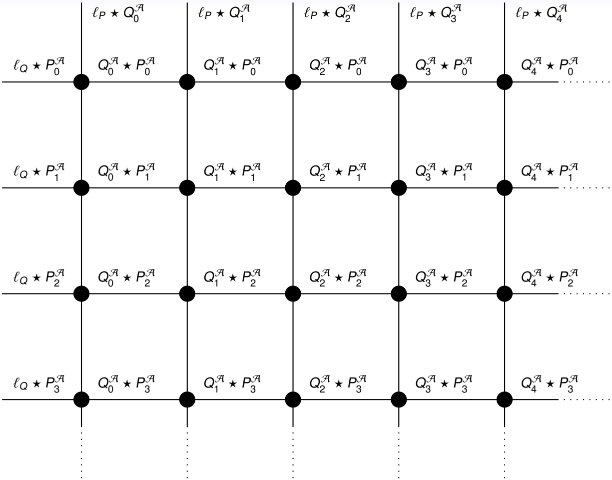
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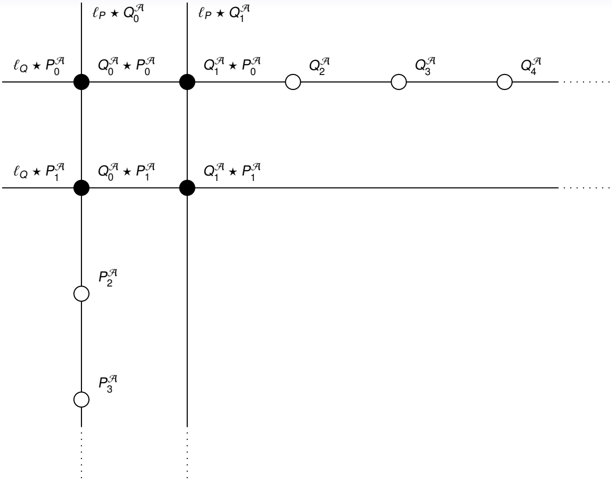
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Denote by $\mathcal{I}(n) = \{i_0, i_1, \ldots, i_{n-1}\}$ a set of nonnegative integers with

$$0 = i_0 < i_1 < \cdots < i_{n-1}.$$

Given positive integers a and b, we define the set of points $Z^{\mathcal{A}}_{a,b}$ in the following way:

$$Z^{\mathcal{A}}_{a,b} = \{P^{\mathcal{A}}_i \star Q^{\mathcal{A}}_j \,:\, i \in \mathcal{I}(a), j \in \mathcal{I}(b)\}.$$

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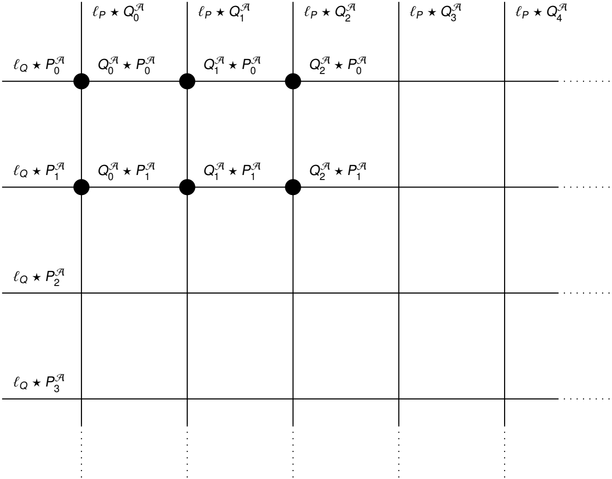
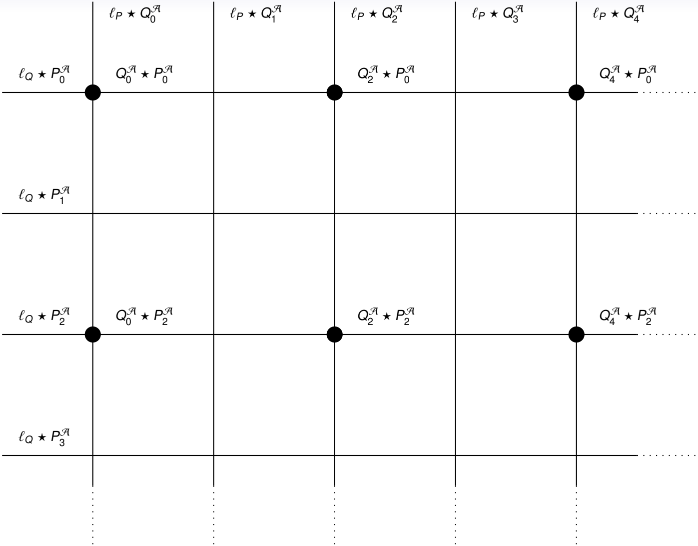
Theorem (B.-Capresi-Carrucoli, 2022)

If the points A_i are distinct and $A_i \notin \Delta_0 \cup \mathcal{W}$, for $i = 0, \dots, 3$, then, for any positive integers a and b, $Z^{\mathcal{A}}_{a,b}$ is a planar complete intersection of ab points in $\mathbb{P}^3$.



$$\mathcal{I}(a) = \{0, 1\} \text{ and } \mathcal{I}(b) = \{0, 1, 2\}$$



$$\mathcal{I}(a) = \{0, 2\} \text{ and } \mathcal{I}(b) = \{0, 2, 4\}$$

Corollary

Suppose that

$$\mathbb{I}(\ell^P) = \langle h, f \rangle \quad \textit{and} \quad \mathbb{I}(\ell^Q) = \langle h, g \rangle$$

Then the ideal of $Z^{\mathcal{A}}_{a,b}$ *is generated by*

$$h, \qquad \prod_{j=0}^{b-1} f^{\star Q^{\mathcal{A}}_{i_j}}, \qquad \prod_{j=0}^{a-1} g^{\star P^{\mathcal{A}}_{i_j}}.$$

$$\begin{matrix}[\alpha_i, \beta_i] \in \mathbb{P}^1 \setminus (\Delta_0 \cup \mathcal{W}) \\ i = 0, \ldots, 3\end{matrix} \longrightarrow \left.\begin{matrix} P_k = \left[\frac{\alpha_0 + k\beta_0}{\alpha_0}, \frac{\alpha_1 + k\beta_1}{\alpha_1}, \frac{\alpha_2 + k\beta_2}{\alpha_2}, \frac{\alpha_3 + k\beta_3}{\alpha_3}\right] \\ Q_k = \left[\frac{k\alpha_0 + \beta_0}{\beta_0}, \frac{k\alpha_1 + \beta_1}{\beta_1}, \frac{k\alpha_2 + \beta_2}{\beta_2}, \frac{k\alpha_3 + \beta_3}{\beta_3}\right]\end{matrix}\right\} \longrightarrow Z_{a,b}^{\mathcal{A}}$$

$$[\alpha_i,\beta_i] \in \mathbb{P}^1 \setminus (\Delta_0 \cup \mathcal{W}) \longrightarrow \left.\begin{matrix} P_k = \left[\frac{\alpha_0+k\beta_0}{\alpha_0}, \frac{\alpha_1+k\beta_1}{\alpha_1}, \frac{\alpha_2+k\beta_2}{\alpha_2}, \frac{\alpha_3+k\beta_3}{\alpha_3}\right] \\ Q_k = \left[\frac{k\alpha_0+\beta_0}{\beta_0}, \frac{k\alpha_1+\beta_1}{\beta_1}, \frac{k\alpha_2+\beta_2}{\beta_2}, \frac{k\alpha_3+\beta_3}{\beta_3}\right] \end{matrix}\right\} \longrightarrow Z^{\mathcal{A}}_{a,b}$$

$$i = 0, \dots, 3$$

$$\downarrow$$

$$L^{\mathcal{A}} : \begin{cases} \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \end{cases}$$

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The condition that the four points A_i are distinct, i.e. $\frac{\alpha_i}{\beta_i} \neq \frac{\alpha_j}{\beta_j}$ for any $0 \leq i < j \leq 3$, is quite important for our constructions since it assures that $L^{\mathcal{A}} \cap \Delta_1 = \emptyset$.

$$[\alpha_i,\beta_i]\in\mathbb{P}^1\setminus(\Delta_0\cup\mathcal{W}) \quad i=0,\dots,3 \longrightarrow \left.\begin{array}{l} P_k=\left[\frac{\alpha_0+k\beta_0}{\alpha_0},\frac{\alpha_1+k\beta_1}{\alpha_1},\frac{\alpha_2+k\beta_2}{\alpha_2},\frac{\alpha_3+k\beta_3}{\alpha_3}\right] \\ Q_k=\left[\frac{k\alpha_0+\beta_0}{\beta_0},\frac{k\alpha_1+\beta_1}{\beta_1},\frac{k\alpha_2+\beta_2}{\beta_2},\frac{k\alpha_3+\beta_3}{\beta_3}\right]\end{array}\right\} \longrightarrow Z^{\mathcal{A}}_{a,b}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$L^{\mathcal{A}}:\begin{cases}\alpha_0x_0+\alpha_1x_1+\alpha_2x_2+\alpha_3x_3=0\\ \beta_0x_0+\beta_1x_1+\beta_2x_2+\beta_3x_3=0\end{cases} \longrightarrow Z^{\mathcal{A}}_{a,b}\star L^{\mathcal{A}}$$

$$[\alpha_i,\beta_i] \in \mathbb{P}^1 \setminus (\Delta_0 \cup \mathcal{W}),\ i=0,\dots,3 \longrightarrow \left. \begin{aligned} P_k &= \left[\tfrac{\alpha_0+k\beta_0}{\alpha_0}, \tfrac{\alpha_1+k\beta_1}{\alpha_1}, \tfrac{\alpha_2+k\beta_2}{\alpha_2}, \tfrac{\alpha_3+k\beta_3}{\alpha_3}\right] \\ Q_k &= \left[\tfrac{k\alpha_0+\beta_0}{\beta_0}, \tfrac{k\alpha_1+\beta_1}{\beta_1}, \tfrac{k\alpha_2+\beta_2}{\beta_2}, \tfrac{k\alpha_3+\beta_3}{\beta_3}\right] \end{aligned} \right\} \longrightarrow Z^{\mathcal{A}}_{a,b}$$

$$\big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow$$

$$L^{\mathcal{A}} : \begin{cases} \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 \\ \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \end{cases} \longrightarrow Z^{\mathcal{A}}_{a,b} \star L^{\mathcal{A}}$$

Theorem (B.-Capresi-Carrucoli, 2022)

Assume that $1 \notin \mathcal{I}(a) \cup \mathcal{I}(b)$. Then $Z^{\mathcal{A}}_{a,b} \star L^{\mathcal{A}}$ is a stick figure of ab lines in $\mathbb{P}^3$. Moreover $Z^{\mathcal{A}}_{a,b} \star L^{\mathcal{A}}$ is a complete intersection.

Theorem

Let C_1, C_2 be two aCM varieties of $\mathbb{P}^n$ of codimension c, with no common components and saturated ideals I_{C_1} and I_{C_2}. If we suppose that $X = C_1 \cup C_2$ is a codimension c arithmetically Gorenstein variety, then $\mathbb{I}(C_1) + \mathbb{I}(C_2)$ is the saturated ideal of a codimension $c + 1$ arithmetically Gorenstein variety Y.

Theorem

Let C_1, C_2 be two aCM varieties of $\mathbb{P}^n$ of codimension c, with no common components and saturated ideals I_{C_1} and I_{C_2}. If we suppose that $X = C_1 \cup C_2$ is a codimension c arithmetically Gorenstein variety, then $\mathbb{I}(C_1) + \mathbb{I}(C_2)$ is the saturated ideal of a codimension $c+1$ arithmetically Gorenstein variety Y.

$$\mathbf{h} = (h_0, h_1, \ldots, h_s) = (1, 3, h_2, \ldots, h_{t-1}, h_t, h_t, \ldots, h_t, h_{t-1}, \ldots, h_2, 3, 1)$$

Define $\mathbf{a} = (a_0, \ldots, a_t)$ and $\mathbf{g} = (g_0, \ldots, g_{s+1})$ as

$$a_i = h_i - h_{i-1} \text{ for } 0 \leq i \leq t$$

and

$$g_i = \begin{cases} i+1 & \text{for } 0 \leq i \leq t \\ t+1 & \text{for } t \leq i \leq s-t+1 \\ s-i+2 & \text{for } s-t+1 \leq i \leq s+1 \end{cases}$$

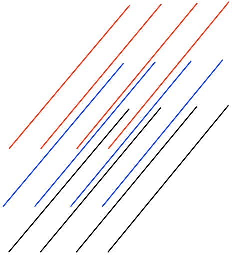
g is h−vector of a Complete Intersection in $\mathbb{P}^3$ of type $(t+1, s-t+2)$.

$$\mathbf{h} = (1, 3, 5, 3, 1)$$

$$\Delta\mathbf{h} = \quad (1, \quad 2, \quad 2 \quad -2, \quad -2, \quad -1)$$

$$\mathbf{h} = (1, 3, 5, 3, 1)$$

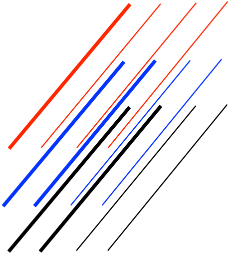
$$\begin{array}{rcccccc} \Delta\mathbf{h} = & (1, & 2, & 2 & -2, & -2, & -1) \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{g} = & (1, & 2, & 3, & 3, & 2, & 1) \end{array}$$



A complete intersection of type $(3, 4)$

$$\mathbf{h} = (1,3,5,3,1)$$

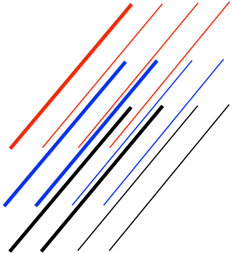
$$\begin{array}{rcccccc} \Delta\mathbf{h} = & (1, & 2, & 2 & -2, & -2, & -1) \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{g} = & (1, & 2, & 3, & 3, & 2, & 1) \end{array}$$



A complete intersection of type $(3,4)$

$$\mathbf{a} = (1,2,2)$$

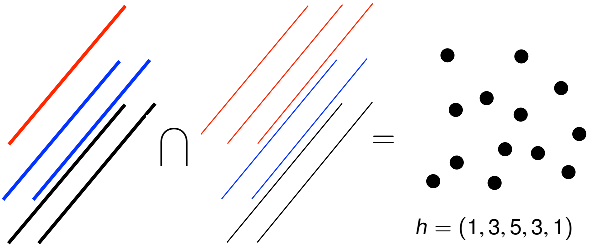
$$\mathbf{h} = (1, 3, 5, 3, 1)$$



$$\begin{array}{lcccccc} \Delta\mathbf{h} = & (1, & 2, & 2 & -2, & -2, & -1) \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{g} = & (1, & 2, & 3, & 3, & 2, & 1) \end{array}$$

A complete intersection of type $(3, 4)$

$$\mathbf{a} = (1, 2, 2)$$



$$h = (1, 3, 5, 3, 1)$$

$$\mathbf{h} = (1, 3, h_2, \dots, h_{t-1}, h_t, h_t, \dots, h_t, h_{t-1}, \dots, h_2, 3, 1)$$
$$\mathbf{a} = (a_0, \dots, a_t) \qquad \mathbf{g} = (g_0, \dots, g_{s+1})$$

g is h−vector of a complete intersection X in $\mathbb{P}^3$ of degrees $t + 1$ and $s - t + 2$.

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g is h–vector of a complete intersection X in $\mathbb{P}^3$ of degrees $t+1$ and $s-t+2$.

We choose $\mathcal{A}$, $\mathcal{I}(t+1)$ and $\mathcal{I}(s-t+2)$ and then we set

$$X = Z^{\mathcal{A}}_{t+1,s-t+2} \star L^{\mathcal{A}}.$$

$$\mathbf{h} = (1,3,h_2,\dots,h_{t-1},h_t,h_t,\dots,h_t,h_{t-1},\dots,h_2,3,1)$$
$$\mathbf{a} = (a_0,\dots,a_t) \qquad \mathbf{g} = (g_0,\dots,g_{s+1})$$

g is $h-$vector of a complete intersection X in $\mathbb{P}^3$ of degrees $t+1$ and $s-t+2$.

We choose $\mathcal{A}$, $\mathcal{I}(t+1)$ and $\mathcal{I}(s-t+2)$ and then we set

$$X = Z^{\mathcal{A}}_{t+1,s-t+2} \star L^{\mathcal{A}}.$$

Thus the aCM scheme C_1 with $h-$vector **a** is given by the following set of lines in $Z^{\mathcal{A}}_{t+1,s-t+2} \star L^{\mathcal{A}}$:

$$P^{\mathcal{A}}_{u_i} \star Q^{\mathcal{A}}_{v_j} \star L^{\mathcal{A}} \text{ for } j = 0,\dots,a_i - 1 \text{ and } i = 0,\dots,t.$$

Corollary (B.-Capresi-Carrucoli, 2022)

Let $\mathbf{h}$ *be an admissible h−vector for a Gorenstein zeroscheme in* $\mathbb{P}^3$ *of the form*

$$\mathbf{h} = (h_0, \ldots, h_s) = (1, 3, h_2, \ldots, h_{t-1}, h_t, h_t, \ldots, h_t, h_{t-1}, \ldots, 3, 1)$$

and let $a_i = h_i - h_{i-1}$ *for* $0 \leq i \leq t$.
Fix four distinct points $A_i = [\alpha_i : \beta_i]$ *in* $\mathbb{P}^1 \setminus (\Delta_0 \cup \mathcal{W})$, *for* $i = 0, \ldots, 3$ *and fix the sets of nonnegative integers*

$$\mathcal{I}(t+1) = \{u_0, \ldots, u_t\} \text{ and } \mathcal{I}(s-t+2) = \{v_0, \ldots, v_{s-t+1}\}$$

with $0 \in \mathcal{I}(t+1) \cap \mathcal{I}(s-t+2)$ *and* $1 \notin \mathcal{I}(t+1) \cup \mathcal{I}(s-t+2)$.
Then the following set of points is a Gorenstein zeroscheme with h−vector $\mathbf{h}$.

$$\left\{\begin{array}{l}\begin{bmatrix} -\dfrac{(\alpha_0+u_i\beta_0)(v_j\alpha_0+\beta_0)(v_l\alpha_0+\beta_0)}{\alpha_0\beta_0(\alpha_0\beta_1-\alpha_1\beta_0)(\alpha_0\beta_2-\alpha_2\beta_0)(\alpha_0\beta_3-\alpha_3\beta_0)} \\[2ex] \dfrac{(\alpha_1+u_i\beta_1)(v_j\alpha_1+\beta_1)(v_l\alpha_1+\beta_1)}{\alpha_1\beta_1(\alpha_0\beta_1-\alpha_1\beta_0)(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_1\beta_3-\alpha_3\beta_1)} \\[2ex] -\dfrac{(\alpha_2+u_i\beta_2)(v_j\alpha_2+\beta_2)(v_l\alpha_2+\beta_2)}{\alpha_2\beta_2(\alpha_0\beta_2-\alpha_2\beta_0)(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_2\beta_3-\alpha_3\beta_2)} \\[2ex] \dfrac{(\alpha_3+u_i\beta_3)(v_j\alpha_3+\beta_3)(v_l\alpha_3+\beta_3)}{\alpha_3\beta_3(\alpha_0\beta_3-\alpha_3\beta_0)(\alpha_1\beta_3-\alpha_3\beta_1)(\alpha_2\beta_3-\alpha_3\beta_2)} \end{bmatrix} \quad \begin{array}{l}\text{with} \\ 0 \le j \le a_i - 1, \\ a_i \le k \le s-t+1, \\ \text{for } i = 0, \ldots, t\end{array} \\[4ex] \begin{bmatrix} -\dfrac{(\alpha_0+u_i\beta_0)(\alpha_0+u_k\beta_0)(v_j\alpha_0+\beta_0)}{\alpha_0\beta_0(\alpha_0\beta_1-\alpha_1\beta_0)(\alpha_0\beta_2-\alpha_2\beta_0)(\alpha_0\beta_3-\alpha_3\beta_0)} \\[2ex] \dfrac{(\alpha_1+u_i\beta_1)(\alpha_1+u_k\beta_1)(v_j\alpha_1+\beta_1)}{\alpha_1\beta_1(\alpha_0\beta_1-\alpha_1\beta_0)(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_1\beta_3-\alpha_3\beta_1)} \\[2ex] -\dfrac{(\alpha_2+u_i\beta_2)(\alpha_2+u_k\beta_2)(v_j\alpha_2+\beta_2)}{\alpha_2\beta_2(\alpha_0\beta_2-\alpha_2\beta_0)(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_2\beta_3-\alpha_3\beta_2)} \\[2ex] \dfrac{(\alpha_3+u_i\beta_3)(\alpha_3+u_k\beta_3)(v_j\alpha_3+\beta_3)}{\alpha_3\beta_3(\alpha_0\beta_3-\alpha_3\beta_0)(\alpha_1\beta_3-\alpha_3\beta_1)(\alpha_2\beta_3-\alpha_3\beta_2)} \end{bmatrix} \quad \begin{array}{l}\text{with} \\ \min\{a_i, a_k\} \le j \le \max\{a_i, a_k\} - 1, \\ \text{for } 0 \le i < k \le t\end{array}\end{array}\right.$$

$h = (1,3,4,3,1)$ $t = 2$, $s = 4$ and $\mathbf{a} = (1,2,1)$.

$$A_0 = [1:1],\ A_1 = [1:2],\ A_2 = [1:3],\ A_3 = [1:4]$$

$$\mathcal{I}(t+1) = \{u_0, u_1, u_2\} \quad = \{0,3,5\}$$

$$\mathcal{I}(s-t+2) = \{v_0, v_1, v_2, v_3\} \quad = \{0,6,7,8\}.$$

$[-\frac{7}{6} : 4 : -\frac{9}{2} : \frac{5}{3}]$, $[-\frac{4}{3} : \frac{9}{2} : -5 : \frac{11}{6}]$, $[-\frac{3}{2} : 5 : -\frac{11}{2} : 2]$,

$[-\frac{16}{3} : \frac{63}{2} : -50 : \frac{143}{6}]$, $[-6 : 35 : -55 : 26]$, $[-\frac{112}{3} : 126 : -150 : \frac{715}{12}]$,

$[-42 : 140 : -165 : 65]$, $[-7 : 44 : -72 : 35]$, $[-8 : \frac{99}{2} : -80 : \frac{77}{2}]$,

$[-9 : 55 : -88 : 42]$, $[-\frac{14}{3} : 14 : -15 : \frac{65}{12}]$, $[-28 : 154 : -240 : \frac{455}{4}]$

$h = (1, 3, 6, 10, 6, 3, 1)$ $t = 3$, $s = 6$ and $\mathbf{a} = (1, 2, 3, 4)$.

$$A_0 = [1 : 1],\ A_1 = [1 : 2],\ A_2 = [1 : 3],\ A_3 = [1 : 4]$$

$$\mathcal{I}(t+1) = \{u_0, u_1, u_2\} \quad = \{0, 2, 4, 6\}$$

$$\mathcal{I}(s-t+2) = \{v_0, v_1, v_2, v_3\} \quad = \{0, 2, 4, 6, 8\}.$$

$[-\frac{1}{2} : 2 : -\frac{5}{2} : 1]$,	$[-\frac{5}{6} : 3 : -\frac{7}{2} : \frac{4}{3}]$,	$[-\frac{7}{6} : 4 : -\frac{9}{2} : \frac{5}{3}]$,
$[-\frac{3}{2} : 5 : -\frac{11}{2} : 2]$,	$[-\frac{5}{2} : 15 : -\frac{49}{2} : 12]$,	$[-\frac{7}{2} : 20 : -\frac{63}{2} : 15]$,
$[-\frac{9}{2} : 25 : -\frac{77}{2} : 18]$,	$[-\frac{15}{2} : 30 : -\frac{245}{6} : 18]$,	$[-\frac{21}{2} : 40 : -\frac{105}{2} : \frac{45}{2}]$,
$[-\frac{27}{2} : 50 : -\frac{385}{6} : 27]$,	$[-\frac{35}{6} : 36 : -\frac{117}{2} : \frac{85}{3}]$,	$[-\frac{15}{2} : 45 : -\frac{143}{2} : 34]$,
$[-\frac{35}{2} : 72 : -\frac{195}{2} : \frac{85}{2}]$,	$[-\frac{45}{2} : 90 : -\frac{715}{6} : 51]$,	$[-\frac{175}{6} : 108 : -\frac{273}{2} : \frac{170}{3}]$,
$[-\frac{75}{2} : 135 : -\frac{1001}{6} : 68]$,	$[-\frac{21}{2} : 65 : -\frac{209}{2} : 50]$,	$[-\frac{63}{2} : 130 : -\frac{1045}{6} : 75]$,
$[-\frac{105}{2} : 195 : -\frac{1463}{6} : 100]$,	$[-\frac{147}{2} : 260 : -\frac{627}{2} : 125]$,	$[-\frac{3}{2} : 5 : -\frac{35}{6} : \frac{9}{4}]$,
$[-\frac{5}{2} : 9 : -\frac{65}{6} : \frac{17}{4}]$,	$[-\frac{25}{6} : \frac{27}{2} : -\frac{91}{6} : \frac{17}{3}]$,	$[-\frac{7}{2} : 13 : -\frac{95}{6} : \frac{25}{4}]$,
$[-\frac{35}{6} : \frac{39}{2} : -\frac{133}{6} : \frac{25}{3}]$,	$[-\frac{49}{6} : 26 : -\frac{57}{2} : \frac{125}{12}]$,	$[-\frac{25}{2} : \frac{135}{2} : -\frac{637}{6} : 51]$,
$[-\frac{35}{2} : \frac{195}{2} : -\frac{931}{6} : 75]$,	$[-\frac{49}{2} : 130 : -\frac{399}{2} : \frac{375}{4}]$,	$[-\frac{245}{6} : 234 : -\frac{741}{2} : \frac{2125}{12}]$

Fat points

CMO Workshop “Ordinary and Symbolic Powers of Ideals” (May 14-19, 2017, Oxaqua, Mexico)

Is it true that for P, Q generic points in $\mathbb{P}^2$, one has

$$\mathbb{I}(P)^r \star \mathbb{I}(Q)^s = \mathbb{I}(P \star Q)^{r+s-1}?$$

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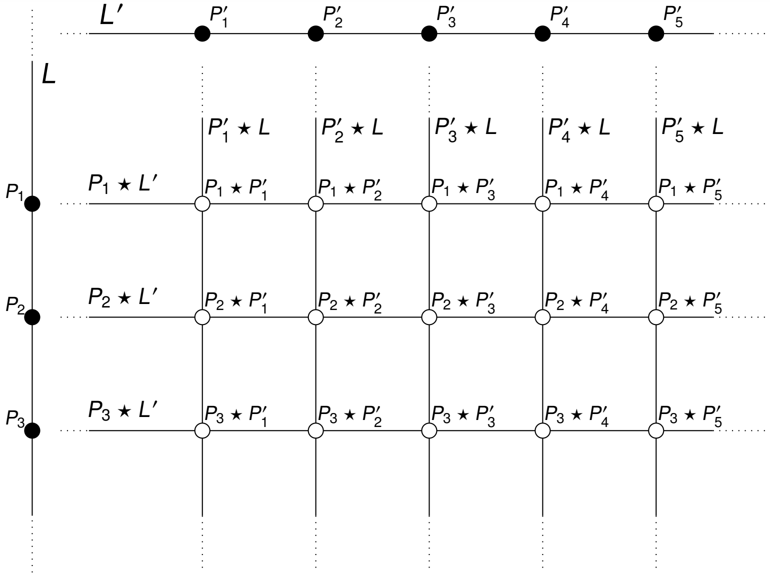
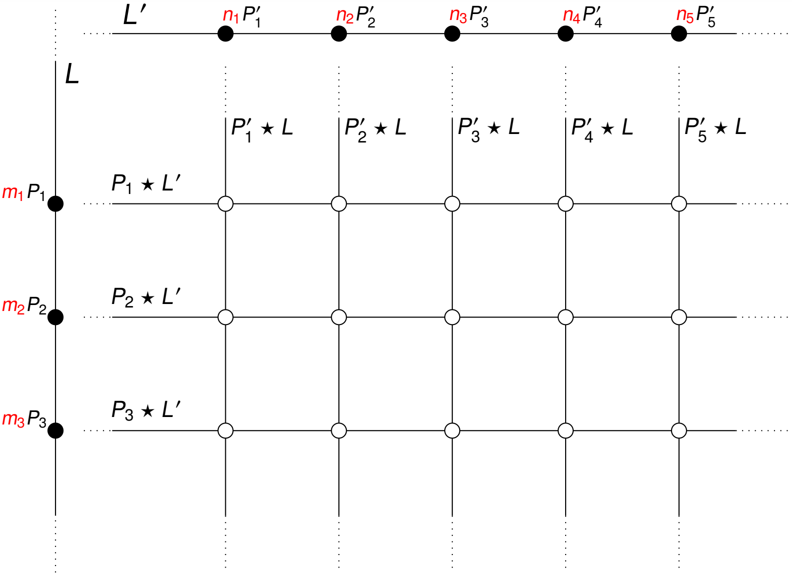
Theorem (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let P and Q be two points in $\mathbb{P}^2 \setminus \Delta_1$. Then for $r, s \geq 1$ one has $\mathbb{I}(P)^r \star \mathbb{I}(Q)^s = \mathbb{I}(P \star Q)^{r+s-1}$.



L′
P′1 P′2 P′3 P′4 P′5
L
P′1 ⋆ L P′2 ⋆ L P′3 ⋆ L P′4 ⋆ L P′5 ⋆ L
P1 P1 ⋆ L′ P1 ⋆ P′1 P1 ⋆ P′2 P1 ⋆ P′3 P1 ⋆ P′4 P1 ⋆ P′5
P2 P2 ⋆ L′ P2 ⋆ P′1 P2 ⋆ P′2 P2 ⋆ P′3 P2 ⋆ P′4 P2 ⋆ P′5
P3 P3 ⋆ L′ P3 ⋆ P′1 P3 ⋆ P′2 P3 ⋆ P′3 P3 ⋆ P′4 P3 ⋆ P′5



L'
$n_1 P'_1$
$n_2 P'_2$
$n_3 P'_3$
$n_4 P'_4$
$n_5 P'_5$
L
$P'_1 \star L$
$P'_2 \star L$
$P'_3 \star L$
$P'_4 \star L$
$P'_5 \star L$
$m_1 P_1$
$P_1 \star L'$
$m_2 P_2$
$P_2 \star L'$
$m_3 P_3$
$P_3 \star L'$

Hadamard fat grids

Let $P_M = \{P_1, \ldots, P_r\}$ and $Q_N = \{Q_1, \ldots, Q_s\}$ be two sets of collinear points in $\mathbb{P}^2 \setminus \Delta_1$ with assigned positive multiplicities, respectively, $M = \{m_1, \ldots, m_r\}$ and $N = \{n_1, \ldots, n_s\}$.

$$I(P_M) = I(P_1)^{m_1} \cap \cdots \cap I(P_r)^{m_r}$$

$$I(Q_N) = I(Q_1)^{n_1} \cap \cdots \cap I(Q_s)^{n_s}.$$

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$$I(Q_N) = I(Q_1)^{n_1} \cap \cdots \cap I(Q_s)^{n_s}.$$

Definition

Assume that $P_i \star Q_j \neq P_k \star Q_l$ for all $1 \leq i < k \leq r$ and $1 \leq j < l \leq s$. Then the set of fat points defined by $I(P_M) \star I(Q_N)$, is called a Hadamard fat grid and it is denoted by $HFG(P_M, Q_N)$.

Proposition (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let I, J be two ideals in $\mathbb{K}[\mathbf{X}]$ with primary decomposition respectively $I = I_1 \cap I_2 \cap \cdots \cap I_s$ and $J = J_1 \cap J_2 \cap \cdots \cap J_t$, then

$$I \star J = \bigcap_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} I_i \star J_j.$$

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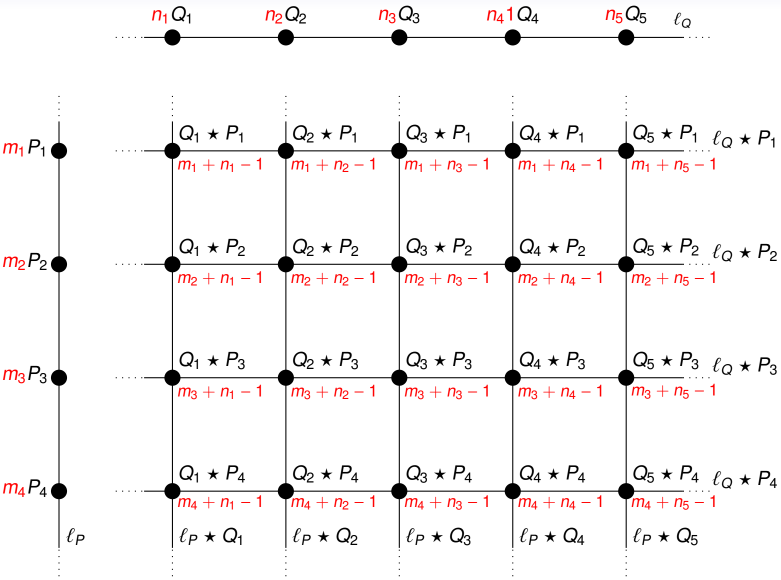
$$I \star J = \bigcap_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} I_i \star J_j.$$

Thus the ideal of $HFG(P_M, Q_N)$ is

$$\mathcal{I}(P_M, Q_N) = \bigcap_{i \in [r]} \bigcap_{j \in [s]} I(P_i)^{m_i} \star I(Q_j)^{n_j} = \bigcap_{i \in [r]} \bigcap_{j \in [s]} I(P_i \star Q_j)^{m_i + n_j - 1}.$$

$HFG(P_M, Q_N)$ has the structure of a planar grid. Specifically, it is a set of fat points whose support is a complete intersection of type (r, s) in $\mathbb{P}^2$.



$n_1 Q_1$
$n_2 Q_2$
$n_3 Q_3$
$n_4 1 Q_4$
$n_5 Q_5$
ℓ_Q
$m_1 P_1$
$m_2 P_2$
$m_3 P_3$
$m_4 P_4$
ℓ_P
$Q_1 \star P_1$
$m_1 + n_1 - 1$
$Q_2 \star P_1$
$m_1 + n_2 - 1$
$Q_3 \star P_1$
$m_1 + n_3 - 1$
$Q_4 \star P_1$
$m_1 + n_4 - 1$
$Q_5 \star P_1$
$m_1 + n_5 - 1$
$\ell_Q \star P_1$
$Q_1 \star P_2$
$m_2 + n_1 - 1$
$Q_2 \star P_2$
$m_2 + n_2 - 1$
$Q_3 \star P_2$
$m_2 + n_3 - 1$
$Q_4 \star P_2$
$m_2 + n_4 - 1$
$Q_5 \star P_2$
$m_2 + n_5 - 1$
$\ell_Q \star P_2$
$Q_1 \star P_3$
$m_3 + n_1 - 1$
$Q_2 \star P_3$
$m_3 + n_2 - 1$
$Q_3 \star P_3$
$m_3 + n_3 - 1$
$Q_4 \star P_3$
$m_3 + n_4 - 1$
$Q_5 \star P_3$
$m_3 + n_5 - 1$
$\ell_Q \star P_3$
$Q_1 \star P_4$
$m_4 + n_1 - 1$
$Q_2 \star P_4$
$m_4 + n_2 - 1$
$Q_3 \star P_4$
$m_4 + n_3 - 1$
$Q_4 \star P_4$
$m_4 + n_4 - 1$
$Q_5 \star P_4$
$m_4 + n_5 - 1$
$\ell_Q \star P_4$
$\ell_P \star Q_1$
$\ell_P \star Q_2$
$\ell_P \star Q_3$
$\ell_P \star Q_4$
$\ell_P \star Q_5$

From the rest of the talk we assume that $s \geq r$ and the multiplicities are ordered in non-decreasing order, that is

$$m_1 \leq m_2 \leq \cdots \leq m_r$$

$$n_1 \leq n_2 \leq \cdots \leq n_s$$

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$$n_1 \leq n_2 \leq \cdots \leq n_s$$

Theorem (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let X be a Hadamard fat grid $HFG(P_M, Q_N)$ in $\mathbb{P}^2$ and Z be an ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ supported on an (r, s)-grid with the same multiplicities m_{ij} as the Hadamard fat grid X. Then X and Z share the same Betti numbers.

Denote by H_i the horizontal lines defining $\ell_Q \star P_{r-i+1}$, and by V_j the vertical lines defining $\ell_P \star Q_{s-j+1}$.

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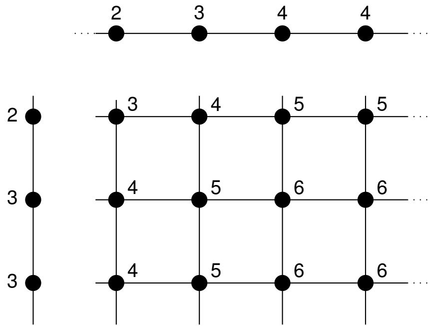
Theorem (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

A minimal set of generators of the ideal $\mathcal{I}(P_M, Q_N)$ *consists of* $m_r + n_s$ *generators of types* $H_1^{a_1-k} \cdots H_r^{a_r-k} \cdot V_1^{b_1+k} \cdots V_s^{b_s+k}$ *for* $k = 0, \ldots, m_r + n_s - 1$ *(*$H_i^{a_i-k} = 1$ *if* $a_i - k \leq 0$ *and* $V_j^{b_j+k} = 1$ *if* $b_j + k \leq 0$*). That is, a minimal set of generators is of type*

$$
\begin{aligned}
&H_1^{m_r+n_s-1} H_2^{m_{r-1}+n_s-1} \cdots H_r^{m_1+n_s-1} \cdot V_1^0 V_2^{n_{s-1}-n_s} \cdots V_{s-1}^{n_2-n_s} V_s^{n_1-n_s},\\
&H_1^{m_r+n_s-2} H_2^{m_{r-1}+n_s-2} \cdots H_r^{m_1+n_s-2} \cdot V_1^1 V_2^{n_{s-1}-n_s+1} \cdots V_{s-1}^{n_2-n_s+1} V_s^{n_1-n_s+1},\\
&\vdots\\
&H_1^0 H_2^{m_{r-1}-m_r} \cdots H_r^{m_1-m_r} \cdot V_1^{m_r+n_s-1} \cdots V_{s-1}^{n_2+m_r-1} V_s^{n_1+m_r-1}.
\end{aligned}
$$

Example

$M = (2, 3, 3)$, $N = (2, 3, 4, 4)$.



$$H_1^6 H_2^6 H_3^5$$
$$H_1^5 H_2^5 H_3^4 \cdot V_1^1 V_2^1$$
$$H_1^4 H_2^4 H_3^3 \cdot V_1^2 V_2^2 V_3^1$$
$$H_1^3 H_2^3 H_3^2 \cdot V_1^3 V_2^3 V_3^2 V_4^1$$
$$H_1^2 H_2^2 H_3^1 \cdot V_1^4 V_2^4 V_3^3 V_4^2$$
$$H_1^1 H_2^1 \cdot V_1^5 V_2^5 V_3^4 V_4^3$$
$$V_1^6 V_2^6 V_3^5 V_4^4 .$$

Example

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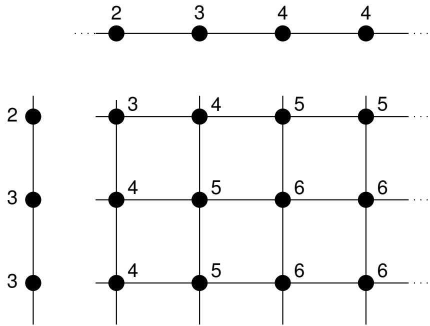

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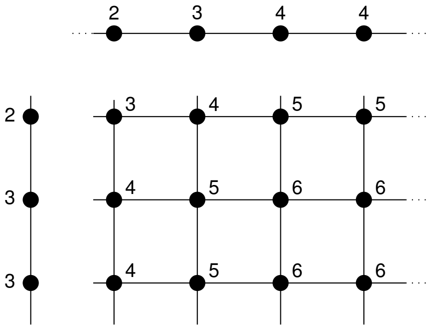
$$H_1^1 H_2^1 \cdot V_1^5 V_2^5 V_3^4 V_4^3$$
$$V_1^6 V_2^6 V_3^5 V_4^4.$$

$$0 \longrightarrow R(-23) \oplus R(-22) \oplus R(-21) \oplus R(-20) \oplus R^2(-19) \longrightarrow$$
$$R(-21) \oplus R(-19) \oplus R(-18) \oplus R^2(-17) \oplus R^2(-16) \longrightarrow I(X) \longrightarrow 0$$

Example

$M = (2, 3, 3)$, $N = (2, 3, 4, 4)$.



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Example

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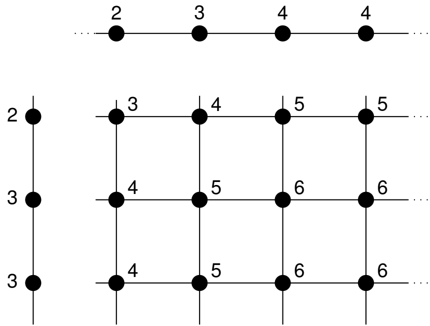

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$$H_1^2 H_2^2 H_3^1 \cdot V_1^4 V_2^4 V_3^3 V_4^2$$
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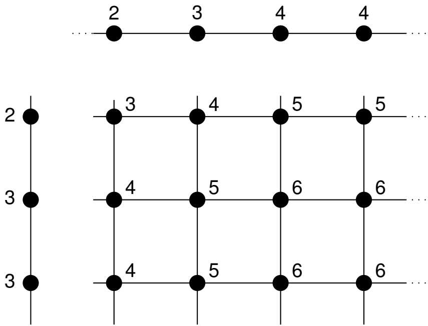
$M = (2,3,3),\ N = (2,3,4,4)$.



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$$\alpha(\mathcal{I}(P_M, Q_N)) = 16 \qquad \hat{\alpha}(\mathcal{I}(P_M, Q_N)) = 16 \quad \rho(\mathcal{I}(P_M, Q_N)) = 1$$

Proposition (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let $\mathcal{I}(P_M, Q_N)$ be the ideal of a Hadamard fat grid. Then

- $\alpha(\mathcal{I}(P_M, Q_N)) = \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} n_{s-i+1} - r.$

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Lemma

The t-th symbolic power of $\mathcal{I}(P_M, Q_N)$ is the ideal of a Hadamard fat grid.

Proof.

$$\mathcal{I}(P_M, Q_N)^{(t)} = \bigcap_{i=1}^{r} \bigcap_{j=1}^{s} I(P_i \star Q_j)^{t(m_i+n_j-1)} = \bigcap_{i=1}^{r} \bigcap_{j=1}^{s} I(P_i \star Q_j)^{((tm_i-(t-1))+tn_j)-1}.$$

$$M' = \{tm_1 - (t-1), \ldots, tm_r - (t-1)\} \qquad N' = \{tn_1, \ldots, tn_s\}$$

$$\boxed{\alpha(\mathcal{I}(P_M, Q_N)^{(t)}) = t\alpha(\mathcal{I}(P_M, Q_N))}$$

□

Proposition (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let $\mathcal{I}(P_M, Q_N)$ be the ideal of a Hadamard fat grid. Then

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Proposition (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let $\mathcal{I}(P_M, Q_N)$ *be the ideal of a Hadamard fat grid, then*

$$\mathcal{I}(P_M, Q_N)^t = \mathcal{I}(P_M, Q_N)^{(t)}$$

for all $t \geq 1$.

Corollary (Bahmani Jafarloo-B.-Guardo-Malara, 2023)

Let $\mathcal{I}(P_M, Q_N)$ *be the ideal of a Hadamard fat grid, then*

$$\rho(\mathcal{I}(P_M, Q_N)) = 1.$$

- C. Bocci and B. Harbourne. *Comparing powers and symbolic powers of ideals*, J. Alg. Geom. **19** (2010), 399–417.

The inspiration for this paper was a question Craig asked Brian: if S is a finite set of points in $\mathbb{P}^2$ with $I = I(S)$, is it true that $I^{(3)} \subseteq I^2$?

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As a stepping stone, we introduce an asymptotic quantity which we refer to as the resurgence, namely $\rho(I) = sup\{m/r : I^{(m)} \not\subseteq I^r\}$.

Harbourne, Brian

MR Author ID	217048
Earliest Indexed Publication	1982
Total Publications	84
Total Related Publications	7
Total Reviews	0
Total Citations	1.647 in 595 publications
Unique Citing Authors	476

Collaboration Distance

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MR2629595 - Comparing powers and symbolic powers of ideals
Bocci, Cristiano; Harbourne, Brian
J. Algebraic Geom. **19** (2010), no. 3, 399–417.
Reviewed
120 citations
MSC 13F20
Article

MR2555949 - A primer on Seshadri constants
Bauer, Thomas; Di Rocco, Sandra; Harbourne, Brian; Kapustka, Michał; Knutsen, Andreas; Syzdek, Wioletta; Szemberg, Tomasz
Contemp. Math., 496
American Mathematical Society, Providence, RI, 2009, 33–70.
ISBN: 978-0-8218-4746-6
Summary
101 citations
MSC 14C20
Article

MR0846019 - The geometry of rational surfaces and Hilbert functions of points in the plane
Harbourne, Brian
CMS Conf. Proc., 6
Published by the American Mathematical Society, Providence, RI; for the , 1986, 95–111.
ISBN: 0-8218-6010-0
Reviewed
81 citations
MSC 14J26
Article

$$\rho(N,d) = \sup\left\{\rho(I) \ : \ 0 \neq I \subsetneq k[\mathbf{P}^N] \text{ homog. of cod. } d.\right\}$$

Corollary

For each $N \geq 1$ and $1 \leq d \leq N$, we have $\rho(N,d) = d$.

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Corollary

For each $N \geq 1$ and $1 \leq d \leq N$, we have $\rho(N,d) = d$.

$$\boxed{I^{(dm)} \subseteq I^m}$$

- L. Ein, R. Lazarsfeld and K. Smith. *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. 144 (2001), p. 241-252.
- M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. **147** (2002), no. 2, 349–369.

Harbourne and Bocci introduced the resurgence of an ideal as an asymptotic measure of the best possible containment

Symbolic Powers of Ideals, Dao, De Stefani, Grifo, Huneke, Núñez-Betancourt

- $\rho(I) = \sup\{m/r : I^{(m)} \not\subseteq I^r\}$
- $\hat{\rho}(I) = \sup\{m/r : I^{(mt)} \not\subseteq I^{rt}\}$ for all $t \gg 0$
  (Guardo, Harbourne and VanTuyl)
- $\rho_{ic}(I) = \sup\{m/r : I^{(m)} \not\subseteq \overline{I^r}\}$
  (Dipasquale, Francisco, Mermim, Schweig)

$$\boxed{I^{(3)} \subseteq I^2}$$

Theorem (B.-Harbourne, 2010)

Let $I = I(S)$, where S is a set of n generic points of $\mathbb{P}^2$. Then I^2 contains $I^{(3)}$ for every $n \geq 1$.

- B. Harbourne and C. Huneke, *Are symbolic powers highly evolved?* J. Ramanujan Math. Soc., 2011

Conjecture 1[Harbourne-Huneke] Let $I = \cap_{i=1}^{n} I(P_i)^{m_i} \subset K[\mathbf{P}^N]$ be any fat points ideal. Then $I^{(rN)} \subseteq M^{r(N-1)} I^r$ holds for all $r > 0$.

Conjecture 2[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^2]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^2$. Then $I^{(m)} \subseteq I^r$ holds whenever $\frac{m}{r} \geq \frac{2\alpha(I)}{\alpha(I)+1}$.

Conjecture 3[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$ holds for all $r \geq 1$.

Conjecture 4[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then

$$\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$$

for every $r > 0$.

Conjecture 5[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \frac{\alpha(I^{(r)})}{r} \qquad \text{for all } r > 0$$

Conjecture 6[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$ for $N \geq 2$. Then $I^{(t(m+N-1))} \subseteq M^t(I^{(m)})^t$.

Conjecture 7[Harbourne-Huneke] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$.

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Conjecture 8[Bauer-Di Rocco-Harbourne- Kapustka-Knutsen-Syzdek-Szemberg] Let $I \subseteq K[\mathbf{P}^N]$ be a homogeneous ideal. Then $I^{(rN-(N-1))} \subseteq I^r$ holds for all r.

Conjecture 9[B.-Cooper-Harbourne] Let $I \subseteq K[\mathbf{P}^N]$ be the radical ideal of a finite set of n points $P_i \in \mathbf{P}^N$. Then $I^{(t(m+N-1)-N+1)} \subseteq (I^{(m)})^t$ and $I^{(t(m+N-1)-N+1)} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t$ hold for all $m \geq 1$.

Conj. 1 $I^{(rN)} \subseteq M^{r(N-1)} I^r$.

Conj. 2 $I^{(m)} \subseteq I^r$ holds whenever $m/r \geq 2\alpha(I)/(\alpha(I)+1)$.

Conj. 3 $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$.

Conj. 4 $\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$.

Conj. 5 $\frac{\alpha(I^{(m)})+N-1}{m+N-1} \leq \frac{\alpha(I^{(r)})}{r}$

Conj. 6 $I^{(t(m+N-1))} \subseteq M^t (I^{(m)})^t$.

Conj. 7 $I^{(t(m+N-1))} \subseteq M^{t(N-1)} (I^{(m)})^t$.

Conj. 8 $I^{(rN-(N-1))} \subseteq I^r$

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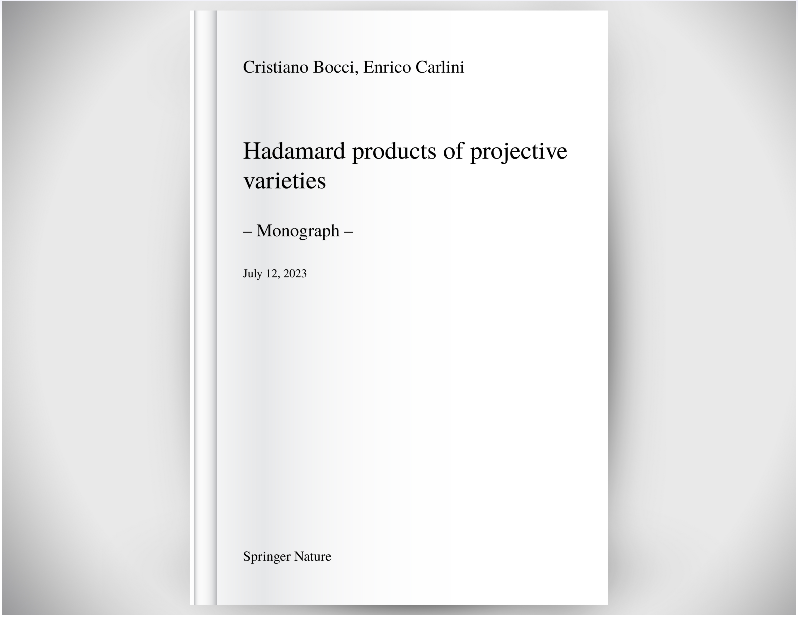
- C. Bocci, S. Cooper and B. Harbourne, *Containment results for ideals of various configurations of points in* $\mathbb{P}^N$, Journal of Pure and Applied Algebra 218 (2014), 65–75.

Thank for your attention


Cristiano Bocci, Enrico Carlini
Hadamard products of projective varieties
– Monograph –
July 12, 2023
Springer Nature