Steiner systems and configurations of points

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Research interests

- Combinatorial Design Theory involves applications in Coding Theory, Cryptography, and Computer Science.

Problem containment problem

Let I be a homogeneous ideal in the standard graded polynomial ring $R := k[x_0, \ldots, x_n]$, where k is a field. Given an integer m, we denote by I^m the regular power of the ideal I. The m-th *symbolic power* of I is defined as

$$I^{(m)} = \bigcap_{\mathfrak{p} \in Ass(I)} (I^m R_{\mathfrak{p}} \cap R)$$

where $Ass(I)$ denotes the set of associated primes of I. If I is a radical ideal (this includes for instance squarefree monomial ideals and ideals of finite sets of points) then

$$I^{(m)} = \bigcap_{\mathfrak{p} \in Ass(I)} \mathfrak{p}^m.$$

Question

(*Containment problem*) Given a homogeneous ideal $I \subseteq k[x_0, \dots, x_n]$, for what pairs $m, r \in \mathbb{N}$, does $I^{(m)} \subseteq I^r$ hold?

Fat Points

Given distinct points $P_1, \dots, P_s \in \mathbb{P}^n$ and nonnegative integers m_i (not all 0), let

$$Z = m_1 p_1 + \cdots + m_s p_s$$

denote the scheme (called a fat point scheme) defined by the ideal $I_Z = \bigcap_{i=1}^{s} (I_{P_i}^{m_i}) \subseteq k[\mathbb{P}^n]$, where I_{P_i} is the ideal generated by all homogeneous polynomials vanishing at P_i.

- Containment problem also helps us to bound certain useful invariants like Waldschmidt constant, $\widehat{\alpha}(I)$ of an ideal I defined as

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m},$$

where $\alpha(I)$ is the minimum integer d such that $I_d \neq (0)$.

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An other tool useful to measure the non containment among symbolic and ordinary powers of ideals is the notion of *resurgence* $\rho(I)$ of an ideal I, introduced by Bocci-Harbourne that gives some notion of how small the ratio m/r can be and still be sure to have $I^{(m)} \subseteq I^r$; specifically,

Definition

Let I be a non zero, proper ideal in a commutative ring R, the *resurgence* of the ideal I is given by

$$\rho(I) = \sup \left\{ \frac{m}{r} \middle| \quad I^{(m)} \not\subseteq I^r \right\}.$$

It always satisfies $\rho(I) \geq 1$. In general, it is extremely difficult to estimate the exact value for $\rho(I)$. An asymptotic versions of the resurgence was introduced in the paper Guardo- Harbourne - Van Tuyl.

Definition

For a non zero, proper homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$, the *asymptotic resurgence* $\rho_a(I)$ is defined as follows:

$$\rho_a(I) = \sup\left\{\frac{m}{r} \mid \quad I^{(mt)} \not\subseteq I^{rt}, \quad \text{for all} \quad t \gg 0\right\}.$$

Harbourne-Huneke conjecture

The following slight different version of the Containement problem was introduced by Harbourne and Huneke. Recall that the *big height* of an ideal I refers to the maximum of all the heights of its associated prime ideals.

Conjecture

(Stable Harbourne Conjecture) Given a non zero, proper, homogeneous , radical ideal $I \subseteq k[x_0, \dots, x_n]$ *with big height* h*, then*

$$I^{(hr-h+1)} \subseteq I^r$$

for all $r \gg 0$.

Conjecture

(Stable Harbourne–Huneke Conjecture) Let $I \subseteq k[x_0, \dots, x_n]$ *be a homogeneous radical ideal of big height* h*. Let* $\mathcal{M} = (x_0, \dots, x_n)$ *be the graded maximal ideal, then for* $r \gg 0$,

1. $I^{(hr)} \subseteq \mathcal{M}^{r(h-1)} I^r$
2. $I^{(hr-h+1)} \subseteq \mathcal{M}^{(r-1)(h-1)} I^r$.

Steiner Systems

Definition

A Steiner system (V, B) of type $S(t, n, v)$, with $t < n \leq v$, is a collection B of n-subsets (or n-tuples called blocks) of V with $v = |V|$ such that each t-tuple of V is contained in a unique block in B.

The elements in V are called vertices or points and those of B are called blocks.

Existence of Steiner Systems

- The existence of a Steiner system strongly depends on the parameters (t, n, v). For instance if $t = 2$ and $n = 3$ then $v \equiv 1, 3 \mod (6)$ must hold.
- There are known necessary conditions for the existence of a Steiner system of type $S(t, n, v)$ that are not in general sufficient.

If a Steiner system (V, B) of type $S(t, n, v)$ exists, then $|B| = \frac{\binom{v}{t}}{\binom{n}{t}}$.

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Example

$S(2,3,7)$

One of the simplest and most known examples of Steiner system is the Fano Plane. It is unique up to isomorphism and it is a Steiner system of type $S(2,3,7)$ with block set

$$B := \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},$$

$$\{3,4,7\},\{3,5,6\}\}.$$

Problem

Problem

Given any Steiner System $S(t, n, v)$ can we construct a suitable configuration of points and its defining ideal?

Results

For a Complement of a Steiner configuration of points in $\mathbb{P}^n_k$,

- We describe its Hilbert Function and Betti numbers, Waldschmidt constant, regularity, bounds on its resurgence and asymptotic resurgence.
- We show that Stable Harbourne Conjecture and Stable Harbourne–Huneke Conjecture.
- We also compute the parameters of linear codes associated to any Steiner configuration of points and its Complement (See Toheneanu and Van Tuyl).

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Star Configurations

We follow Geramita-Harbourne-Migliore. Let $\mathcal{H} = \{H_1, \ldots, H_v\}$ be a collection of $v \geq 1$ distinct hyperplanes in $\mathbb{P}^n$, $n \leq v$, and suppose $H_i = V(\ell_i)$, $i = 1, \ldots, v$, where $\ell_1, \ldots, \ell_v \in R := K[x_1, \ldots, x_n]$ are some linear forms. Suppose that the hyperplanes *meet properly*, that is the intersection of any j of these hyperplanes is either empty or has codimension j. For any $1 \leq c \leq min\{v, n\}$, the codimension c star configuration with skeleton $\mathcal{H}$ is the union of the codimension c linear varieties defined by the intersections of these hyperplanes, taken c at a time, $V_c(\mathcal{H}) := \bigcup_{1 \leq i_1 < \cdots < i_c \leq v} H_{i_1} \cap \cdots \cap H_{i_c}$. This variety has defining ideal in R:

$$I(V_c(\mathcal{H})) = \bigcap_{1 \leq i_1 < \cdots < i_c \leq v} \langle \ell_{i_1}, \ldots, \ell_{i_c} \rangle$$

Steiner Configurations of points

Let (V, B) be a Steiner system of type $S(t, n, v)$ with $t < n \leq v$. We associate to B the following set of points in $\mathbb{P}^n$:

- Let $\mathcal{H} := \{H_1, \dots H_v\}$ be a collection of $v \geq 1$ distinct hyperplanes $H_i = V(\ell_i)$ of $\mathbb{P}^n$ defined by the linear forms ℓ_i for $i = 1, \dots, v$. Assume that any n hyperplanes, $n \leq v$, in $\mathcal{H}$ meet in a point.
- Given a n-subset $\sigma := \{\sigma_1, \dots, \sigma_n\} \in B$, we denote by $P_{\mathcal{H},\sigma}$ the point intersection of the hyperplanes $H_{\sigma_1}, \dots, H_{\sigma_n}$.
- Then the ideal $I_{P_{\mathcal{H},\sigma}} = (\ell_{\sigma_1}, \dots, \ell_{\sigma_n}) \subseteq k[\mathbb{P}^n]$ is the vanishing ideal of the point $P_{\mathcal{H},\sigma}$.

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- Given a n-subset $\sigma := \{\sigma_1, \dots, \sigma_n\} \in B$, we denote by $P_{\mathcal{H},\sigma}$ the point intersection of the hyperplanes $H_{\sigma_1}, \dots, H_{\sigma_n}$.
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Steiner Configurations of points and its ideal

Definition

Let (V, B) be a Steiner system of type $S(t, n, v)$ with $t < n \leq v$. We associate to B the following set of points in $\mathbb{P}^n$

$$X_{\mathcal{H},B} := \bigcup_{\sigma \in B} P_{\mathcal{H},\sigma}$$

and its defining ideal

$$I_{X_{\mathcal{H},B}} := \bigcap_{\sigma \in B} I_{P_{\mathcal{H},\sigma}}.$$

We call $X_{\mathcal{H},B}$ the Steiner configuration of points associated to the Steiner system (V, B) of type $S(t, n, v)$ with respect to $\mathcal{H}$ (or just X_B if there is no ambiguity).

Complement of Steiner Configurations of points and its ideal

Definition

Let (V, B) be a Steiner system of type $S(t, n, v)$ with $t < n \leq v$. Denote by $C_{(n,v)}$ the set containing all the n-subsets of V, we associate to $C_{(n,v)} \setminus B$ the following set of points in $\mathbb{P}^n$

$$X_{\mathcal{H}, C_{(n,v)} \setminus B} := \cup_{\sigma \in C_{(n,v)} \setminus B} \; P_{\mathcal{H},\sigma}$$

and its defining ideal

$$I_{X_{\mathcal{H}, C_{(n,v)} \setminus B}} := \cap_{\sigma \in C_{(n,v)} \setminus B} \; I_{P_{\mathcal{H},\sigma}}.$$

We call $X_{\mathcal{H}, C_{(v,n)} \setminus B}$ the Complement of a Steiner configuration of points with respect to $\mathcal{H}$ (or C-Steiner X_C if there is no ambiguity).

Steiner Configurations of points and Star Configurations

- We have that $X_{\mathcal{H},C_{(n,v)}}$ is a special type of the so called *star configuration* of $\binom{v}{n}$ points in $\mathbb{P}^n$, i.e., the codimension n star configuration in $\mathbb{P}^n$.
- Thus, a Steiner configuration of points and its Complement are subschemes of a star configuration of $\binom{v}{n}$ points in $\mathbb{P}^n$.

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- Thus, a Steiner configuration of points and its Complement are subschemes of a star configuration of $\binom{v}{n}$ points in $\mathbb{P}^n$.

Remark

Since the set $X_{\mathcal{H},B}$ contains $|B|$ points, we have that

$$|X_{\mathcal{H},C_{(n,v)}\setminus B}| = \deg X_{\mathcal{H},C_{(n,v)}\setminus B} = \binom{v}{n} - |B| = \binom{v}{n} - \frac{\binom{v}{t}}{\binom{n}{t}}.$$

Example

Consider the Steiner configuration associated to (V, B) of type $S(2,3,7)$. Take $\mathcal{H} := \{H_1, \dots, H_7\}$ a collection of 7 distinct hyperplanes H_i in $\mathbb{P}^3$ $i = 1, \dots, 7$ with the property that any 3 of them meet in a point $P_{\mathcal{H},\sigma} = H_{\sigma_1} \cap H_{\sigma_2} \cap H_{\sigma_3}$, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3\} \in B$.

We get

- $X_{\mathcal{H},B}$ is a Steiner configuration consisting of 7 points in $\mathbb{P}^3$
- $X_{\mathcal{H},C_{(3,7)} \setminus B}$ is a C-Steiner configuration consisting of $\binom{7}{3} - 7 = 28$ points in $\mathbb{P}^3$.

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Monomial ideals and Graph Theory

- Given a graph G with vertices $\{x_1, \dots, x_v\}$, we associate the ideal I_G in $k[x_1, \dots, x_v]$ generated by the quadratic monomials x_ix_j such that x_i is adjacent to x_j.
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- From known results in the literature, ideals generated by squarefree monomials have a beautiful combinatorial interpretation in terms of simplicial complexes.

Simplicial Complexes and Matroids

Definition

A simplicial complex Δ over a set of vertices $V = \{x_1, \ldots, x_v\}$ is a collection of subsets of V satisfying the following two conditions:

1. $\{x_i\} \in \Delta$ for all $1 \leq i \leq v$
2. if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

An element F of Δ is called a face, and the dimension of a face F of Δ is $|F| - 1$, where $|F|$ is the number of vertices of F.

- The Alexander dual of a simplicial complex Δ on $V = \{x_1, \ldots, x_v\}$ is the simplicial complex $\Delta^\vee$ on V with faces $V \setminus \sigma$, where $\sigma \notin \Delta$.
- The Stanley-Reisner ideal of Δ is the ideal $I_\Delta := (x^\sigma \mid \sigma \notin \Delta)$ of $R = k[x_1, \ldots, x_v]$, where $x^\sigma = \prod_{i \in \sigma} x_i$.
- It is well known that the Stanley-Reisner ideals are squarefree monomial ideals. The quotient ring $k[\Delta] := R/I_\Delta$ is the Stanley-Reisner ring of the simplicial complex Δ.

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If V is a set v points, we denote by $k[V] := k[x_1, \ldots, x_v]$ the standard graded polynomial ring in v variables. Given a n-subset of V, $\sigma := \{i_1, i_2, \ldots, i_n\} \subseteq V$, we will write

$$\mathfrak{p}_\sigma := (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \subseteq k[V]$$

for the prime ideal generated by the variables indexed by σ, and

$$M_\sigma := x_{i_1} x_{i_2} \cdots x_{i_n} \in k[V]$$

for the monomial given by the product of the variables indexed by σ.

Let $n \leq v$ be positive integers, and V a set of v points; recall that $C_{(n,v)}$ is the set containing all the n-subsets of V.

Definition

If $T \subset C_{(n,v)}$, we define two ideals

$$I_T := (M_\sigma \mid \sigma \in T) \subseteq k[V]$$

and

$$J_T := \bigcap_{\sigma \in T} \mathfrak{p}_\sigma \subseteq k[V]$$

called the *face ideal* of T and the *cover ideal* of T, respectively.

Since J_T is a squarefree monomial ideal, the m-th symbolic power of J_T (Theorem 3.7 in Cooper-Ha) is

$$J_T^{(m)} := \bigcap_{\sigma \in T} \mathfrak{p}_\sigma^m .$$

Matroids

Definition

A matroid Δ on a vertex set $\{1, \ldots, v\}$ is a non-empty collection of subsets of $\{1, \ldots, v\}$ that is closed under inclusion and satisfies the following property: If F and G are Δ and $|F| > |G|$ then there exists $i \in F \setminus G$ such that $G \cup \{i\} \in \Delta$.

Equivalently, a matroid is a simplicial complex Δ such that, for every subset $F \subset \{1, \ldots, v\}$, the restriction $\Delta|F = \{G \in \Delta | G \subset F\}$ is pure, that is, all its facets have the same dimension.

Varbaro and Minh - Trung have independently shown

Theorem

Let Δ be a simplicial complex on $\{1,\ldots,v\}$. Then $k[V]/I_{\Delta}^{(m)}$ is Cohen-Macaulay for each $m \geq 1$ if and only if Δ is a matroid.

Main results

Theorem

Let (V, B) be a Steiner system of type $S(t, n, v)$. Then Δ_C is a matroid.

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Let (V, B) be a Steiner system of type $S(t, n, v)$. Then $k[V]/I_{\Delta_C}^{(m)}$ is Cohen-Macaulay for each $m \geq 1$.

Main result for symbolic powers

Theorem

$I^{(m)}_{X_{\mathcal{H},C}} \subseteq k[\mathbb{P}^n]$ and $I^{(m)}_{\Delta_C} \subseteq k[V]$ share the same homological invariants.

The Cohen-Macaulay property of $k[V]/I^{(m)}_{\Delta_C}$ also allows us to look at $I^{(m)}_{X_{\mathcal{H},C}}$ as a proper hyperplane section of $I^{(m)}_{\Delta_C}$. This construction is quite standard but is very useful.

Using the previous results, we have the following theorem:

Theorem

Let (V, B) be a Steiner system of type $S(t, n, v)$. Then

i) $\alpha(I_{X_C}) = v - n$;

ii) $\alpha(I_{X_C}^{(q)}) = v - n + q$, *for* $2 \leq q < n$;

iii) $\alpha(I_{X_C}^{(m)}) = \alpha(I_{X_C}^{(q)}) + pv$, *where* $m = pn + q$ *and* $0 \leq q < n$ *and* $\alpha(I_{X_C}^{(n)}) = \alpha(I_{X_C}^{(0)}) + v = v$.

Corollary

If (V, B) is a Steiner system of type $S(t, n, v)$, then the Waldschmidt constant of I_{X_C} is

$$\widehat{\alpha}(I_{X_C}) = \frac{v}{n}.$$

Proposition

If (V, B) is a Steiner system $S(t, n, v)$, then the h-vector of X_C is

$$h_{X_C} = \left(1, n, \binom{n+1}{n-1}, \cdots, \binom{v-2}{n-1}, \binom{v-1}{n-1} - |B|\right).$$

The regularity of I_{X_C} is an easy consequence of the previous result:

Corollary

$$\mathrm{reg}(I_{X_C}) = \alpha(I_{X_C}) + 1 = v - n + 1.$$

Resurgence

Corollary

Let $I \subseteq k[x_0, \ldots, x_n]$ be the ideal defining complement of a Steiner Configuration of points in $\mathbb{P}^n_k$. Then, $\rho(I) < n$.

Containment

Theorem

[Ballico, Favacchio, -, Milazzo, Thomas] Let $I \subseteq k[x_0, \ldots, x_n]$ be the ideal defining complement of a Steiner Configuration of points in $\mathbb{P}^n_k$. Then I satisfies

1. *Stable Harbourne–Huneke Conjecture;*
2. *Stable Harbourne Conjecture.*

Application to coding theory

- There are several ways to compute the minimum distance. One of them comes from linear algebra.
- Let k be any field and $X = \{P_1, \dots, P_r\} \subseteq \mathbb{P}^n$ a not degenerate finite set of reduced points. The linear code associated to X denoted by $\mathcal{C}(X)$ is the image of the injective linear map $\varphi : k^{n+1} \to k^r$.

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Parameters of a linear code

- We are interested in three parameters $[|X|, k_X, d_X]$ that we use to evaluate the goodness of a linear code.
- The first number $|X|$ is the cardinality of X. The number k_X is the dimension of the code as k-linear vector space, that is the rank of the matrix associated to φ.
- The number d_X denotes the minimal distance of $\mathcal{C}(X)$, that is the minimum of the Hamming distance of two elements in $\mathcal{C}(X)$.

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- Given a set of points $X = \{P_1, \ldots, P_r\} \subseteq \mathbb{P}^n$, the linear code associated to X has generating matrix of type $(n+1) \times r$

$$A(X) = [c_1 \ldots c_r]$$

where c_i are the coordinates of P_i.

- Assume that $A(X)$ has no proportional columns is equivalent to say that the points P_i are distinct points in $\mathbb{P}^n$ and $n < r$
- Then $|X| = r$, $Rank(A(X)) = n + 1$ and $r - d_X$ is the maximum number of these points that fit in a hyperplane of $\mathbb{P}^n$.

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Geometric interpretation of d_X

- From Toheneanu and Van Tuyl's papers we know that the minimum distance d_X is also the minimum number such that $r - d_X$ columns in $A(X)$ span an n-dimensional space.
- The generating matrix $A(X)$ of an $[|X|, n+1, d_X]$-linear code $\mathcal{C}$ naturally determines a matroid $M(\mathcal{C})$.
- Denoted by $hyp(X)$ the maximum number of points contained in some hyperplane, d_X has also geometrical interpretation, that is

$$d_X = |X| - hyp(X)$$

.

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- The generating matrix $A(X)$ of an $[|X|, n+1, d_X]$-linear code $\mathcal{C}$ naturally determines a matroid $M(\mathcal{C})$.
- Denoted by $hyp(X)$ the maximum number of points contained in some hyperplane, d_X has also geometrical interpretation, that is

$$d_X = |X| - hyp(X)$$

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Parameters of a Steiner System

Theorem

Let (V, B) be a Steiner system $S(t, n, v)$ with $|V| = v$. Then the parameters of the linear code defined by a Steiner configuration of points X_B are $[|B|, n+1, d_{X_B}]$ where

$$d_{X_B} = \frac{\binom{v}{t}}{\binom{n}{t}} - \frac{\binom{v-1}{t-1}}{\binom{n-1}{t-1}}.$$

Parameters of the Complement

With the above results, we have

Theorem

Let (V, B) be a Steiner system $S(t, n, v)$ with $|V| = v$. Then the parameters of the linear code defined by a Complement of a Steiner configuration of points X_C are $\left[\binom{v}{n} - |B|, n+1, d_{X_C}\right]$ where

$$d_{X_C} = \binom{v}{n} - \frac{\binom{v}{t}}{\binom{n}{t}} - \binom{v-1}{n-1} + \frac{\binom{v-1}{t-1}}{\binom{n-1}{t-1}}.$$

Computing the linear code associated to the Steiner system $S(2,3,7)$.

Consider the Steiner system $S(2,3,7)$.
For $i = 1, \ldots, 7$, let $H_i \subseteq \mathbb{P}^3$ be the hyperplane defined by

$$\ell_i := x + 2^i y + 3^i z + 5^i w$$

$$A(X_{\mathcal{H},B}) := \begin{pmatrix} -15 & -1983 & -438045 & -350 & -639000 & 9315 & 104625 \\ 20 & 1576 & 269060 & 160 & 240075 & -2610 & -25875 \\ -10 & -418 & -34230 & -35 & -37550 & 470 & 4250 \\ 1 & 17 & 523 & 1 & 666 & -9 & -99 \end{pmatrix}$$

The parameters of the code $\mathcal{C}(X_{\mathcal{H},B})$ are $[7,4,4]$ and the parameters of the code $\mathcal{C}(X_{\mathcal{H},C_{(3,7)}\setminus B})$ are $[28,4,16]$.

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Thank you