Star configurations and their progenitors and descendants

Conference on Unexpected and Asymptotic Properties of Algebraic Varieties

A conference to celebrate Professor Brian Harbourne

Juan Migliore
University of Notre Dame

August 11-13, 2023
University of Nebraska

Slides available by emailing migliore.1@nd.edu

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Later, I made a short visit to Lincoln in 1999:

July 16, 1999

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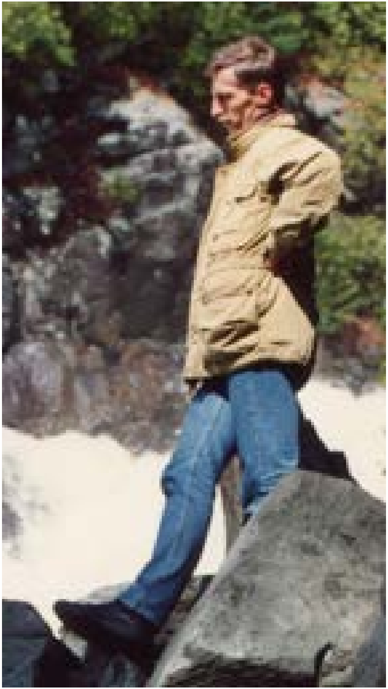
Tony Geramita (August 4, 1942 – June 22, 2016)

and

Uwe Nagel (presumably somewhere in the room but I can't see you guys...)

June, 1986, Kingston, Ontario



April 17, 1993
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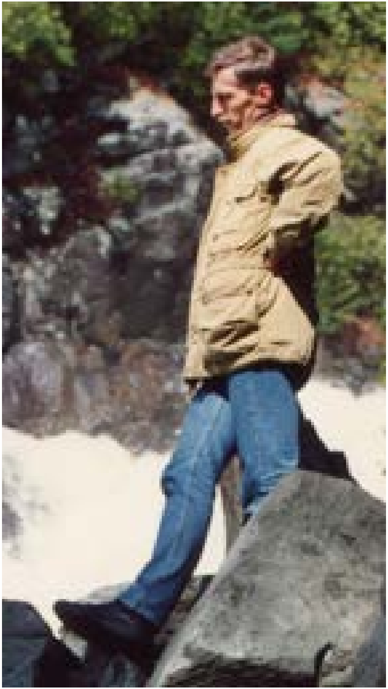

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Who knows what theorem he was thinking about in those days...

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To “complete the picture,” here is a picture of me from a few years ago, working on a theorem about geproci sets on a quadric surface. (I’m still working on that problem.)

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This talk centers around the paper [GHM2013]:

Star Configurations in $\mathbb{P}^n$:
A.V. Geramita, B. Harbourne, J. Migliore
Journal of Algebra 376 (2013), 279–299

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Many extremely interesting papers have been written in which star configurations have played a prominent role. Lacking time, this talk will focus on a small subset (with apologies).

I mostly want to talk about a useful tool to study star configurations and related problems.

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In this paper the authors start a systematic study of the ideals of star configurations.

A star configuration is constructed as follows.

Given a collection of properly intersecting hyperplanes, one takes all possible intersections of them in groups of c.

The variety obtained in this way is called a star configuration and it has codimension c.

He goes on to give some citations of related work. He continues...

The authors provide many interesting results on the ideal of a star configuration.

More precisely, they consider the following:

- Hilbert functions;
- minimal free resolutions;
- symbolic powers;
- arithmetic Cohen-Macaulayness;
- primary decompositions;
- minimal degree of a generator;
- maximal degree of a minimal generator;
- resurgence.

We won't talk about most of these today.

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We'll discuss soon how to relax the non-concurrence condition.

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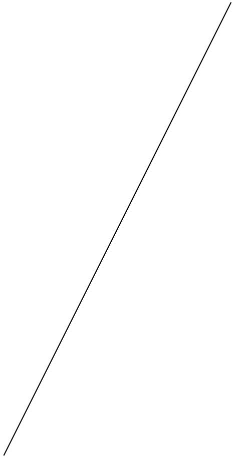

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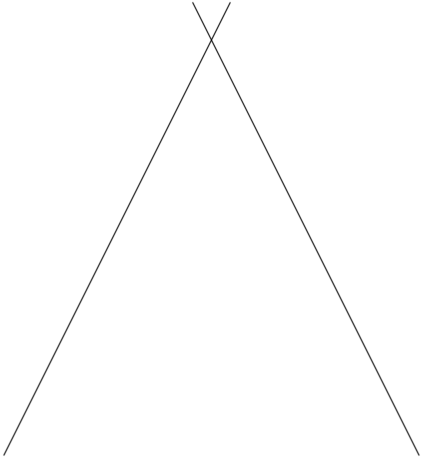

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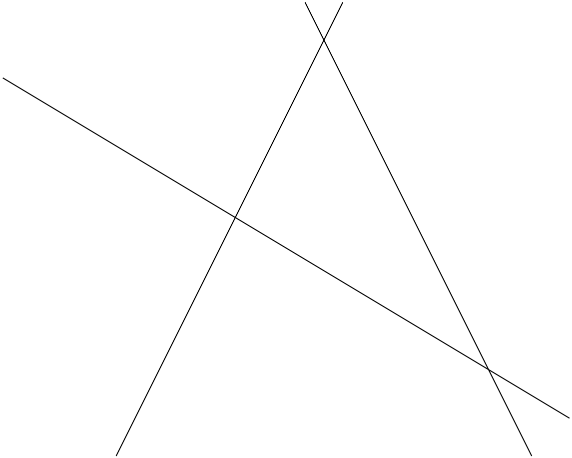

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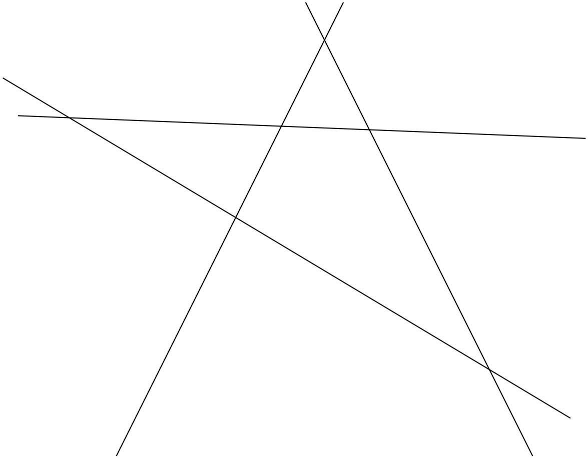

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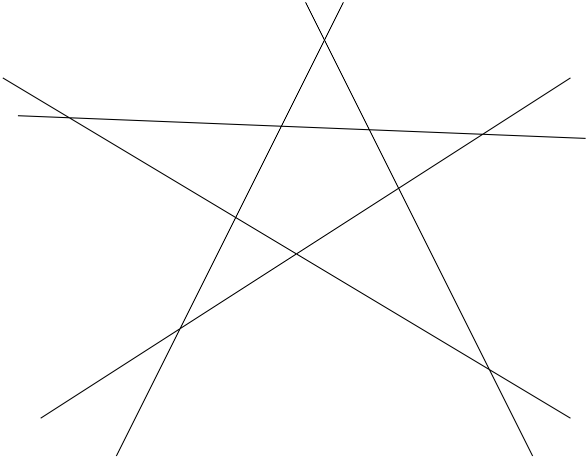

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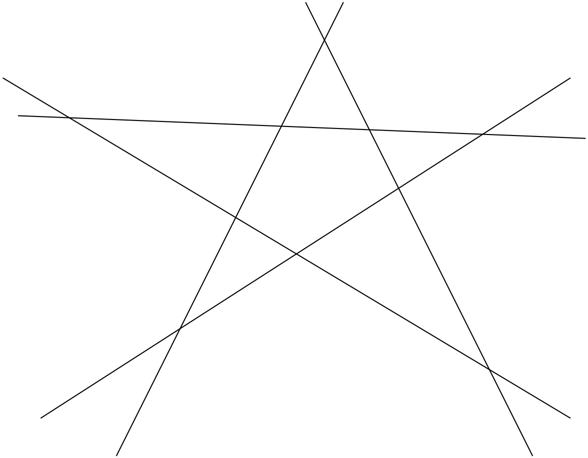

Hence the name!

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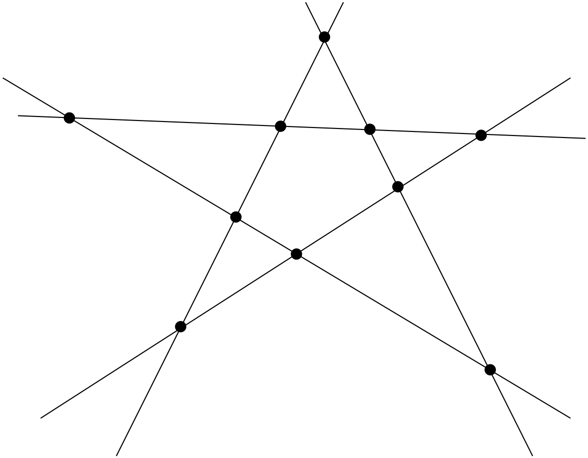

$\binom{5}{2} = 10$ pairwise intersections of the lines
(since the codimension is $c = 2$)

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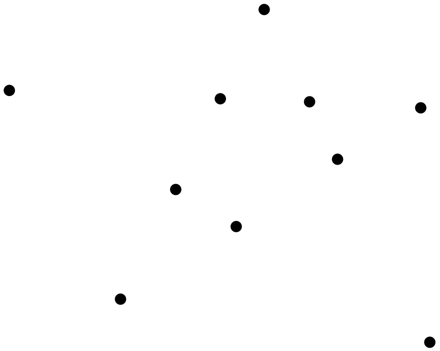

Erase the lines.

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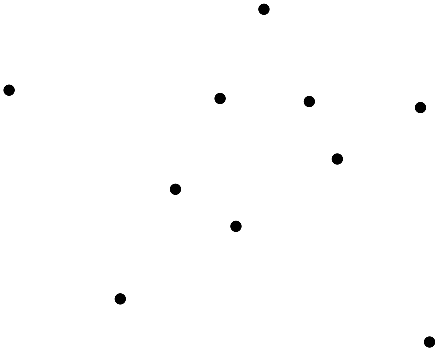

The intersection points, Z, form a star configuration with 10 points, defined by $r = 5$ lines.

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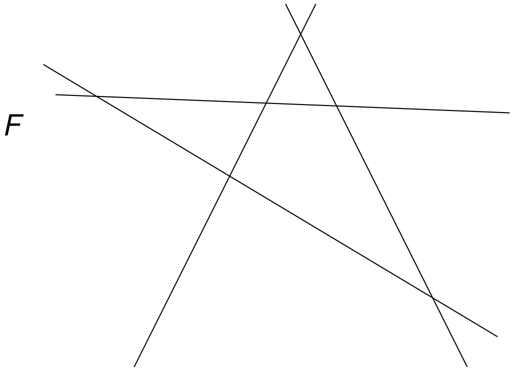

Let F be the curve defined by the union of the first four lines.

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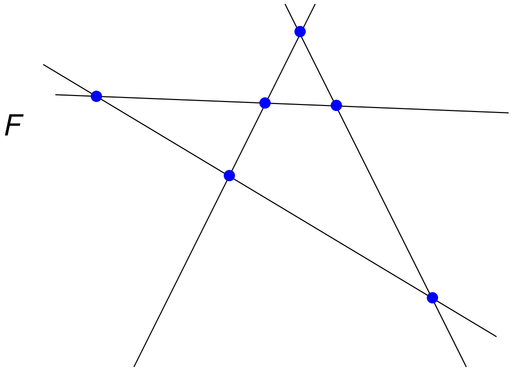

Let Z_1 be the corresponding $\binom{4}{2} = 6$ points.

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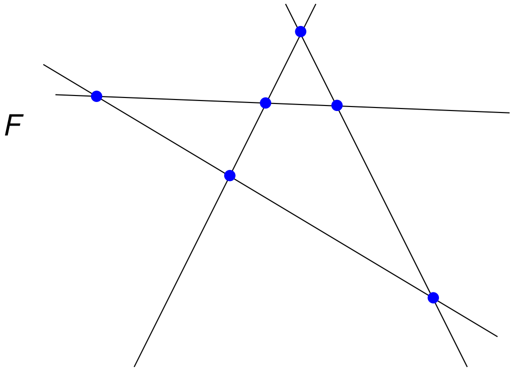

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Note that Z_1 is contained in F, i.e. $F \in I_{Z_1}$.

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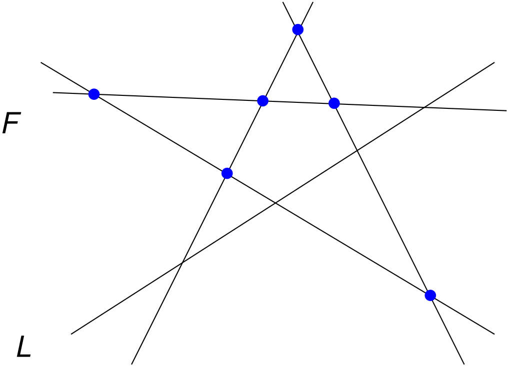

Let L be the fifth line.

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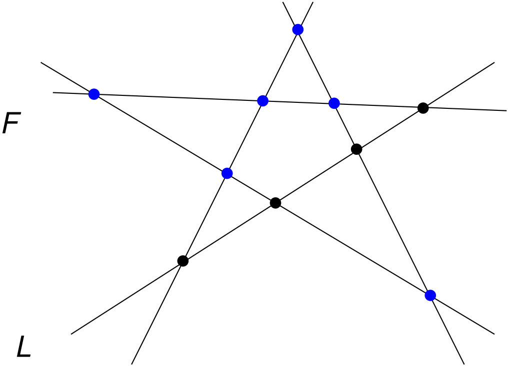

Then $Z = Z_1 \cup (F \cap L)$.

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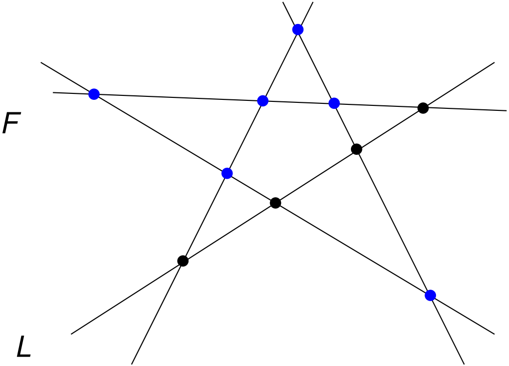

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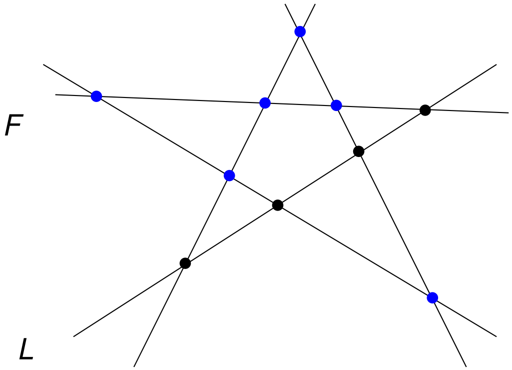

Then $Z = Z_1 \cup (F \cap L)$. This is an example of a basic double link (BDL). Key Fact: $I_Z = L \cdot I_{Z_1} + (F)$

A star is born – where did the name come from?

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In [GMS2006] Tony, Sindi Sabourin and I introduced a set of points $C_t \subset \mathbb{P}^2$ as follows.

> *Let* $\lambda_1, \ldots, \lambda_t$ *be a set of* t *distinct lines in* $\mathbb{P}^2$ *such that each* λ_i *meets the remaining* $t-1$ *lines in* $t-1$ *distinct points.*
>
> *We denote by* C_t *the configuration consisting of the* $\binom{t}{2}$ *pairwise intersections of these lines.*

But we didn't call them star configurations, and our picture didn't look anything like a star!

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The paper [GHM2013] starts off indicating that star configurations

> *“have arisen as objects of study in numerous research projects lately”*

and suggests that their properties were not well understood, and “it is of interest to understand them better,” as Enrico also mentioned.

Since then, many papers have focused on star configurations from different points of view.

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Example. How do we produce the $\binom{5}{3} = 10$ points of intersection of 5 planes in $\mathbb{P}^3$, taken 3 at a time? (Codimension 3.)

Assume no 4 of the planes are concurrent.

We'll build up the points inductively, but with a bit of care.

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- Let $C(L_1, L_2)$ be the star configuration gotten with L_1, L_2 (it is a line).

- Similarly produce additional curves (codimension 2 star configurations):
    - $C(L_1, L_2, L_3)$

      $\deg C(L_1, L_2, L_3) = \binom{3}{2} = 3$ ("coordinate axes"),

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- We'll see shortly that these curves are all ACM (thanks to the theory of basic double links).

Step 3: Now produce finite sets of points by adding hyperplane sections.

- $Z(L_1, L_2, L_3)$ is the hyperplane section of $C(L_1, L_2)$ by L_3.

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This process, e.g.

$$Z(L_1, L_2, L_3, L_4, L_5) = \\ Z(L_1, L_2, L_3, L_4) \cup \Big[C(L_1, L_2, L_3, L_4) \cap L_5 \Big]$$

takes a divisor on an ACM curve and adds to that divisor a hyperplane section of that curve.

This is a fancier version of basic double linkage called basic double G-linkage.

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The evolution of basic double linkage, and the appearance of many applications, emerged over the decades with work of many authors, including:

- Schwartau (1982 Ph.D. thesis)
- Lazarsfeld and Rao (1983)
- Bolondi and M. (many, between 1987 and 1993)
- Martin-Deschamps and Perrin (1990)
- Ballico, Bolondi and M. (1991)
- Geramita and M. (1994)
- Nollet (1996)
- Nagel (1998)
- Kleppe, M., Miró-Roig, Nagel and Peterson [KMMNP2001]
- M. and Nagel (many, e.g. [MN2002], [MN2003])

Essential facts for us, glossing over details:

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Let $C \subset S \subset \mathbb{P}^n$ be schemes. Let A be a form.

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  Remark: For codimension 2 star configurations, S is a hypersurface (union of planes).

  As we saw, for higher codimension star configurations, S is not a hypersurface but still needs to be ACM.

  C does not need to be ACM.

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  (C is a divisor on S.)

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Let $C \subset S \subset \mathbb{P}^n$ be schemes. Let A be a form. Assume

- *S is ACM;*
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- *A does not vanish on any component of S, so A cuts out a (hypersurface section) divisor, on S. Call it Y.*

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- *S is ACM;*
- $\dim C + 1 = \dim S$.
- *A does not vanish on any component of S, so A cuts out a (hypersurface section) divisor, on S. Call it Y.*

  Remark: it's OK if A vanishes on a component of C! But we need to be careful with "union" below. Example coming.

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(a) *$I_{C \cup Y} = A \cdot I_C + I_S$ (as saturated ideals), and you can get lots of information about $I_{C \cup Y}$ from knowledge of I_C and I_S.*

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(b) *$C \cup Y$ is Gorenstein-linked to C in two steps. In particular, one is ACM iff the other is. I.e. ACMness is preserved.*

Corollary. [MN2002]

Let $V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{P}^n$ be ACM schemes of the same dimension.

Let $H_1, \ldots, H_r$ be hypersurfaces, defined by forms $F_1, \ldots, F_r$, such that for each i, H_i contains no component of V_j for $j \leq i$.

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Remark. This is exactly what we used in our example.

Back to star configurations

Corollary. [GHM2013]

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(This is Enrico's "properly intersecting.")

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i.e. $X_c(\mathcal{L})$ is the union of all the linear varieties defined by intersections of c elements of $\mathcal{L}$.

Then

(a) *$X_c(\mathcal{L})$ is ACM;*

(b) *the minimal generators, Hilbert function and Betti numbers of $X_c(\mathcal{L})$ can be computed in terms of r and c.*

Corollary. (M.-Nagel-Schenck 2022)

Let $\mathcal{L} = \{\ell_1, \ldots, \ell_r\}$ be hyperplanes in $\mathbb{P}^n$, $r \geq n$, defined by linear forms L_i, no 3 meeting in codimension 2.

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Let $\mathcal{L} = \{\ell_1, \ldots, \ell_r\}$ be hyperplanes in $\mathbb{P}^n$, $r \geq n$, defined by linear forms L_i, no 3 meeting in codimension 2.

Let $F = \prod_{i=1}^{r} L_i$. Let J be the Jacobian ideal of F:

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This scheme is precisely the codimension 2 star configuration defined by $\mathcal{L}$.

Remark. The bulk of the paper aimed to relax the condition "no 3 meeting in codimension 2." We omit details here.

Remark. The fact that the Jacobian gives the codimension 2 star configuration is intuitively clear from basic double linkage,

since the star configuration is the singular locus of the hypersurface defined by $F = \prod L_i$, and Jacobian ideals give you the singular locus [MNS].

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since the star configuration is the singular locus of the hypersurface defined by $F = \prod L_i$, and Jacobian ideals give you the singular locus [MNS].

But a rigorous ideal-theoretic proof directly using the Jacobian takes some extra work [MN].

The point is to relate Jacobian ideals to basic double links (and liaison addition).

Sidenote: From hyperplanes to hypersurfaces

A paper with Tony, Brian and Uwe (2017) began a study extending the basic double link approach (and much more) to hypersurface configurations.

(Not the Jacobian approach.)

I.e. extend star configurations to "hypersurface configurations."

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The bulk of this work was done during a visit to Kingston in 2014:

June, 2014, Kingston, Ontario

Shin and others also explored these configurations (also not from a Jacobian point of view).

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The main goal is to extend the work with Uwe and Hal mentioned above, but again making rigorous the connection to Jacobians.

There's not enough time in this talk to discuss those results.

The two-fold way*

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* This is the title of a different paper with Brian and Tony, but is used here in a slightly different context.

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So far: In codimension 2 we get the same star configuration, whether you use a Jacobian ideal or BDL.

To do that we needed no three (i.e. $c + 1$) of the hyperplanes to meet in codimension 2 (i.e. c).

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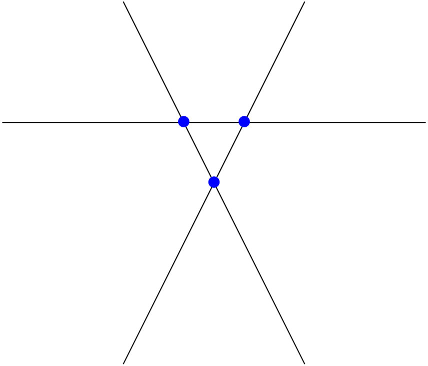
To do that we needed no three (i.e. $c+1$) of the hyperplanes to meet in codimension 2 (i.e. c).

What if we relax this genericity assumption?

Example.



Basic double linkage and Jacobian ideals both give three points. But move the “horizontal” line down...

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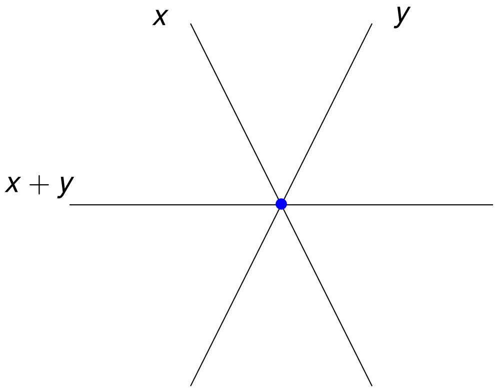

What does the Jacobian give and what does BDL give?

Jacobian ideal:

$F = xy(x+y) = x^2y + xy^2$, so

$$J(F) = \langle F_x, F_y, F_z \rangle = \langle 2xy + y^2, x^2 + 2xy \rangle$$

a non-reduced complete intersection scheme of degree 4 supported at a point. (This approach played a major role in [MNS] and in [MN].)

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Basic Double Link:

$$(x+y) \cdot \langle x, y \rangle + \langle xy \rangle = \langle x^2, xy, y^2 \rangle = \langle x, y \rangle^2,$$

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No longer the same scheme! The twofold way!

Two roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could
To where it bent in the undergrowth ...

Robert Frost
The Road Not Taken

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Robert Frost
The Road Not Taken

What, you have an appointment somewhere? Let's go down both roads and see what we see.

Brian Harbourne
Just about any time he's in a new city,

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Robert Frost
The Road Not Taken

What, you have an appointment somewhere? Let's go down both roads and see what we see.

Brian Harbourne
Just about any time he's in a new city, or while doing mathematics.

The Wager (with apologies to David Grann)

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(Easiest pizza Tony ever won!)

This led to a 2006 paper with Tony and Sindi Sabourin.

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Given any Hilbert function $\underline{h}$ for a reduced set of points in $\mathbb{P}^2$, we

- constructed a specific set of reduced points X with Hilbert function $\underline{h}$;
- produced the double point scheme $2X$ supported on X using a sequence of basic double links (based on the above example);
- gave a description of the Hilbert function and Betti numbers of $2X$.

We also discussed further questions, including some about "star configurations."

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Anyway,

Happy Birthday, Brian,

and all the best!!