Expect to be Surprised: Brian Harbourne's Contributions on Unexpected Properties

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Interpolation

Assumption: K denotes an alg closed field of characteristic zero.

Interpolation: Given distinct elements $a_1, \dots, a_r \in K$ and any $b_1, \dots, b_r \in K$, find a polynomial $f \in K[x_1]$ with $f(a_i) = b_i$ for $i = 1, \dots, r$.

Questions:

(i) Minimum degree α of such a polynomial?

(ii) How many?

(iii) Maximum α when varying the input data?

Dependency on position of the points $(a_i, b_i) \in K^2$.
Generically (general points), the same answer.

Note: $f(a_i) = b_i \Leftrightarrow g(a_i, b_i) = 0$, where $g(x_1, x_2) = f(x_1) - x_2$.

Interpolation

More generally: Given a set Z of r distinct points $P_1, \ldots, P_r \in K^2$, find an algebraic curve C passing through Z.

Answers:

(i) $\alpha(Z) \leq \sqrt{2r}$.

(iii) $\alpha(Z) \approx \sqrt{2r}$ if Z is general.

Move to $\mathbb{P}^n = \mathbb{P}^n_K$: point $P = (a_0 : a_1 : \ldots : a_n) \in \mathbb{P}^n$.
$f \in K[x_0, \ldots, x_n] = R$ homogeneous.
So, $f(P) = f(a_0, \ldots, a_n) = 0$ makes sense.

Interpolation: Given a set $Z = \{P_1, \ldots, P_r\} \subset \mathbb{P}^n$ of r distinct points, find homog. $0 \neq f \in R$ with $f(P_i) = 0$ for $i = 1, \ldots, r$, or, equivalently, a hypersurface passing through Z.

Interpolation

Quantify:

$\dim_K[R]_j = \binom{n+j}{n}$ (monomial basis of $[R]_j$).

Vanishing at a point imposes one condition on $0 \neq f \in [R]_j$.

$f(P_i) = 0$ for every $P_i \in Z \Leftrightarrow f \in I_Z = I_{P_1} \cap \cdots \cap I_{P_r}$

Answers:

(i) $\alpha(Z) \leq \min\{j \in \mathbb{Z} \mid \binom{n+j}{n} > |Z| = r\}$.

(iii) Equality if Z is general.

(ii) $\dim_K[I_Z]_j \geq \min\{0, \binom{n+j}{n} - |Z|\}$
and equality if Z is general.

Hermite Interpolation

More smoothness: require vanishing to higher order.

A homog. pol f vanishes at a point P to order m (or with multiplicity m) if

$$\frac{\partial^k}{\partial_{x_{i_1}} \dots \partial_{x_{i_k}}}(f) = 0 \text{ for any } i_1, \dots, i_k \text{ and } k < m$$
$$\Leftrightarrow \frac{\partial^k}{\partial_{x_{i_1}} \dots \partial_{x_{i_k}}}(f) = 0 \text{ for any } i_1, \dots, i_k \text{ and } k = m - 1.$$

So, vanishing of f to order m at P imposes $\binom{n+m-1}{n}$ conditions on f.

Artin-Nagata: f has multiplicity m at $P \Leftrightarrow f \in I_P^m$.

Hermite Interpolation

Question: Given $Z = \{P_1, \ldots, P_r\} \subset \mathbb{P}^n$ and $m_1, \ldots, m_r \in \mathbb{N}$, define a fat point scheme X supported on Z by

$$I_X = I_{P_1}^{m_1} \cap \cdots \cap I_{P_r}^{m_r} \subset K[x_0, \ldots, x_n].$$

Symbolically, $X = m_1 P_1 + \cdots + m_r P_r$.

$$\dim_K [I_X]_j = ?$$

Expectations: $V \subseteq [R]_j$ a subspace

$f \in V$ vanishes at P to order $m \Leftrightarrow f \in V \cap I_P^m$. So,

$$\dim_K \left(V \cap [I_P^m]_j \right) \geq \min \left\{ 0,\ \dim_K V - \binom{n+m-1}{n} \right\}.$$

Expect equality if P is general in $\mathbb{P}^n$.

Repeat. If $P_1, \ldots, P_r$ are general, then one *expects* for $X = m_1 P_1 + \cdots + m_r P_r$,

$$\dim_K [I_X]_j = \max \left\{ 0,\ \binom{n+j}{n} - \sum_{i=1}^{r} \binom{n+m_i-1}{n} \right\}.$$

Alexander-Hirschowitz Theorem

Example 1

Consider cubics passing through 7 general double points in $\mathbb{P}^4$, that is, $X = 2P_1 + \cdots + 2P_7 \subset \mathbb{P}^4$ and $j = 3$. The expected dimension of $[I_X]_3$ is

$$\max\left\{0,\ \binom{4+3}{4} - \sum_{i=1}^{7} \binom{4+2-1}{4}\right\} = \max\{0, 35 - 7 \cdot 5\} = 0.$$

So, we do not expect any such cubic to exist. **But** there is one: Any 7 points in $\mathbb{P}^4$ lie on a rational normal curve, C. In convenient coordinates, C is defined by the 2-minors of $\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$.

The variety of secant lines to C is defined by the cubic polynomial $f = det \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$. This hypersurface is singular along C, and so $f \in I_X$.

Alexander-Hirschowitz Theorem

Theorem (Alexander-Hirschowitz, 1995)

If $X = 2P_1 + \cdots + 2P_r \subset \mathbb{P}^n$ is a subscheme of r general double points then $[I_X]_j$ has the expected dimension, that is,

$$\dim_K [I_X]_j = \max\left\{0, \binom{j+n}{n} - \sum_{i=1}^{r} \binom{n+1}{n}\right\} = \max\left\{0, \binom{j+n}{n} - r\cdot(n+1)\right\},$$

except in the following cases:

(i) $j = 2$ and $2 \leq r \leq n$;

(ii) $j = 3$, $n = 4$ and $r = 7$; or

(iii) $j = 4$, $2 \leq n \leq 4$ and $r = \binom{n+2}{2} - 1$.

In the exceptional cases, the actual dimension is one more than the expected dimension.

Higher Multiplicities

General case: $X = m_1 P_1 + \cdots + m_r P_r$ with $m_i \geq 2$ and $P_1, \ldots, P_r \in \mathbb{P}^n$ general. - Open!

Theorem (Alexander-Hirschowitz, 2000)

Given any integer $m \geq 1$, there is an integer $j(m)$ such that, for any $X = m_1 P_1 + \cdots + m_r P_r \subset \mathbb{P}^n$ with $m_i \leq m$ and every $j \geq j(m)$, one has

$$\dim_K [I_X]_j = \max \left\{ 0, \binom{j+n}{n} - \sum_{i=1}^{r} \binom{m_i + n - 1}{n} \right\}.$$

Note: $j(m)$ is independent of r, the number of points supporting X.

Initial Degree

Fix n and consider $X = m_1P_1 + \cdots + m_rP_r$ with $m_i \geq 1$ supported on general points $P_1, \ldots, P_r \in \mathbb{P}^n$.

Problem 1

For each such X, i.e. for any $(m_1, \ldots, m_r) \in \mathbb{N}^r$, determine the Hilbert function of X, i.e., for each $j \in \mathbb{N}$, determine $\dim_K[I_X]_j$.

Problem 2

For each such X, i.e. for any $(m_1, \ldots, m_r) \in \mathbb{N}^r$, determine the initial degree of I_X, that is,

$$\alpha(X) = \min\{j \in \mathbb{Z} \mid [I_X]_j \neq 0\}.$$

Apparently, Problem 2 is easier than Problem 1.

Initial Degree

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Apparently, Problem 2 is easier than Problem 1.

Theorem (Harbourne, 2005)

Problems 1 and 2 are equivalent.

Reference: The (unexpected) importance of knowing α.

SHGH Conjecture

SHGH Conjecture ($n = 2$)

For $X = m_1 P_1 + \cdots + m_r P_r$ with general points $P_i \in \mathbb{P}^2$, $[I_X]_j$ fails to have the expected dimension only if $[I_X]_j \neq 0$ and the base locus of $[I_X]_j$ contains a multiple of a rational curve of a prescribed kind.

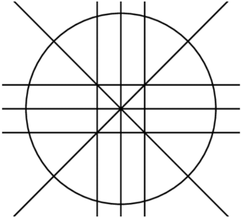
Remark:

- 4 equivalent versions of the Conjecture: Segre, 1961; Harbourne, 1986; Gimigliano, 1987; Hirschowitz, 1989.
- Ciliberto, Miranda, 2001: The above necessary criterion provides a full quantitative conjectural answer:
  (i) a complete list of all $(m_1, \ldots, m_r)$ and j for which $[I_X]_j$ conjecturally fails to have the expected dimension; and
  (ii) a prediction for the actual value of $\dim_K [I_X]_j$.

Unexpected Curves

Example (Di Gennaro, Ilardi and Vallés, 2014)

Let $Z \subset \mathbb{P}^2$ be a set of nine points dual to the so-called B_3 line arrangement. It has the property that, for every point $P \in \mathbb{P}^2$, there is a degree four curve passing through Z and vanishing to order three at P.



Expectation: For $X = Z + 3P$, i.e., $I_X = I_Z \cap I_P^3$, one has

$$\dim_K [I_X]_4 = \max\left\{0, \binom{2+4}{2} - 9 - \binom{2+3-1}{2}\right\}$$
$$= 0.$$

Unexpected Curves

Set-up: $Z \subset \mathbb{P}^2$ *any* finite set of points, $P \in \mathbb{P}^2$ a *general* point, $X = Z + mP$, so $I_X = I_Z \cap I_P^m$.

Problem 3

For which Z and m, d is the actual dimension of $\dim_K [I_Z \cap I_P^m]_d$ different from the expected dimension, that is, when is

$$\max \left\{ 0, \dim [I_Z]_d - \binom{m+1}{2} \right\} \le \dim_K \left[I_Z \cap I_P^m \right]_d$$

not an equality?

Note: $\left[I_Z \cap I_P^m \right]_d = 0$ if $d \le m < |Z|$.

Definition 1: Let $P \in \mathbb{P}^n$ be a general point.

- Z has an unexpected curve (of degree $m+1$) if

$$\max \left\{ 0, \dim [I_Z]_{m+1} - \binom{m+1}{2} \right\} < \dim_K \left[I_Z \cap I_P^m \right]_{m+1}$$

- The multiplicity index of Z is

$$m_Z = \min \{ m \in \mathbb{Z} \mid \left[I_Z \cap I_P^m \right]_{m+1} \neq 0 \}.$$

Unexpected Curves

Theorem (Cook II, Harbourne, Migliore, N., 2018)

- If no 3 points of Z are collinear then $m_Z = \frac{|Z|-1}{2}$ and Z has no unexpected curves.
- Z has a unexpected curve $\Leftrightarrow m_Z < \frac{|Z|-1}{2}$ and no $m_Z + 2$ points of Z are collinear.
- If Z has any unexpected curve then it has an unexpected curve of degree d iff $m_Z < d \leq |Z| - m_Z - 2$.
- Any unexpected curve of Z is the union of an irreducible rational curve C and $|Z \setminus Z'|$ lines, where C has degree $m_{Z'} + 1 \geq 2$ and is the unique unexpected curve of a subset $Z' \subseteq Z$.

Example (Fermat Configuration, CHMN, 2018)

For any integer $t \geq 3$, let $Z \subset \mathbb{P}^2$ be the set of $3t$ points defined by

$$\begin{cases} x^t + y^t + z^t = xyz = 0 & \text{if } t \text{ is odd} \\ x^t - y^t - z^t = yz = 0 \text{ or } y^t - z^t = x = 0 & \text{if } t \text{ is even.} \end{cases}$$

The multiplicity index of Z is $m_Z = t + 1$, and
if $t \geq 5$ then Z has unexpected curves of degree $t + 2$, $t + 3, \ldots, 2t - 3$.
The unexpected curve of degree $t + 2$ is irreducible. (Not expected for general points according to the SHGH Conjecture.)

Note: Connection to line arrangements in $\mathbb{P}^2$ dual to Z.

Unexpected Hypersurfaces

More general set-up: $Z \subset \mathbb{P}^n$ any projective subscheme,
$P \in \mathbb{P}^n$ a *general* point,
$X = Z + mP$, so $I_X = I_Z \cap I_P^m$.

Problem 4

For which Z and m, j is the actual dimension of $\dim_K[I_Z \cap I_P^m]_j$ different from the expected dimension, that is, when is

$$\max\left\{0, \dim[I_Z]_d - \binom{n+m-1}{n}\right\} \leq \dim_K \left[I_Z \cap I_P^m\right]_d$$

not an equality?

Definition 2: Z admits an unexpected hypersurface of degree d with a general point P of multiplicity m if

$$\max\left\{0, \dim[I_Z]_d - \binom{n+m-1}{n}\right\} < \dim_K \left[I_Z \cap I_P^m\right]_d.$$

Note: Szpond (2022) has results on hypersurfaces that vanish with some multiplicity at more than one general point.

Unexpected Hypersurfaces

Theorem (Harbourne, Migliore, N., Teitler, 2021)

Given positive integers (n, d, m) with $n \geq 2$, there exists an unexpected hypersurface of degree d with a general point of multiplicity m for some finite subset $Z \subset \mathbb{P}^n$ if and only if one of the following conditions holds true:

(a) $n = 2$ and (d, m) satisfies $d > m > 2$; or

(b) $n \geq 3$ and (d, m) satisfies $d \geq m \geq 2$.

Key: Many curves admit unexpected hypersurfaces that are cones.

Unexpected Hypersurfaces

A subscheme X is a cone with vertex P if, for every point Q in X, the line joining P and Q is in X.

Note: By Bézout's Theorem, every hypersurface of degree d with a point P of multiplicity d is a cone with vertex P.

Proposition (HMNT, 2021)

Let V be a reduced, equidimensional, non-degenerate subvariety of $\mathbb{P}^n$ ($n \geq 3$) of codimension 2 and degree d. Let $P \in \mathbb{P}^n$ be a general point. Then the cone S_P over V with vertex P is an unexpected hypersurface for V of degree d and multiplicity d at P. It is the unique unexpected hypersurface of this degree and multiplicity.

Quantifying Unexpected Hypersurfaces

Set-up: $Z \subset \mathbb{P}^n$ any subscheme, $j \geq m \geq 1$ integers
Always have

$$\max\left\{0, \dim[I_Z]_j - \binom{n+m-1}{n}\right\} \leq \dim_K \left[I_Z \cap I_P^m\right]_j.$$

Following Favacchio, Guardo, Harbourne, Migliore, fix any integer $d \geq 0$, and define a sequence $AV_{Z,d} = (AV_{Z,d}(m))_{m \in \mathbb{N}}$ by

$$AV_{Z,d}(m) = \dim_K \left[I_Z \cap I_P^m\right]_{m+d} - \dim_K [I_Z]_{m+d} + \binom{n+m+d-1}{n}.$$

Lemma (Favacchio, Guardo, Harbourne, Migliore, 2021)

If $P \in \mathbb{P}^n$ is a general point then

$$AV_{Z,d}(m) = \dim_K [R/(I_Z + I_P^m)]_{m+d}.$$

Quantifying Unexpected Hypersurfaces

Note: If $\left[I_Z \cap I_P^m\right]_j \neq 0$ then $X = Z + mP$ admits an unexpected hypersurface of degree j and multiplicity m iff $AV_{Z,d}(m) = \dim_K[R/(I_Z + I_P^m)]_{m+d} > 0$ for $d = j - m$.

Notation: $\mathrm{gin}(I)$, generic initial ideal of I with respect to the lexicographic order with $x_0 > \cdots > x_n$.

Lemma (FGHM, 2021)

(a) $\dim_K \left[I_Z \cap I_P^m\right]_j = \dim_K \left[\mathrm{gin}(I_Z) \cap I_Q^m\right]_j$,
where $Q = (1 : 0 : \cdots : 0) \in \mathbb{P}^n$.

(b) $AV_{Z,d}(m) = \dim_K[R/(\mathrm{gin}(I_Z) + I_Q^m)]_{m+d}$.

Quantifying Unexpected Hypersurfaces

Theorem (FGHM, 2021)

For any $d \geq 0$, the sequence $AV_{Z,d}$ is a shifted Hilbert function. More precisely, setting

$$J = \text{gin}(I_Z) : x_0^{d+1},$$

one has

$$AV_{Z,d}(m+1) = \dim_K [R/J]_m \quad (m \geq 0).$$

Corollary (FGHM, 2021)

If Z is contained in a hypersurface of degree $d+1$ then $AV_{Z,j}(m+1) = 0$ whenever $j \geq d$ and $m \geq 0$.
In particular, if Z is degenerate ($d = 0$) then Z admits no unexpected hypersurfaces.

Proof: By assumption, $x_0^{d+1} \in \text{gin}(I_Z)$, and so $1 \in \text{gin}(I_Z) : x_0^{j+1}$ if $j \geq d$. Hence, $AV_{Z,j}(m+1) = 0$ for every $m \geq 0$.

Quantifying Unexpected Hypersurfaces

An SI-sequence is a finite, nonzero, symmetric O-sequence whose first half is a differentiable O-sequence.

SI-sequences are precisely the Hilbert functions of Artinian Gorenstein algebras with the weak Lefschetz property.

Conjecture (FGHM, 2021)

Let $Z \subset \mathbb{P}^3$ be a smooth aCM curve. If Z is not contained in a quadric hypersurface then the sequence $AV_{Z,1}$ is an SI-sequence (shifted by 1).

Recall

$$AV_{Z,1}(m) = \dim_K [R/(I_Z + I_P^m)]_{m+1}.$$

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Happy Birthday, Brian!