On the BOUNDARY of the MORI CONE of GENERAL BLOWUPS of $\mathbb{P}^2$

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joint work with Ciro Ciliberto and Rick Miranda

Unexpected and Asymptotic Properties of Algebraic Varieties (BrianFest  2023)

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Main Theorem

Let X_n be the blow-up of $\mathbb{P}^2$ at n *very general* points.

1. For every $n \geqslant 10$, in the space $N = \mathrm{Pic}(X_n) \otimes \mathbb{R}$ of $\mathbb{R}$-divisors, there exist 8-dimensional spheres S such that cone(S) lies on the boundary of the Mori cone (and of the nef cone) of X_n.
2. For every $n \geqslant 13$, there exist such spheres contained in $K_{X_n}^+$.
3. For every $n \geqslant 13$ such that $n - 4$ is a square, there exist such spheres meeting $F_{X_n}^-$, where $F_{X_n} = \sqrt{n-1}H - \sum_{i=1}^n E_i$

General Blowups of $\mathbb{P}^2$

- $\operatorname{Pic} X_n = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n$
- Riemann-Roch problem:
  given $L = dH - \sum_i m_i E_i \in \operatorname{Pic} X_n$, determine $h^0(X_n, L)$
  expected : $\max\left(0, \frac{(d-1)(d-2)}{2} - \sum \frac{m(m+1)}{2}\right)$
- Effectivity problem: When is $L \neq 0$? (Dually: nef classes)

Anticanonical cases:  1982, 1985, 1994, 1996, 1997...

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ANTICANONICAL RATIONAL SURFACES

BRIAN HARBOURNE

Either h^0 = expected or there is (-1)-curve C with $C \cdot L < 0$

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Cremona-Kantor group  1985

Our results are stated using the following terminology. Let $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ be an exceptional configuration for V, and denote by r_i, $i = 1, \ldots, n-1$, the class $\mathcal{E}_i - \mathcal{E}_{i+1}$; also, $r_{-2} = \mathcal{E}_0 - \mathcal{E}_1$, $r_{-1} = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$ and $r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$. The classes $r_0, \ldots, r_{n-1}$ are the simple roots of a root system in $\mathrm{Pic}\, V$ [**5**].

To each is associated the reflection s_i: $\mathrm{Pic}\, V \to \mathrm{Pic}\, V$ defined by $s_i(x) = x + (x \cdot r_i) r_i$. (The significance of these operations seems first to

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SHGH conjecture  1986

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THE GEOMETRY OF RATIONAL SURFACES AND HILBERT FUNCTIONS OF POINTS IN THE PLANE

Brian Harbourne

In particular, let Y be the blowing up of sufficiently general points $p_1, \ldots, p_n$ of $\mathbf{P}^2$. Our first conjecture is that a class F of Pic(Y) is numerically effective if and only if $F \cdot F \geq 0$ and F is a standard class, i.e., for some exceptional configuration $E'_O, \ldots, E'_n$ of Y F is a nonnegative sum of the classes E'_0, $E'_0 - E'_1$, $2E'_0 - E'_1 - E'_2$ and $3E'_0 - E'_1 - \cdots E'_i$, $i \geq 3$. Our second conjecture is that if F is standard then $h^0(Y, F) h^1(Y, F) = 0$.

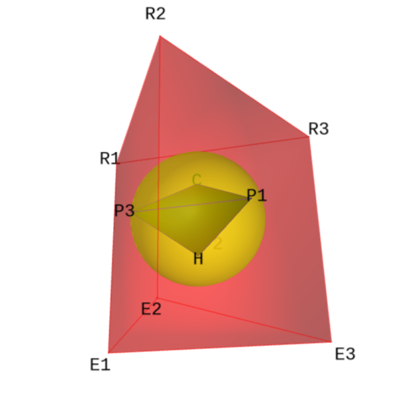
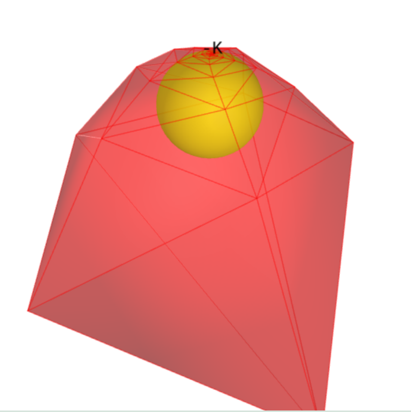
RAYS AND CONES (ASYMPTOTICS IN Pic X_n)

- $N = N^1(X_n) = \mathrm{Pic}(X_n) \otimes_{\mathbb{Z}} \mathbb{R}$
- For $L \in N$, $\overset{\bullet\!-}{L} = \mathbb{R}_{\geqslant 0} \cdot L \subset N$ is a ray
- $\overset{\bullet\!-}{L_1} \cdot \overset{\bullet\!-}{L_2} := \mathrm{sign}(L_1 \cdot L_2) \in \{-, 0, +\}$
- $\deg \overset{\bullet\!-}{L} := \overset{\bullet\!-}{L} \cdot \overset{\bullet\!-}{H}$
- R ray effective if $R = \overset{\bullet\!-}{L}$ for some effective L
  eg, if R rational, $\deg R > 0$ and $R^2 > 0$ (Riemann-Roch)
- Mori cone $\overline{\mathrm{NE}}(X_n)$: closed convex cone generated by effective rays
- Nef cone $\mathrm{Nef}(X_n)$: dual of Mori cone

Mori's Cone Theorem

$$\overline{\mathrm{NE}}(X_n) = \overline{\mathrm{NE}}(X_n)^{K_n^{\geqslant 0}} + \sum_{E\ (-1)\text{-curve}} \overset{\bullet\!-}{E}$$

CONE SHAPES (ANTICANONICAL CASES)

$n < 8$	$n = 9$
	

$\mathcal{Q}_n^\perp = \{L \in N(X_n) \mid L^2 = LK = 0\}$

- If E is (-1)-curve, $E^\perp \cap \mathcal{Q}_n^\perp$ is a single ray
- (de Fernex) $\mathcal{Q}_n^\perp$ is part of the boundary of $\overline{\mathrm{NE}}(X_n)$ and $\mathrm{Nef}(X_n)$
- $\{(-1)\text{-curves}\} \leftrightarrow \{\text{rational rays on } \mathcal{Q}_n^\perp\}$
- Every rational ray is CK-equivalent to $3H - E_1 - \cdots - E_9$

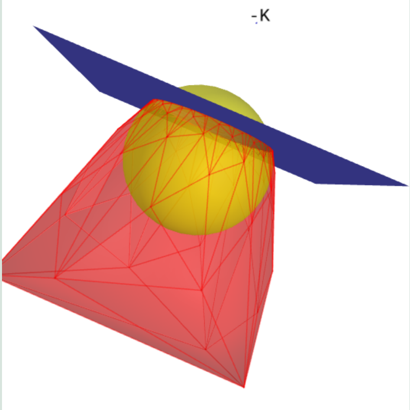
NEFNESS ON $K^\perp$, $n > 10$

$\mathcal{W}_n = \langle H, E_1, \ldots, E_{10} \rangle \cap \mathcal{Q}_n^\perp \subset N(X_n)$
(cone over 8-dim sphere)

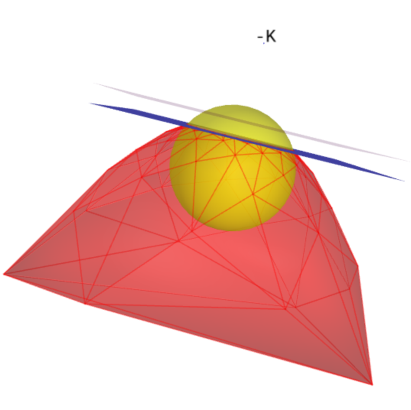
- $\mathcal{Q}_n^\perp$ covered by subcones $\mathcal{Q}_E$ where E are (-1)-curves E on X_n
- $\mathrm{int}(\mathcal{Q}_E)$: non-nef classes
- $\partial \mathcal{Q}_E = \mathcal{Q}_n^\perp \cap E^\perp \cong \mathcal{Q}_{n-1}^\perp$
- Nef locus Nef^0 on $\mathcal{Q}_n^\perp$ = (CK)-translates of $3H - E_1 - \cdots - E_9$
- $\overline{\mathrm{Nef}^0}$ = (CK)-translates of $\mathcal{W}_n$
- At each rational ray of Nef^0, exactly $n - 9$ translates of $\mathcal{W}_n$ meet.

CONE SHAPES ($n > 9$)

$n = 10$



$n > 10$

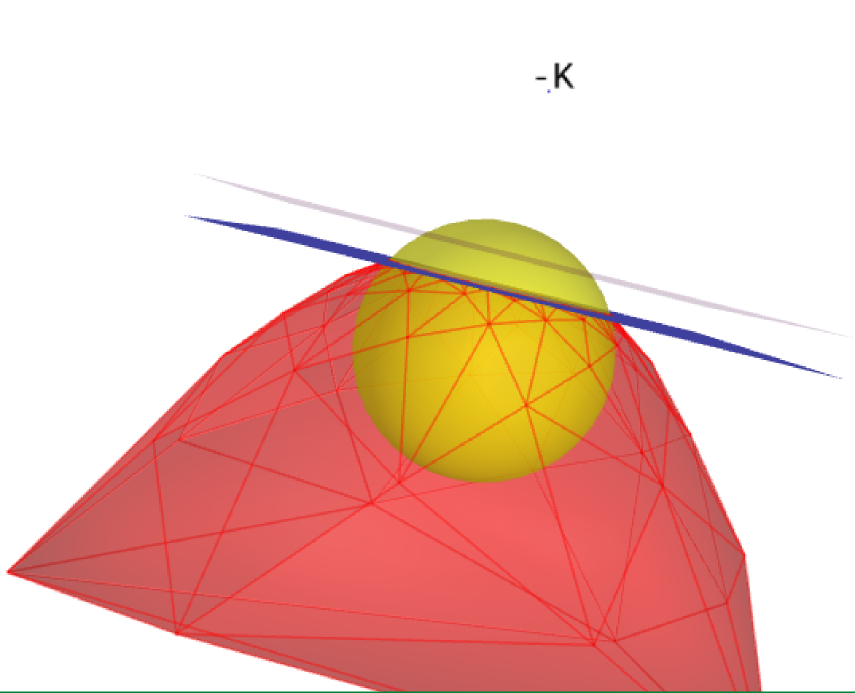


- Good ray [C  MR, 2013]: non-effective rational ray R with $\deg R > 0$, $R^2 = 0$ and zero self-intersection.
- Wonderful ray: irrational nef ray with $R^2 = 0$

Lemma (Application of Ciliberto-Miranda degeneration)

Fix $r \geqslant 2$, $n \geqslant r^2+1$, and multiplicities $m, m_{r^2+1}, \ldots, m_n$.
If $h^0(X_n, dH - rmE_1 - m_{r^2+1}E_{r^2+1} - m_nE_n) \leqslant m$ then
$dH - m(E_1 + \cdots + E_{r^2}) - m_{r^2+1}E_{r^2+1} - m_nE_n$ is non-effective

- Both classes have the same self-intersection
- The nef classes on $Q_n^{\perp}$ have $h^0 = 1$
- Each 8-dimensional sphere of nef classes on X_{n-r^2+1} gives a 8-dimensional sphere of noneffective boundary classes on X_n
- These intersect K positively, and sometimes F negatively


-K

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