BRIANFEST

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Through base loci towards positivity

University of Nebrasca - Lincoln
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UNIWERSYTET PEDAGOGICZNY W KRAKOWIE

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Seshadri constants

Definition 1 (Demailly 1990)

Let X be an algebraic variety and let L be a nef line bundle on X. The *Seshadri constant of* L *at a point* $x \in X$ is the real number

$$\varepsilon(L,x) = \inf_{C \ni x} \frac{L.C}{\text{mult}_x(C)}.$$

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Definition 2

Let X be an algebraic variety and let L be a nef line bundle on X. The *Seshadri constant of L at a finite set of points* $x_1, \ldots, x_r \in X$ is the real number

$$\varepsilon(L, x) = \inf_{C \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L.C}{\sum_{i=1}^{r} \text{mult}_{x_i}(C)}.$$

Nagata's Conjecture

Conjecture 1 (Nagata 1959)

Let $x_1, \ldots, x_r$ *be general points in the complex projective plane* $\mathbb{P}^2$ *with* $r \geq 10$ *and let* $m_1, \ldots, m_r$ *be positive integers. Then the minimal degree* d *of a curve passing through the points* $x_1, \ldots, x_r$ *with multiplicities at least* m_i *at* x_i *for* $i = 1, \ldots, r$ *is subject to the restriction*

$$d > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i.$$

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$$d > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i.$$

Theorem 3 (TS, 2001)

If there is a curve violating the Nagata Conjecture, then its multiplicities in all but one point are equal.

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Seshadri fibrations

Theorem 4 (Wioletta Syzdek, TS)

Let X be a smooth projective surface, L be a nef and big line bundle on X and $r \geq 2$ be a fixed integer. If

$$\varepsilon(L; r) < \sqrt{\frac{r-1}{r}} \cdot \sqrt{\frac{L^2}{r}},$$

then there exists a fibration $f : X \longrightarrow B$ over a curve B such that given $P_1, \dots, P_r \in X$ very general, for arbitrary $i = 1, \dots, r$ the fiber $f^{-1}(f(P_i))$ computes $\varepsilon(L; P_1, \dots, P_r)$ i.e. the fiber is a Seshadri curve of L.

Seshadri fibrations

Theorem 5 (Wioletta Syzdek, TS)

Let $r \geq 2$ be a given integer, X a smooth projective surface and L a nef and big line bundle on X such that

$$\varepsilon(L; r) = \sqrt{\frac{r-1}{r}} \cdot \sqrt{\frac{L^2}{r}} \, .$$

If X is not fibred by Seshadri curves for L, then

a) *either $r = 2$, $X = \mathbb{P}^2$ and $L = \mathcal{O}(1)$,*

b) *or X is a surface of minimal degree in $\mathbb{P}^r$ and $L = \mathcal{O}_X(1)$.*

The Primer

Bauer, Thomas; Di Rocco, Sandra; Harbourne, Brian; Kapustka, Michał; Knutsen, Andreas; Syzdek, Wioletta; Szemberg, Tomasz
A primer on Seshadri constants. (English) Zbl 1184.14008
Bates, Daniel J. (ed.) et al., Interactions of classical and numerical algebraic geometry. A conference in honor of Andrew Sommese, Notre Dame, IN, USA, May 22–24, 2008. Providence, RI: American Mathematical Society (AMS) (ISBN 978-0-8218-4746-6/pbk). Contemporary Mathematics 496, 33-70 (2009).

Seshadri constants express the local positivity of a line bundle on a projective variaty. They were introduced by [*J.-P. Demailly*, Singular Hermitian metrics on positive line bundles. Complex algebraic varieties, Proc. Conf., Bayreuth/Ger. 1990, Lect. Notes Math. 1507, 87–104 (1992; Zbl 0784.32024)]. The original hope of using them towards a proof of the Fujita conjecture was too optimistic, but it soon became clear that they are interesting invariants quite in their own right. The subject witness intense development producing strong results with interesting connections. A beautiful introduction the subject and presentation of its connections with local positivity can be found in *R. Lazarsfeld*'s book [Positivity in Algebraic Geometry. Berlin: Springer (2004; Zbl 1066.14021)].

The article is an excellent review of Seshadri constants, it provides a complete account of recent progress, it discuss many open question. The note also contains a lot of interesting key examples too. (The following section-titles suggest rather well the completeness of the discussion: basic properties, lower bounds, weakly-submaximal curves, special focus on toric case and surfaces, slope stability, symbolic powers.)

Recent Developments

Bauer, Thomas; Bocci, Cristiano; Cooper, Susan; Di Rocco, Sandra; Dumnicki, Marcin; Harbourne, Brian; Jabbusch, Kelly; Knutsen, Andreas; Küronya, Alex; Miranda, Rick; Roé, Joaquim; Schenck, Hal; Szemberg, Tomasz; Teitler, Zach
[Hwang, J.-M.]
Recent developments and open problems in linear series. (English) Zbl 1254.14001
Pragacz, Piotr (ed.), Contributions to algebraic geometry. Impanga lecture notes. Based on the Impanga conference on algebraic geometry, Banach Center, Będlewo, Poland, July 4–10, 2010. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-114-9/hbk). EMS Series of Congress Reports, 93-140 (2012).

From the introduction: These notes contain problems, examples and theorems that were prepared for and grew out of the Oberwolfach Mini-Workshop "Linear series on algebraic varieties" (Oct. 3–9, 2010), thereby giving a useful collection of results which are either scattered through the literature or are considered to be folklore.

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The Conjecture

Conjecture 2 (Segre 1961, Harbourne 1984, Gimigliano 1987, Hirschowitz 1989)

Let X_r be the blowup of $\mathbb{P}^2$ at r general points $x_1, \dots, x_r$ with exceptional divisors $E_1, \dots, E_r$. Let

$$M = dH - \sum_{i=1}^{r} m_i E_i,$$

where H is the pull-back of the hyperplane bundle to X_r and $m_1, \dots, m_r$ are positive integers. Then

M is special if and only if it is (-1)-special.

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Definition 6

Using the above notation, a linear system M is (-1)-special if there exists a (-1)-curve $C \subset X_r$ contained in the base locus of M with multiplicity at least 2.

Linking to the Nagata Conjecture

Okounkov bodies enter the stage

Conjecture 3 (SHGH, second formulation)

Let X_r be the blow up of the projective plane P^2 in r general points with exceptional divisors $E_1, \dots, E_r$. Let H denote the pullback to X_r of the hyperplane bundle. Let the integers $d, m_1 \geq \cdots \geq m_s \geq -1$ with $d \geq m_1 + m_2 + m_3$ be given. Then the line bundle

$$dH - \sum_{i=1}^{r} m_i E_i$$

is non-special.

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is non-special.

Theorem 7 (Dumnicki, Küronya, Maclean, TS 2013)

Let $r \geq 9$ be an integer for which the SHGH Conjecture holds true. Then

a) *either there exists on X_r an ample line bundle whose Seshadri constant at a very general point is irrational;*

b) *or the SHGH Conjecture fails for $r + 1$ points.*

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Motivation and notation

1 (Elena Guardo, Brian Harbourne, Adam Van Tuyl 2011)

Symbolic powers versus regular powers of ideals of general points in $\mathbb{P}^1 \times \mathbb{P}^1$.

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Notation 1

Let $P_{n,r,s,m}$ be the Hilbert polynomial of the union of s disjoint r-dimensional flats in $\mathbb{P}^n$ taken with multiplicity m.
Let $\Lambda_{n,r,s}(\tau)$ be the leading term of $P_{n,r,s,m}(m\tau)$. Let

$$e_{n,r,s} = \inf\left\{\frac{t}{m} :\ t \geq m \geq 1,\ P_{n,r,s,m}(t) > 0\right\},$$

be the expected Waldschmidt constant of the ideal of s disjoint r-dimensional flats in $\mathbb{P}^n$.

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Theorem 8 (Dumnicki, Harbourne, Tutaj-Gasińska, TS)

Let n, r, s be integers with $n \geq 2r+1$, $r \geq 0$ and $s \geq 1$. Let I be the ideal as above. Then the polynomial $\Lambda_{n,r,s}(\tau)$ has a single real root $g_{n,r,s}$ bigger than or equal to 1 and

$$\hat{\alpha}(I) \leq e_{n,r,s} \leq g_{n,r,s}.$$

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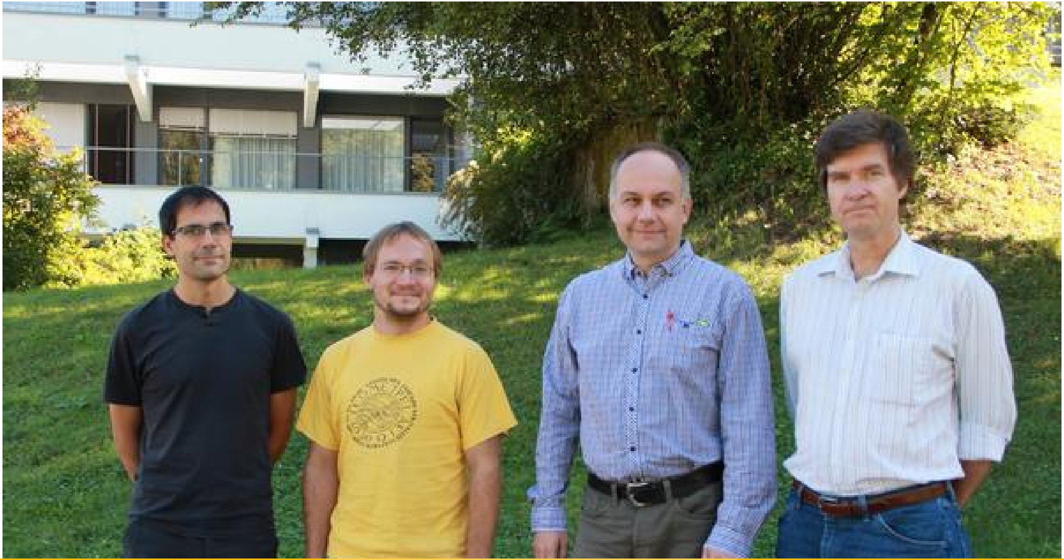
MFO Research in Pairs 2013



Monomial valuations and Nagata type Conjecture

Definition 9

Let ν be a rank 1 valuation on the field of functions of an algebraic surface S.
For a divisor $D \subset S$, define $\nu(D)$ as the value of ν on any equation of an affine part of D.
Let $\mu_D(\nu) = \max\{\nu(D') : \ D' \in |D|\}$ and $\hat{\mu}_D(\nu) = \lim_{k\to\infty} \frac{\mu_{kD}(\nu)}{k}$.
Let $I_m = \{f \in \mathcal{O}_S : \ \nu(f) \geq m\}$.
For ν centred at a point of $S = \mathbb{P}^2$ there is

$$\mathrm{vol}(\nu) = \lim_{m\to\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_S/I_m)}{m^2/2}.$$

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Let D be a big divisor and ν a real valuation centered at a point $P \in S$. Then

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Definition 11

For $\xi(x) \in \mathbb{C}[[x]]$ with $\xi(0) = 0$ and $t \geq 1$ let

$$\nu(\xi, t; f) = \operatorname{ord}_x(f(x, \xi(x) + \theta x^t)),$$

where θ is a transcendental over $\mathbb{C}$.

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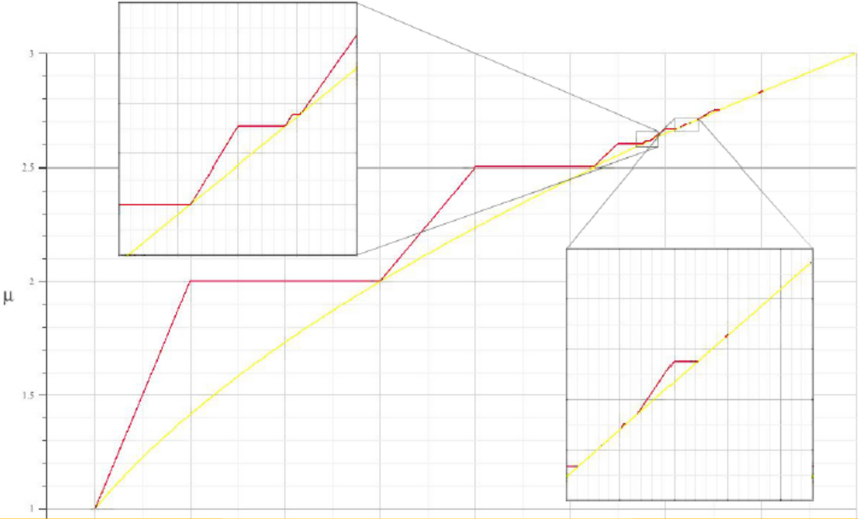
Conjecture 4 (Dumnicki, Harbourne, Küronya, Roe, TS 2016)

For sufficiently general ξ the valuation $\nu = \nu(\xi, t)$ is minimal for $t \geq 8 + 1/36$.

The graph of $\hat{\mu}(t)$



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