Some constructions of unexpected hypersurfaces

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BrianFest

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Definition of an unexpected curve on $\mathbb{P}^2$

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- A hypersurface defined by a form from $L_d(L \cup Z)$ is unexpected with respect to Z if the space $L_d(L \cup Z)$ has
  - dimension greater than 0 and
  - codimension in $L_d(Z)$ less than is expected

How to construct/find such a hypersurface?

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$(g_{k,0,0}, \dots, g_{0,0,k}, g)$ a syzygy of $(f_a, f_b, f_c)^k + (L)$ and $\deg(g_{i_1,i_2,i_3}) = d$	$S_Q(x, y, z) :=$ $g_{k,0,0}(Q)x^k + g_{k-1,1,0}(Q)x^{k-1}y + \cdots + g_{0,0,k}(Q)z^k$

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$$\{L_Q = 0\} \cap \{S_Q = 0\}$$

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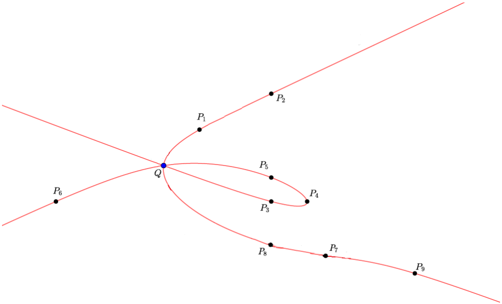
- C passes through points of Z and $\mathrm{mult}_{P_L} C = d$

B3 example



- iff conditions?

Theorem

Let $Z \subset \mathbb{P}^2$ be a finite set of points whose dual is a line arrangement with splitting type (a, b). Let P be a general point. Then the subscheme $X = mP$ fails to impose the expected number of conditions on $[I_Z]_{m+1}$ if and only if

(i) $a \leq m \leq b - 2;$ *and*

(ii) $h^1(\mathcal{I}_Z(t_Z)) = 0,$

where $t_Z := \min\left\{ j \geq 0 \ : \ h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} > 0 \right\}.$

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- *$C_\lambda(V)$ is unexpected.*

Veneroni map

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6. Base locus of v_n consists of all the Π_j and all common transversals to them

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Veneroni – Cremona map, example in $\mathbb{P}^4$

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- $D := \phi^*(T \cap Y)$ is the duodectic.

Only a few

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Thank you

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All the best, Brian!