SHEAR BANDING—A STUDY IN NONLOCAL PROBLEMS

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Dedicated to Gary Meisters in honor of his sixty-fifth anniversary.

1 Introduction

The formation of shear bands in materials has important implications to a variety of physical processes. These bands are observed in very thin zones and are generally regarded as a precursor to material failure. Shear band formation is caused by the heat generated in regions with highest strain rate. With insufficient time for diffusion of this heat, a localized thermal softening of the material occurs which enhances plastic flow in a thin zone. This adiabatic strain localization can be modelled as nonlinear thermally-activated reaction-diffusion equations. This leads to a class of nonlocal parabolic problems and their associated time-independent steady-state counterparts.

This lecture describes and explains some of the results found in Bebernes-Talaga [1], Bebernes-Lacey [2], and Bebernes-Li-Talaga [3]. The modelling discussed in the following is based on the work of Burns ([4],[5]).

Consider loading a thin-walled tube of metal of length d in torsion with ends held at constant temperature T_0 and the tube having initial temperature T_0. One end is fixed and the other end is twisted at a constant rate $v = v_0$. If z denotes the axial coordinate, t time, $w(z,t)$ the linear displacement, $v = w_t$ the velocity, $\gamma(z,t) = w_z(z,t)$ the shear strain, and $\tau(z,t)$ the shear stress, then the thermoviscoplastic shear model is given by the following system of conservation laws:

$$\begin{array}{ll} \varphi v_t = \tau_z & \text{(Momentum)} \\ b\tau_t = v_z - \gamma_t & \text{(Elasticity)} \\ T_z = \lambda T_{zz} + \mu^{p-1} \cdot \tau \cdot \gamma_t & \text{(Energy)} \\ \gamma_t = \Phi(\tau, \gamma, T) & \text{(Constitutive)} \end{array} \tag{1.1}$$

where φ, b, λ, μ, and p are constants.

If $\varphi << 1$ and $b << 1$, then the model simplifies to the quasi-static model:

$$\begin{array}{l} \tau_z = 0 \\ v_z = \gamma_t \\ T_t = \lambda T_{zz} + \mu^{p-1} \cdot T \cdot \gamma_t \\ \gamma_t = \Phi(\tau, \gamma, T) \end{array} \tag{1.2}$$

from which we observe that the stress is only time-dependent, $\tau = \tau(t)$. When the stress-strain law is in the plastic regime (Marchand-Duffy, [6]), $\tau = \tau(t) = \tau_0$ is approximately constant. If the (plastic) strain rate is given by the Arrhenius law:

$$\gamma_t = v_z = \mu \exp\left(\frac{-\Delta H(\tau)}{KT}\right) \tag{1.3}$$

where ΔH is the activation enthalpy and K is Boltzman's constant, then, from (1.2),(1.3) the mathematical model for the shearing process reduces to a reaction-diffusion equation which describes the energy balance coupled with a compatibility equation

$$\begin{array}{cc} \multicolumn{2}{c}{T_t - \lambda T_{zz} = \tau \mu^p \exp\left(\frac{-\Delta H}{KT}\right)} \\ \multicolumn{2}{c}{v_z = \mu \exp\left(\frac{-\Delta H}{KT}\right)} \\ T(0,t) = T(d,t) = T_0 & v(0,t) = 0 \\ T(z,0) = T_0 & v(d,t) = v_0 \end{array} \tag{1.4}$$

By integrating the compatibility equation, (1.4) reduces to the nonlocal problem

$$\begin{gathered} T_z = \lambda T_{zz} + \tau \mu^p \frac{\exp\left(\frac{-\Delta H}{KT}\right)}{\left(\int_0^d \exp\left(\frac{-\Delta H}{KT}\right) dz\right)^p} \\ T(0,t) = T(d,t) = T_0, \\ T(z,0) = T_0. \end{gathered} \tag{1.5}$$

This reduces to the nondimensional model

$$\begin{gathered} \theta_t - \theta_{xx} = \frac{\delta}{\left(\int_{-1}^1 \exp\left(\frac{-\beta}{1+\theta}\right) dx\right)^p} \cdot \exp\left(\frac{-\beta}{1+\theta}\right) \\ \theta(-1,t) = 0 = \theta(1,t) \\ \theta(x,0) = \theta_0(x) \geq 0 \end{gathered} \tag{1.6}$$

where $\beta = \dfrac{\Delta H(\tau_0)}{KT_0}$ and $\delta = \tau_0 \mu^p \geq 0$.

It has been experimentally verified ([6]) that a typical value of β is 40 so $\varepsilon = \beta^{-1} << 1$ and the reciprocal of the strain rate sensitivity can play a role in shear banding similar to the activation energy in combustion theory. Setting $\theta(x,t) = \varepsilon u(x,t) + O(\varepsilon^2)$, then to first-order, (1.6) becomes

$$\begin{aligned} u_t - u_{xx} &= \frac{\delta}{\left(\int_{-1}^{1} e^u dx\right)^p} \cdot e^u \\ u(-1,t) &= 0 = u(1,t) \\ u(x,0) &= u_0(x) \geq 0 \end{aligned} \tag{1.7}$$

where $\delta > 0$ and $p \geq 0$. Thus, the problem of shear band localization can be modelled by a nonlocal parabolic problem.

In the experimental study of shear band formation by Marchand and Duffy [6], a thin-walled tubular specimen of steel was loaded at a strain rate large enough to produce shear banding. During the shear band formation, temperature measurements were made and photographs taken of the specimens to provide strain measurements at several locations along the tube at different times. A narrow shear band is seen to form. This narrow band of high strain which often precedes failure in materials was seen to form near the axial midpoint on the surface of the tube as the temperature there increased dramatically.

The question of this lecture is does the nonlocal model (1.7) predict these experimental observations? Mathematically this would be answered affirmatively if solution to (1.7) blows up in finite time and if information can be given about the blow-up set.

2 Nonlocal problems

We are led to consider nonlocal problems of the form

$$\begin{aligned} u_t - \Delta u &= \frac{\delta f(u)}{\left(\int_\Omega f(u) dx\right)^p} \quad , \qquad x \in \Omega,\ t > 0 \\ u(x,0) &= u_0(x) \geq 0 \quad , \qquad x \in \Omega \\ u(x,t) &= 0 \quad , \qquad x \in \partial\Omega,\ t > 0 \end{aligned} \tag{2.1}$$

and the associated steady-state problem

$$\begin{aligned} -\Delta u &= \frac{\delta f(u)}{\left(\int_\Omega f(u) dx\right)^p} \quad , \quad x \in \Omega \\ u(x) &= 0 \quad , \qquad x \in \partial\Omega \end{aligned} \tag{2.2}$$

where $p \geq 0$, $\delta > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, f is positive and Lipschitz continuous and $u_0(x) \geq 0$ is in $L^2(\Omega)$ with $u_0(x) = 0$ on $\partial\Omega$. The following standard results for classical nonlinear partial differential equations carry over to the nonlocal problems (2.1) and (2.2):

1) Any solution of IBVP(2.1) or BVP(2.2) is positive for $x \in \Omega$ with outer normal derivative $\frac{\partial u}{\partial N} \leq 0,\ x \in \partial\Omega$.
2) For $u_0 \in L^2(\Omega)$, $\sup u_0(x) < \infty$, IBVP(2.1) has a unique, nonextendable classical solution $u(x,t)$ on $\Omega \times [0,T)$ where either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \sup_{x\in\Omega} u(x,t) = +\infty$.
3) For $\Omega = B_1(0)$, (a) any solution of BVP(2.2) is radially symmetric and radially decreasing; (b) if $u_0(x)$ is radially symmetric and radially decreasing, then the solution of $u(x,t)$ of IBVP(2.1) is also for each $t \in [0,T)$.

3 Existence-nonexistence for BVP(2.2).

Let $f(u) = e^u$. The following theorems are proven in [2]. Nonexistence for BVP(2.2) should give information about nonexistence of global solutions for IBVP(2.1).

Theorem 1 *For $\Omega = B_1(0) \subset \mathbb{R}^1$:*

a) if $p \geq 1$, BVP (2.2) has a unique solution for all $\delta > 0$;
b) if $0 \leq p < 1$, then there exists $\delta^* > 0$ such that BVP (2.2) has: i) two solutions for $\delta < \delta^*$; ii) one solution for $\delta = \delta^*$; and iii) no solution for $\delta > \delta^*$.

Theorem 2 *For $\Omega = B_1(0) \subset \mathbb{R}^2$,*

a) if $p > 1$, BVP(2.2) has a unique solution for all $\delta > 0$.
b) if $p = 1$, BVP(2.2) has a unique solution for $\delta < \delta^* = 8\pi$ and no solution for $\delta \geq \delta^*$.
c) if $0 \leq p < 1$, then there exists $\delta^* > 0$ such that BVP(2.2) has: i) two solutions for $\delta < \delta^*$, ii) a unique solution for $\delta = \delta^*$, and iii) no solution for $\delta > \delta^*$.

By using Pohozaev's identity, nonexistence results can be extended to strictly star-shaped domains. A domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is *strictly star-shaped* (containing 0) if there exists $a > 0$ such that

$$x \cdot N \geq a \int_{\partial\Omega} ds, \quad \text{for } x \in \partial\Omega,\ N \text{ unit outer normal.}$$

Theorem 3 *For $p \leq 1$, spatial dimension $n \geq 2$, BVP(2.2) has no solution for $\delta > \dfrac{2n}{a}|\Omega|^{p-1}$.*

4 Finite time blowup

Consider IBVP(2.1) and the associated steady-state problem BVP(2.2) when $f(u) = e^u$, $p < 1$, $n = 1$ or 2, and $\Omega \subset \mathbb{R}^n$ such that Theorem 1, 2, or 3 is valid. Then there exists a critical $\delta^* > 0$ such that for $\delta > \delta^*$, no solution of BVP(2.2) exists.

For spatial dimensions $n = 1$ or 2, IBVP(2.1) defines a local semiflow in $H_0^1(\Omega)$. For $0 < p < 1$, this local semiflow has a Lyapanov functional given by

$$V[u](t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p} \tag{4.1}$$

and the semiflow is gradient-like in the sense that for any $t \in [0, T)$

$$\int_0^t ||u_t||_2^2 + V[u](t) = V[u_0]. \tag{4.2}$$

Theorem 4 *For $\delta > \delta^*$, the solution $u(t, u_0)$ of IBVP(2.1) blows up in finite time $T < \infty$.*

The proof is given in [2] and is based on ideas of Marek [7]. An outline of the proof can be given by stating the following five lemmas.

Lemma 1 *If u is a global solution of IBVP(2.1), then there exists $\kappa = \kappa(u_0)$ such that*

$$||u(t, u_0)||_2 = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \leq \kappa \quad \textit{for all } t \geq 0.$$

Lemma 2 *If* $||u(t, u_0)|| \equiv \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \to \infty$ *as* $t \to t_m$, *then* $t_m < \infty$.

Lemma 3 *If $u(t, u_0)$ is a global solution with the ω-limit set $\omega(u_0) \neq \varphi$, and if $w \in \omega(u_0)$ is any equilibrium solution, then $||w|| \leq K(u_0)$.*

Lemma 4 *If $u(t, u_0)$ is a global solution with*

$$\liminf_{t\to\infty} ||u(t, u_0)|| < \infty, \quad \limsup_{t\to\infty} ||u(t, u_0)|| = \infty,$$

then for any B sufficiently large, there exists an equilibrium solution w with $||w|| = B$.

Lemma 5 *If $u(t, u_0)$ is global, then $\sup_{t\geq 0} ||u(t, u_0)|| < \infty$ and $\sup_{t\geq\tau} |u(t, u_0)| < \infty$ for any $\tau > 0$.*

Lemma 5 follows from the previous lemmas. For if $u(t, u_0)$ is global, then, by Lemma 1, $||u(t, u_0)|| \not\to \infty$. By Lemmas 3 and 4, $\limsup_{t\to\infty} ||u|| \neq +\infty$ so $\sup_{t\geq 0} ||u(t, u_0)|| < \infty$. But then Lemma 5 implies solution of BVP(2.2) exists. But, by assumption $\delta > \delta^*$; therefore no such solution exists. Thus $u(t, u_0)$ must blow up in finite time.

5 Single point blowup

Theorem 4 for $\Omega = (-1,1) \subset \mathbb{R}^1$, $\delta > \delta^*$, $p < 1$ where $u(x,t)$ represents the temperature perturbation for the shear banding model (1.7) tells us that thermal runaway or blowup occurs in finite time T. The question of where blowup occurs is answered by the following theorem.

Theorem 5 *For $\Omega = B_1(0)$, initial data $u_0(x)$ radially symmetric and decreasing, $f(u) = e^u$, $n = 1$ or 2, $p < 1$, and $\delta > \delta^*$, then the blowup set for IBVP(2.1) consists of a single point $x = 0$.*

The details of the proof of this theorem appear in [3]. Here we will sketch an outline of the main ingredients of the proof.

The equation (2.1) for radially symmetric and radially decreasing initial data $u_0(x)$ becomes, with $r = |x|$,

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u_r)_r + \delta k(t)e^u \tag{5.1}$$

where $k(t) = \left(\int_{B_1(0)} e^u dx\right)^{-p}$ with $u_r < 0$ on $\{0 < r < 1\} \times [0,T)$.

Assume there exists $\bar{x} \neq 0$ in $B_1(0)$ such that

$$\lim_{t \to T^-} u(\bar{x},t) = +\infty,$$

then $k(t) \to 0$ as $t \to T^-$ and $k'(t) \leq 0$ for t sufficiently near T.

On $[0,1] \times [t^*,T)$ for t^* sufficiently near T,

$$J(r,t) \equiv r^{n-1}u_r(r,t) + \eta c(r)F(u,t) \tag{5.2}$$

for $\eta \in (0,1)$ satisfies

$$J_t + \frac{n-1}{r}J_r - J_{rr} - AJ = D. \tag{5.3}$$

For $F(u,t) = e^{\beta u} \cdot k(t)$, any $\beta < 1$, and $c(r) = r^n$, $D \leq 0$ so Maximum Principle arguments can be used to show $J(r,t) \leq 0$ and hence

$$u_r \leq -\eta r e^{\beta u} k(t). \tag{5.4}$$

Let φ be the normalized positive eigenfunction for the first eigenvalue of

$$\begin{aligned} -\Delta\varphi = \lambda\varphi\ , \quad & x \in B_r(0)\ , \quad 0 < r \leq 1, \\ \varphi = 0\ , \quad & x \in \partial B_r(0)\ , \end{aligned}$$

with $\int_{B_r} \varphi dx = 1$.

Using Jensen's inequality, one can prove

Lemma 6 *For each $r \in (0,1]$, there exists $C(r) > 0$ such that the solution $u(x,t)$ of IBVP (2.1) satisfies:*

$$\int_{B_r} u\varphi dx \le \frac{1}{\alpha} \ln \frac{1}{T-t} + C(r) \tag{5.5}$$

where $\alpha = 1 - p$.

Let $0 < r_1 < |\bar{x}| < r_2 < 1$, $A = \{r_1 < |x| < r_2\}$, and $B = B_b(\bar{x})$ where $b = 1/2 \min[r_2 - |\bar{x}|, |\bar{x}| - 1]$, then $B \subset A \subset B_1(0)$. Let ψ be the normalized first eigenfunction of

$$\begin{aligned} -\Delta\psi = \lambda\psi \ , \quad & x \in B \\ \psi = 0 \ , \quad & x \in \partial B \end{aligned}$$

with $\int_B \psi dx = 1$. Then the following dichotomy holds: Either

a) there exists $\varepsilon_0 \in (0,1)$ such that

$$\frac{\int_B e^u \psi dx}{\left(\int_\Omega e^u dx\right)^p} \le C(\bar{x}) \left(\frac{1}{T-t}\right)^{1-\varepsilon_0} \tag{5.6}$$

for all t sufficiently near T, or

b) for any $\varepsilon \in (0,1)$, there exists a sequence $\{t_n\}$, $t_n \to T^-$, such that

$$\frac{\int_B e^{u(x,t_n)} \psi(x) dx}{\left(\int_\Omega e^{u(x,t_n)} dx\right)^p} \ge \left(\frac{1}{T-t_n}\right)^{1-\varepsilon}. \tag{5.7}$$

If a) holds, multiply (2.1) by ψ and integrate over B. Using Green's Second Identity, (6.6), and Lemma 6, we arrive at

$$\frac{d}{d_t}\left(\int_B u\psi dx\right) \le C_1 \ln\left(\frac{1}{T-t}\right) + C_2 \left(\frac{1}{T-t}\right)^{1-\varepsilon_0} \tag{5.8}$$

which implies $\int_B uxdx \le C$ for all $t \in [0,T]$. But $\bar{x} \in B$ is assumed to be a blowup point and $u(r,t)$ is radially decreasing. This is a contradiction. We conclude b) must hold. From (5.7), we have

$$\left(\int_\Omega e^u dx\right)^{2p} \le C \left(\frac{1}{T-t_n}\right)^{\frac{2p}{\alpha}\left(1+\frac{\alpha\varepsilon}{p}\right)} \tag{5.9}$$

and, for any $\beta \in (0,1)$,

$$\left(\int_B e^u \psi dx\right)^{2\beta} \ge \frac{1}{C} \left(\frac{1}{T-t_n}\right)^{\left(\frac{1-\varepsilon}{2}\right)2\beta}. \tag{5.10}$$

Using the Lyapunov functional (4.1) and (4.2) we have

$$V[u_0] \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\delta}{\alpha}\left(\int_\Omega e^u dx\right)^\alpha . \tag{5.11}$$

Also,

$$J(r,t) = r^{n-1}u_r + \eta r^n e^{\beta u} k(t) \leq 0 \tag{5.12}$$

for $\eta > 0$ sufficiently small and $\beta \in (0,1)$.

From (5.11) and (5.12), we have

$$V[u_0] \geq \frac{1}{2}\eta^2 k^2(t)\int_\Omega r^2 e^{2\beta u} dx - \frac{\delta}{\alpha}\left(\int_\Omega e^u dx\right)^\alpha . \tag{5.13}$$

Therefore, for any $\varepsilon > 0$ there exists $\{t_n\}, t_n \to T^-$ such that

$$V[u_0] \geq C\left(\frac{1}{T-t_n}\right)^{\frac{1-\varepsilon}{\alpha}2\beta - \frac{2p}{\alpha} - 2\varepsilon} - D\left(\frac{1}{T-t_n}\right)^{1+\frac{\varepsilon\alpha}{p}} \tag{5.14}$$

with $\beta \in (0,1)$. for $\varepsilon > 0$ sufficiently small and $\beta < 1$ sufficiently near 1, since $\frac{2}{\alpha} - \frac{2p}{\alpha} < 1$, we see that the right hand side of (5.13) tends to $+\infty$ as $t_n \to T$. This is a contradiction and we must conclude that the only blowup point is the origin.

For the shear banding model (1.7), Theorem 5 tells us that the temperature perturbation $u(x,t)$ becomes unbounded only at the origin.

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