THE KING OF THE TALKING FROGS AND POLYNOMIAL AUTOMORPHISMS

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In honour of Gary Meisters on the occassion of his retirement

1 Preface

Maybe you don't understand the first half of the title, so let me first explain this to you.
In April 1992 my wife Sandra, our daughter Raissa (then five years old) and I visited Gary and Mary Ellen in Lincoln. During the very pleasant stay at their house Raissa found out that Gary talked very much. On the airport of Lincoln, waiting for our flight to Holland Raissa suddenly said to Gary: "you are the King of the talking frogs!" As you can imagine Gary was very pleased with this name. And to show her his appreciation he has sent her later several frogs, varying from very small to very large!
The name **King** of the **talking** frogs is, in my opinion, a very good characterization of Gary. In an always **humoristic** style he discusses mathematics and other things of life with everyone who wants to listen and more importantly he spreads his ideas

around.
In this way he makes many people aware what is going on in our field. His well-known very long list of "polynomial mapping papers" is a beautiful example of his "missionary" work. Also his recent idea to award prizes for interesting questions has shown to be effective (the final solution of the Markus-Yamabe Conjecture has its origin in Gary's $100 prize which he announced at the Curaçao Conference in July 1994: I will come back to this point in the next sections).
Finally I like to mention another important characteristic of Gary, namely that he always comes up with many questions and is not afraid to make conjectures!
Let me conclude this short preface by giving my interpretation of his initials G.H.M. namely. He is a

Great Humoristic Mathematician

2 A short survey of Gary's work on polynomial automorphisms

The first paper of Gary I could track back concerning polynomial automorphisms is his 1982 paper in [30], in which he discusses both Jacobian problems: the Jacobian Problem from algebraic Geometry and the Jacobian Conjecture from differential equations, also known as the Markus-Yamabe Conjecture.
For the sake of completeness I will recall briefly both conjectures (see [2, 9, 12, 27, 29]).

Conjecture 1 (Jacobian Conjecture, 1939) *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial map with* $\det JF \in \mathbb{C}^*$, *then* F *is invertible.*

Conjecture 2 (Markus-Yamabe Conjecture, 1960) *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1-*vector field satisfying the Markus-Yamabe Assumption (MYA) i.e.*

(MYA) For all $x \in \mathbb{R}^n$ *the real parts of all eigenvalues of* $JF(x)$ *are negative*

then 0 *is a global attractor of the system*

$$\dot{x}(t) = F(x(t))$$

i.e. every solution of this system converges to 0 *if* t *tends to infinity.*

Gary's 1982-paper mentioned above is "classical" by now: it is very clearly written and describes the state of the art around 1982 concerning both conjectures. The paper is full of questions and nice examples. It should be read by everyone interested in polynomial automorphisms and related topics, in particular various connections with differential equations are given. In this paper he already introduces the notion of **Polynomial flows**, or as Gary likes it better **Polyflows**.

2.1 Polynomial flows

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field and consider the system of ordinary differential equations

$$\dot{x}(t) = F(x(t))$$

The unique solution which at $t = 0$ has a given value $x_0 \in \mathbb{R}^n$ is denoted by $x(t, x_0)$. The system above is called a **Polynomial flow system** and F a **Polynomial flow vector field** if and only if the map

$$x_0 \to x(t, x_0)$$

is a polynomial map. In other words, the solution depends polynomially on the initial condition. In fact the map above is a polynomial automorphism for every t where the solution is defined.

In [3] Bass and Meisters study such systems and amongst other things they show that a polynomial flow vector field F is automatically polynomial (this seems obvious, but it is not!), the solutions are complete i.e. are defined for all $t \in \mathbb{R}$ and that there exist an integer d and real analytic functions $a_\alpha : \mathbb{R} \to \mathbb{R}^n$ such that

$$x(t, x_0) = \sum_{|\alpha| \leq d} a_\alpha(t) {x_0}^\alpha$$

Using these results one can prove a very nice characterization of polynomial vector fields, due to Coomes and Zurkowski [6], namely that F is a polynomial flow vectorfield if and only if the associated derivation $D := \sum F_i \partial_i$ is **locally finite** on $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ (i.e. for every element $g \in \mathbb{R}[X]$ there exists an integer m such that

$$\deg D^q(g) \leq m$$

for all $q \in \mathbb{N}$. Furthermore in [3] Bass and Meisters give a complete classification of all polynomial vector fields in dimension two (see also the papers [10] and [40] for alternative proofs).

Polynomial flow vector fields in dimension ≥ 3 remain still to be understood.

2.2 Polynomial flows and the Jacobian Conjecture

In [35] Meisters and Olech related polynomial flows to the Jacobian Conjecture, namely let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $\det JF = 1$. For each $v \in \mathbb{R}^n$ and each $x_0 \in \mathbb{R}^n$, consider the system

$$\dot{x}(t) = JF(x)^{-1} v, \; x(0) = x_0$$

Then Meisters and Olech proved

Proposition 1 *The Jacobian Conjecture is equivalent to the following statement: For each $v \in \mathbb{R}^n$ the solution $x(t, x_0, v)$ of the system above depends polynomially on both x_0 and t.*

This result was also obtained by Adjamagbo in 1986, but remained unpublished; in fact it was this result which Adjamagbo explained to me in June 1986 when I heard about the Jacobian Conjecture for the first time.

In [10] I used this polyflow result to give an inversion formula for polynomial automorphisms. Later I realised that this result can be easily extended to arbitray $\mathbb{Q}$-algebras and we get the following result (the proof is left to the reader: just use the formal inverse function theorem and Taylor expansion)

Theorem 1 (Inversion Formula) *Let R be any commutative $\mathbb{Q}$-algebra and $F \in Aut_R R[X_1, \ldots, X_n]$ with inverse $G = (G_1, \ldots, G_n)$, then*

$$G_{i(d)} = \frac{1}{d!} D^d(X_i)_{|X=0}$$

for all $d \geq 1$, all i.

Here $_{|X=0}$ means substitute $X_1 = 0, \ldots, X_n = 0$, $G_{i(d)}$ is the homogeneous component of degree d of G_i and finally D is the derivation

$$D = \sum Y_i \frac{\partial}{\partial F_i}$$

on the polynomial ring $R[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.

2.3 The solution of the 2-dimensional Markus-Yamabe Conjecture for polynomial vector fields

In 1987 there was the first break-through concerning the Markus-Yamabe conjecture: Meisters and Olech established the 2-dimensional Markus-Yamabe Conjecture for polynomial vector field in the plane (see [34]). (A maybe less known result is that in [33] Meisters and Olech showed that for polynomial flow vector fields in any dimension the Markus-Yamabe Conjecture is true!) The proof given in [34] is based on two results

1. A result of Olech ([37]) stating that to show that the Markus-Yamabe Conjecture in the plane is true, it suffices to show that the Markus-Yamabe assumption implies that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is injective.
2. If det $JF(x)$ is non-zero for all $x \in \mathbb{R}^2$, then the number of elements in the fibre $F^{-1}(x)$ is bounded by a constant N which does not depend on x.

(This result was later generalised by the author in [11] to arbitrary dimension and one year later improved in [1]).

In 1993 the general C^1-case of the two dimensional Markus-Yamabe Conjecture was proved independently by Fessler and Gutierrez (in [19] and [24] respectively). In 1994 another proof was given by Glutsuk in [20].

2.4 Strong nilpotence

In 1991 Meisters and Olech invented a new notion: **strong nilpotence**. In their paper [36] they studied invertible quadratic homogeneous polynomial maps $X+Q : \mathbb{R}^n \to \mathbb{R}^n$ i.e. $Q = (Q_1, \ldots, Q_n)$ and each Q_i is a homogeneous polynomial map of degree 2. The Jacobian matrix JQ is called **strongly nilpotent** if for every n-tuple of vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ we have

$$JQ(v_1) \cdots JQ(v_n) = 0.$$

They introduced this notion because they observed that the expressions describing formulas for the inverse of the map $X+Q$ could be simplified if strong nilpotence holds. In the paper [36] it is shown that if $n \leq 4$ nilpotence of JQ (which is equivalent to $X+Q$ is invertible) is indeed equivalent to strong nilpotence. However if $n \geq 5$ counterexamples to this equivalence are given.
In [15] Hubbers and the author generalise the notion of strong nilpotence to arbitrary polynomial maps $H : k^n \to k^n$, where k is any field of characteristic zero. If k is an infinite field the definition agrees with the one given above. The main result of [15] is

Theorem 2 *Let $H : k^n \to k^n$ be a polynomial map. Then JH is strongly nilpotent if and only if there exists $T \in Gl_n(k)$ such that $T^{-1} \cdot JH \cdot T$ is an uppertriangular matrix with zeros on the main diagonal.*

2.5 Power Similarity

Another notion invented by Meisters is **power similarity**. He invented it in order to study the so-called Drużkowski-forms, or cubic-linear mappings.
Consider two matrices $A, B \in M_n(\mathbb{C})$ and their respective Drużkowski forms

$$F_A(X) := X + (AX)^3 \text{ and } F_B(X) := X + (BX)^3$$

Then A and B are called **power-3-similar** (or for short **power similar**) if there exists $T \in Gl_n(k)$ such that

$$F_B = T^{-1} \cdot F_A \cdot T$$

or equivalently

$$(BX)^3 = T^{-1}(ATX)^3$$

i.e.

$$T(BX)^3 = (ATX)^3$$

In the paper [31] a complete set of representatives for power similarity in dimension 3 is given as well as a list of 6 representatives fot the case $n = 4$. It was later shown by Hubbers in [25] that this list is complete! The case $n = 5$ was started by Meisters and completed by Hubbers in [26]. We refer to Hubbers' paper in this proceedings.

3 The DMZ-Conjecture and the solution of the Markus-Yamabe Conjecture

In this section I will describe how a conjecture due to Deng, Meisters and Zampieri has led to the final solution of the Markus-Yamabe Conjecture.
The story started in 1992 when David Wright proved the following result (see [38]).

Proposition 2 *Let $F = (X_1+H_1, X_2+H_2, X_3+H_3)$ where each H_i is homogeneous of degree three or $H_i = 0$. If det $JF = 1$, then there exists $T \in Gl_n(\mathbb{C})$ such that*

$$T^{-1} \cdot F \cdot T = (X_1 + a(X_2, X_3), X_2 + b(X_3), X_3)$$

In particular F is invertible.

In februari 1994 Engelbert Hubbers in his Masters thesis ([25]) completely classified the cubic homogeneous case in dimension four. His result is:

Theorem 3 (Hubbers, 1994) *Let $F = X - H$ be a cubic homogeneous polynomial map in dimension four, such that* $\det(JF) = 1$*. Then there exists some $T \in GL_4(\mathbb{C})$ with $T^{-1} \circ F \circ T$ being one of the following forms:*

1. $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \quad -a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad -h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 \\ x_4 \quad -x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix}$$

3. $$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ \quad +\frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 \\ x_4 \quad -x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$

4. $$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 \\ x_3 \quad -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 \quad -e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix}$$

5. $$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 \quad -x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ \quad -s_3x_3x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

$$6.\ \begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ & -p_3x_2x_4^2 - t_3x_4^3 \\ x_4 & \end{pmatrix}$$

$$7.\ \begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 & -x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ & -n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

$$8.\ \begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ x_4 & -x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ & -\frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}$$

If we look at this result we make an astonishing discovery:
at least one of the H_i is zero! Equivalently, the hypothesis $\det JF = 1$, which is equivalent to JH is nilpotent, implies that $H_1, \ldots, H_4$ are **linearly dependent over** k**!** This leads to

Problem 1 (Dependence Problem, DP) Let k be a field of characteristic zero. Let $H = (H_1, \ldots, H_n)$, with each H_i homogeneous of degree 3 or $H_i = 0$. Does the hypothesis JH is nilpotent imply that the H_i are linearly dependent over k?

This problem is still **open** for all $n \geq 5$. In my opinion it is the most important open problem related to the Jacobian Conjecture.
The assumption that an even stronger version of the dependence problem would be true, has led to the discovery of a large class of polynomial automorphisms over any commutative ring A. (see [16, 17] for more details).
Without going into any detail let us say the following: let A be a commutative ring and $n \in \mathbb{N}$. We define a subset $H_n(A)$ of $A[X]^n$ with the property that if $H \in H_n(A)$ then JH is nilpotent. Furthermore the corresponding map $F := X + H$ is a polynomial automorphism over A, in fact F is stably tame! The elements of $H_2(A)$ are of the following form:
Let $H = (H_1, H_2) \in A[X_1, X_2]^2$. Then $H \in H_2(A)$ if and only if

$$\begin{aligned} H_1 &= a_2 f(a_1X_1 + a_2X_2) + c_1 \\ H_2 &= -a_1 f(a_1X_1 + a_2X_2) + c_2 \end{aligned}$$

for some a_i, c_i in A and some $f(T) \in A[T]$ with $f(0) = 0$.

Then in March 1994 Deng, Meisters and Zampieri proposed a new attack to the Jacobian Conjecture, inspired by an old result of Poincare and Siegel.
They conjecture the following (see [8]).

Conjecture 3 (DMZ-Conjecture) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map of the form $F = X + H$, where each H_i is homogeneous of some degree $d \geq 2$ and det $JF = 1$. Then for all $s \in \mathbb{C}$, $|s|$ sufficiently large, sF is* ***global analytic linearisable*** *i.e. there exists an analytic isomorphism $h_s : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$h_s{}^{-1} \circ sF \circ h_s = sX.$$

Because of the result of Poincare and Siegel mentioned above one knows that h_s exists locally in a neighborhood of 0 and that h_s is unique if one assumes that $h_s(0) = 0$ and $Jh_s(0) = I_n$. Furthermore, if DMZ is true then so is the Jacobian Conjecture: namely by a classical result of Bass, Connell, Wright and Yagzhev (see [2] resp. [39]) it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps. Now if $h_s{}^{-1} \circ sF \circ h_s = sX$ for some non-zero $s \in \mathbb{C}$ it follows that sF and hence F is injective, which by another classical result implies that F is invertible.

In [8] the authors proved that $h_s{}^{-1}$ is entire, but could not prove it for h_s. Meisters was very sceptical about the new approach and decided to do some computer experiments about the structure of h_s in case $F = X+H$ with H cubic homogeneous. **To his own surprise** he found that in all the cases he computed h_s was **much nicer** as expected: all h_s were **polynomial automorphisms!** His sceptism changed into optimism and he formulated

Conjecture 4 (Meisters' Linearisation Conjecture, MLC) *Let $F = X + H$ be cubic homogeneous with JH nilpotent (or equivalently det $JF = 1$). Then for every $s \in \mathbb{C}^*$ (except a finite number of roots of unity) there exists an invertible* ***polynomial*** *map $h_s : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$h_s{}^{-1} \circ sF \circ h_s = sX.$$

Meisters formulated this conjecture at the Curaçao Conference in July 1994, (see [12]), where he offered a \$100 reward for the first person to find a counterexample to his conjecture.
Some weeks later Hubbers and I found the following partial confirmation of MLC (see [15]).

Proposition 3 *Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with JH* ***strongly nilpotent****. Then $F = X + H$ satisfies MLC, and hence F is invertible.*

Combining this with Wrights result mentioned above we get

Corollary 1 *MLC is true if $n \leq 3$.*

Inspired by these results my aim was to prove MLC for all F of the form $F = X + H$ with $H \in H_n(\mathbb{C})$. However in september 1994 I found a counterexample in $H_n(\mathbb{C})$ for all $n \geq 4$ (see [14]), namely

Theorem 4 *Let $n \geq 4$. Put $d := X_3X_1 + X_4X_2$, then*

$$F = (X_1 + X_4d, X_2 - X_3d, X_3 + X_4{}^3, X_4, \ldots, X_n)$$

is a counterexample to MLC.

Now immediately the question was raised: is this F also a counterexample to the DMZ-Conjecture?
In July 1995 Gorni and Zampieri showed in [22] that the answer is **no!** They showed that for all complex s with $|s| \neq 1$ the map $h_s : \mathbb{C}^n \to \mathbb{C}^n$ is global analytic linearisable. So DMZ and hence the approach to the Jacobian Conjecture remained open.
In August 1995 I also received a preprint ([7]) of Bo Deng in which he showed that F is not a counterexample to the DMZ-Conjecture. His proof was based on

Lemma 1 *Let$F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial automorphism with $F(0) = 0$ of the form $F = X + H$ with JH nilpotent. Let $0 < |s| < 1$. Then sF is global analytic linearisable if and only if 0 is a global attractor of sF (i.e. for every $x \in \mathbb{C}^n$ we have that $(sF)^m(x)$ tends to 0 if m tends to infinity).*

When I saw this lemma I realised that we now had a way to investigate the DMZ-Conjecture: so I said to Engelbert, just take some complicated F of the form $F = X + H$ with $H \in H_5(\mathbb{C})$, take some $0 < s < 1$ and check if 0 is an attractor of sF by iterating sF in some arbitrary points, and then start proving!
I was convinced that for all $F = X + H$ with $H \in H_n(\mathbb{C})$ 0 would be a global attractor of sF if $|s| < 1$. In fact in 1976 LaSalle had made the following stronger conjecture, see ([28]), which was reinvented (independently) in 1994 by Cima, Gasull and Mañosas. They called it the Discrete Markus-Yamabe Problem (see [5])

Conjecture 5 (LaSalle Conjecture (Discrete Markus-Yamabe Problem)) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $F(0) = 0$ and such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ are smaller than 1 in absolute value. Then 0 is a global attractor of F.*

So Hubbers started to look at some complicated examples in dimension 5. A few days later he came to me with the following example

Example 1 Let $s = (1/100)$ and $F = (X_1 + 3X_1X_2{}^2 + X_4{}^4 + X_1{}^3 - 3X_4{}^2X_1X_2{}^2 - 3X_4{}^2X_1X_2{}^2 - 2X_1{}^3X_4{}^3 - 3X_4{}^3X_1{}^2X_2 - X_4X_2{}^3 - X_4{}^4X_1{}^3 + 3X_1{}^2X_4X_2 + 2X_4X_1{}^3 - X_4{}^3 + X_2{}^3 + 3X_1{}^2X_2,\ -3X_4{}^2X_1X_2{}^2 - 6X_4{}^3X_1{}^2X_2 - 3X_4{}^4X_1{}^2X_2 - 3X_4{}^3X_1X_2{}^2 - X_2 + 3X_1X_4X_2{}^2 - X_4{}^3 + 3X_1X_2{}^2 - X_4{}^5X_1{}^3 - X_4{}^2X_2{}^3 + X_2{}^3 + 6X_1{}^2X_2X_4 - 3X_4{}^4X_1{}^3 + X_1{}^3 - 2X_1{}^3X_4{}^3 + 3X_1{}^2X_2 + 2X_4{}^2X_1{}^3 + 3X_1{}^3X_4 + X_4{}^5,\ 3X_4{}^2X_1X_2{}^2 + 3X_4{}^2X_1{}^2X_2 + 3X_4{}^3X_1{}^2X_2 + X_3 - 2X_1X_2{}^2 + X_4X_2{}^3 - X_4{}^4 - X_2{}^3 - X_1{}^2X_4X_2 + X_4{}^4X_1{}^3 + 2X_1{}^3X_4{}^3 - X_1{}^2X_2 + X_4{}^2X_1{}^3,\ -X_4,\ X_5)$.

If $v := (0,0,0,3.6314,0)$, then computer calculations indicate that

$$\lim_{m\to\infty}(sF)^m(v) = 0.$$

Furthermore if $w := (0,0,0,3.6315,0)$, then computations indicate that

$$\lim_{m\to\infty}(sF)^m(w) = \infty.$$

The next day I went to Poland for two weeks. On the airport I started to think about this phenomenon. Hubbers' example was far to complicated to prove anything, so I had to find an easier example.
Then I remembered that both in the Gorni-Zampieri paper and the Deng-paper they **essentially used** that both X_1 and X_2 appeared **linearly** in the F giving the counterexample to MLC. So I had to find a "better" automorphism F in which both X_1 and X_2 do not appear linearly. Having the whole class $H_4(\mathbb{C})$ at my disposal I simply took the simplest possibility

$$F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + {X_4}^3, X_4)$$

which is of the form $X + H$ with $H \in H_4(\mathbb{C})$. So F is an automorphism and since JH is nilpotent all eigenvalues of $J(sF) = sI + s(JH)$ are equal to s. So if $0 < s < 1$ and if the LaSalle Conjecture is true then 0 should be a global attractor! However during my visit at Torun I showed that for all $a \in \mathbb{R}$ and all $0 < s < 1$ such that $as > 1$ we have that $(sF)^m(a, a, \ldots, a)$ tends to infinity if m tends to infinity! So sF gave a counterexample to the LaSalle Conjecture. By also considering real numbers s with $s > 1$ and considering $(sF)^{-1}$ we finally got (see [18])

Theorem 5 (van den Essen, Hubbers) *Let* $n \geq 4$, $m \geq 1$. *Put* $d := X_3X_1 + X_4X_2$ *and*

$$F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + {X_4}^m, X_4, \ldots, X_n)$$

Then

1. *For all* $0 < s < 1$ sF *gives a counterexample to the LaSalle Conjecture.*
2. $F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + X_4^5, X_4, \ldots, X_n)$ *gives a counterexample to the DMZ-Conjecture.*

After having found the above counterexamples to the Dicrete Markus-Yamabe Problem, it was natural to ask: does the following system give a counterexample to the Markus-Yamabe Conjecture?

$$\begin{aligned}
\dot{x}_1(t) &= -x_1 + x_4 d(x)^2 \\
\dot{x}_2(t) &= -x_2 - x_3 d(x)^2 \\
\dot{x}_3(t) &= -x_3 + {x_4}^m \\
\dot{x}_4(t) &= -x_4
\end{aligned}$$

More precisely, does there exist solutions which tend to infinity if t tends to infinity? (remember that according MYC all solutions should tend to 0!)

So we conjectured that such a solution should exist!
Again we asked the computer for help, however this time the computer could not help us at all.
In the second week of November 1995 Anna Cima visited Nijmegen. I described her our 4-dimensional candidate counterexamples. About one week after she left I received an email confirming that there do exist solutions which tend to infinity! One week later Cima, Gasull and Mañosas where able to modify the 4-dimensional counterexample into a 3-dimensional counterample. So we finally obtained the following result (see [4])

Theorem 6 (Cima, van den Essen, Gasull, Hubbers, Mañosas) *Let* $n \geq 3$, $d(X) := X_3 X_1 + X_2$ *and* $F : \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$F(x_1, \ldots, x_n) = (-x_1 + d(x)^2, -x_2 - x_3 d(x)^2, -x_3, \ldots, -x_n).$$

Then F *gives a counterexample to MYC. More precisely*

$$x_1(t) = 12e^{2t},\ x_2(t) = -18e^t,\ x_3(t) = \cdots = x_n(t) = e^{-t}$$

is a solution of $\dot{x}(t) = F(x)$ *which tends to infinity if* t *tends to infinity!*

Remark 1 After we sent out our counterexample to several people we received, by email, a preprint of Glutsuk, ([21]), in which he constructs a C^1-counterexample to the Markus-Yamabe Conjecture in dimension three.

4 Meisters' Cubic Linear Linearisation Conjecture and new counterexamples to the Markus-Yamabe Conjecture

In the previous section we saw how the MYC was completely solved. However the story of the DMZ-Conjecture was not finished yet. There would still be the possibility that the DMZ-Conjecture is true for all Drużkowski forms, leaving open a proof for the Jacobian Conjecture. This lead Meisters to the following conjecture (see [32]).

Conjecture 6 (Meisters' Cubic Linear Linearisation Conjecture, CLLC) *If* F *is of the form* $(X_1 + {L_1}^3, \ldots, X_n + {L_n}^3)$ *with* $\det JF = 1$ *and each* L_i *is a linear form, then for all* $|s| \neq 1$ *there exists* $h_s : \mathbb{C}^n \to \mathbb{C}^n$, *a global analytic automorphism such that*

$${h_s}^{-1} \circ sF \circ h_s = sX$$

This time Gary offered a \$200 reward for a counterexample!

The problem here is that it is hard to find cubic linear forms which are invertible: in fact in the literature there was only one interesting example in dimension 16, due to

Drużkowski (see [9]). So Hubbers and I tried to prove that it did not satisfy DMZ. Simultaneously Gorni and Zampieri were looking at the same example and tried to prove that it does satify DMZ. Therefore they developped a very elegant theory of **pairing between Cubic Linear forms and Cubic Homogeneous forms** i.e. to every cubic linear map F they associated a cubic homogeneous map f (in less variables) and to every cubic homogeneous map f a cubic linear map F (in more variables) in such a way that one of them satisfies DMZ if and only if the other one does! (see [23])
They calculated the cubic homogeneous map f associated to the 16-dimensional cubic linear example F mentioned above and found that f was even polynomially linearisable, hence the same holds for F. So this F is not a counterexample to DMZ, so the approach to prove the Jacobian Conjecture via CLLC remained open!

However in November 1996 I found a 5-dimensional counterexample to DMZ which was cubic homogeneous. So using the Gorni-Zampieri theory this finally leads to a 17-dimensional counterexample to CLLC, which kills this approach to prove the Jacobian Conjecture! More precisely we get the following result (see [14])

Theorem 7 *Let $n \geq 5$ and $F : \mathbb{C}^n \to \mathbb{C}^n$ be defined by*

$$\begin{aligned} F &= (X_1 + X_2{X_5}^2, X_2 + {X_1}^2 X_5 - X_4{X_5}^2, \\ &\quad X_3 + {X_2}^2 X_5, X_4 + 2X_1X_2X_5 - X_3{X_5}^2, X_5, \ldots, X_n) \end{aligned}$$

Then F is invertible and for every non-zero $s \in \mathbb{C}$ with $|s| \neq 1$ the map sF is not global analytic linearisable.

Corollary 2 *There exists a counterexample in dimension 17 to Meisters' cubic linear linearisation conjecture!*

To conclude this section we will show that the cubic homogeneous map described above can also be used to give both **cubic** as well as **quadratic homogeneous counterexamples to MYC**. Namely let

$$Q = (X_2X_5, {X_1}^2 - X_4X_5, {X_2}^2, 2X_1X_2 - X_3X_5, 0, \ldots, 0)$$

Then we have (see [14])

Theorem 8 *Let $n \geq 5$ and $F = -X + Q$ (resp. $F = -X + X_5 Q$). Then F gives a counterexample to MYC. More precisely*

$$x(t) = (30e^t, 60e^{2t}, 720e^{4t}, 720e^{3t}, e^{-t}, \ldots, e^{-t})$$

(resp.

$$x(t) = (120e^{3t}, 480e^{5t}, 23040e^{9t}, 11520e^{7t}, e^{-t}, \ldots, e^{-t}))$$

is a solution of $\dot{x}(t) = F(x)$ which tends to infinity if t tends to infinity.

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