G_A-ACTIONS OBTAINED BY LOCAL SLICE CONSTRUCTIONS

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Abstract. Let k be a field of characteristic 0, and let B denote the polynomial ring in three variables over k. We describe local slice constructions, a procedure by which new locally nilpotent k-derivations of B can be obtained from given derivations of a certain type. Geometrically, a local slice construction corresponds to an elementary Cremona transformation of affine 3-space. In this way, one obtains the recently discovered rank-3 examples. In fact, every known locally nilpotent k-derivation of B can be from a partial derivative via a sequence of local slice constructions.

1 Introduction

In this article, we summarize the ideas and results found in [3], to which the reader is referred for details.

Let k be a field of characteristic zero, and let $k^{[n]}$ denote the polynomial ring in n variables over k. Recently, in [4], the first examples of locally nilpotent derivations of $k^{[n]}$ having maximal rank n were constructed. In an attempt to understand and generalize these examples, our present aim is to describe *local slice constructions*, a general procedure by which new locally nilpotent derivations can be constructed from given derivations of a certain type. The examples of [4] can be obtained from familiar derivations via local slice constructions, though they were not originally discovered in this way. In fact, we will see in what follows that every known locally nilpotent derivation of $k^{[3]}$ can be obtained from a partial derivative via a (finite) sequence of local slice constructions. Whether any others exist is the content of *Questions 1* below.

The following notation and definitions will be used. The word *derivation* will mean k-derivation. Let R be an integral k-domain, and let $R^{[n]}$ denote the polynomial ring in n variables over R. Let D be a derivation of R, and let $A = Ker(D)$, the kernel of D. D is *irreducible* if its image is contained in no proper principal ideal of R. D is *locally nilpotent* if, to each $f \in R$, there is an $n \geq 0$ such that $D^n f = 0$.

We say $s \in R$ is a *slice* for D if $Ds \in R^*$. (It is well known that, when D is locally nilpotent, A is factorially closed, and thus $R^* \subset A$.) The main fact concerning slices is the following.

Proposition 1 ([7], Proposition 2.1) *Suppose R is an integral k-domain, D is a locally nilpotent derivation of R having kernel A, and $s \in R$ is such that $Ds = 1$. Then $R = A[s] \cong A^{[1]}$, and D is given by $D = d/ds$.*

When D is locally nilpotent (and non-zero), we can find $r \in R$ such that $Dr \in A = Ker(D)$, but $Dr \neq 0$. If $Dr = f$, then D extends to a locally nilpotent derivation D_f on the localization R_f, and r becomes a slice for D_f. Moreover, the above theorem shows $R_f = A_f[r] = A_f^{[1]}$. This gives rise to the following.

Definition 1 Let R be an integral k-domain, and let D be a locally nilpotent derivation of R. An element $r \in R$ is called a *local slice* for D if $Dr \in Ker(D)/\{0\}$.

Of particular interest is the case $R = k^{[n]}$. In this case, for any locally nilpotent derivation D of R, the *rank* of D is defined to be the least integer $r \geq 0$ for which there exists a system of variables $(X_1, ..., X_n)$ of $k^{[n]}$ satisfying $k[X_{r+1}, ..., X_n] \subset Ker(D)$.

2 Local Slice Constructions

From now on, B will denote the polynomial ring $k[X, Y, Z] = k^{[3]}$. Given $f, g \in B$, define a derivation $D_{(f,g)}$ of B by

$$D_{(f,g)}(h) = \frac{\partial(f,g,h)}{\partial(X,Y,Z)} \ .$$

The following theorem of Miyanishi is required (c.f. [5]).

Theorem 1 *If D is any non-zero locally nilpotent derivation of $k^{[3]}$, then $Ker(D) \cong k^{[2]}$.*

Suppose D is a locally nilpotent derivation of B, with kernel $A = k[f, g]$. Then, up to A-multiples, $D = D_{(f,g)}$ (c.f. [2]). Let S denote the set $(k[f] - 0)$, and define

$$\Omega(f,g) = D^{-1}(gS) \ ,$$

a subset (possibly empty) of the set of local slices of D.

Proposition 2 *Given* $r \in \Omega(f,g)$, $\Omega(f,g) = \{ r' \in B \mid S^{-1}A[r'] = S^{-1}A[r] \}$

Suppose $\Omega(f,g)$ is not empty. Let $\bar{B}$ denote the domain B modulo (g), and let $\bar{D}$ denote the induced locally nilpotent derivation on $\bar{B}$. Then for every $r \in \Omega(f,g)$, $\bar{r} \in Ker(\bar{D})$. Since $Ker(\bar{D})$ is the algebraic closure in $\bar{B}$ of $\bar{A} = k[\bar{f}] \cong k^{[1]}$, there exists $\varphi \in k[f][r]$ such that $\varphi(r) \in (g)$. If we choose φ to be of minimal r-degree in $k[f][r]$, and irreducible as an element of $k[f,r] = k^{[2]}$, then φ is unique up to non-zero constant multiples.

Define $h = \varphi(r)/g$, and let $\Delta(f,g,r)$ (or simply Δ) denote the derivation on B defined by $D_{(f,h)}$. (Up to non-zero constant multiples, Δ is uniquely determined by f, g, and r.) The crux of the matter is the following fact.

Theorem 2 *If* $K = k(f,h)$ *and* $B_K = K[X,Y,Z]$, *then* $K[r] = B_K$. *Consequently,* Δ *is locally nilpotent.*

We say Δ is the derivation obtained from D via the *local slice construction* using f, g, and r. Note that, if $Ker\Delta = k[f,h]$, then $r \in \Omega(f,h)$, so we can carry out a further local slice construction. However, since $g = (1/h)\varphi(r)$, this simply results in reversing the process: $\Delta(f,h,r) = D$. To continue the process inductively, we may, by *Proposition 2* above, replace r with any r' for which $S^{-1}A[r'] = S^{-1}A[r]$.

It may also happen that the original derivation, D, admits a local slice r such that $Dr = fg$. Then $\Delta r = -fh$, so both $\Omega(f,h)$ and $\Omega(h,f)$ contain r. Thus, to continue the process inductively, we may also use $\Delta(h,f,r)$ instead of $\Delta(f,h,r)$.

In order to determine the kernel of Δ, the following criterion is quite useful.

Proposition 3 (Kernel Criterion) *Suppose* $a, b \in B = k[X,Y,Z]$ *are such that* $\delta := D_{(a,b)}$ *is locally nilpotent and non-zero. Then the following are equivalent.*

(i) $k[a,b] = Ker(\delta)$

(ii) δ *is irreducible, and* $Ker(\delta) \subset k(a,b)$

Proof. The implication $(i) \Rightarrow (ii)$ follows from [2], *Corollary 2.5.* Conversely, assume (ii) holds. By *Theorem 1* above, there exist $u, v \in B$ such that $Ker(\delta) = k[u,v] \cong k^{[2]}$. It follows that

$$\delta = D_{(a,b)} = \frac{\partial(a,b)}{\partial(u,v)} \cdot D_{(u,v)} \ .$$

Since δ is irreducible, $\frac{\partial(a,b)}{\partial(u,v)} \in k^*$, i.e., (a,b) is a "Jacobian pair" for $k[u,v]$. Since $k(a,b) = k(u,v)$, the inclusion $k[a,b] \hookrightarrow k[u,v]$ is birational. It is well known that the Jacobian Conjecture is true in the birational case, and we thus conclude $k[a,b] = k[u,v]$.[1] ∎

[1] The same reasoning yields yet another equivalent formulation of the two-dimensional Jacobian Conjecture: Given $a, b \in B$, if $D_{(a,b)}$ is irreducible, then $Ker\, D_{(a,b)} = k[a,b]$.

Remark 1 The reader will probably have noticed that, geometrically, passage from $k[f,g,r] \cong k^{[3]}$ to $k[f,h,r] \cong k^{[3]}$ corresponds to a birational transformation of k^3 of a particularly elementary sort. Thus, algebraic passage from D to Δ via a local slice construction may be thought of geometrically as a sequence blow-ups of 3-space, followed by a sequence of blow-downs.

3 Rank Three Examples

The examples discussed in this section are homogeneous in the standard sense (as maps of B). These tend to be easier to work with. It should be noted, however, that local slice constructions can be used to construct rank-3 examples which are weighted-homogeneous, or which are not homogeneous in any (non-trivial) grading of B.

Given any locally nilpotent derivation D of B, D is homogeneous iff there exist homogeneous polynomials $f, g \in B$ such that $Ker(D) = k[f,g]$ [8]. In this case, we say D is homogeneous of *type* (e_1, e_2), where $e_1 = \min\{\deg f, \deg g\}$, and $e_2 = \max\{\deg f, \deg g\}$. As noted in [4], the rank of D is 3 iff $e_1 > 1$. Following the appearance of the $(2,5)$ example in [4], Daigle gave the following beautiful geometric characterization of the homogeneous locally nilpotent derivations of B. ($\mathbf{P}^2$ denotes the projective plane over k.)

Theorem 3 ([1]) *Let f_1 and f_2 be homogeneous elements of $k^{[3]}$, and let C_1 and C_2 be the corresponding projective plane curves which they define. The following are equivalent.*

- *There exists a locally nilpotent derivation D of $k^{[3]}$ such that $Ker(D) = k[f_1, f_2]$.*

- *$\mathbf{P}^2$ minus $(C_1 \cup C_2)$ is isomorphic to $\mathbf{P}^2$ minus 2 lines.*

In other words, f_1 and f_2 define a locally nilpotent derivation precisely when there exists a plane Cremona transformation which is an isomorphism away from C_1 and C_2, and which transforms C_1 and C_2 into a pair of lines.

In order to construct examples, let D be the rank-2 locally nilpotent (linear) derivation on B defined by $DX = 0$, $DY = X$, and $DZ = 2Y$. Then D is homogeneous, and the kernel of D is $k[X,F]$, where $F = XZ - Y^2$. Since $D(FY) = XF$, we see that neither $\Omega(X,F)$ nor $\Omega(F,X)$ is empty.

3.1 Examples of Type (2,4n+1)

Given $n \in \mathbf{Z}^+$, set $r_n = (X^{2n+1} + F^n Y) \in \Omega(F,X)$, and let $\Delta_n = \Delta(F,X,r_n)$. By *Theorem 2*, Δ_n is locally nilpotent. Now $F \equiv (-Y^2)$ and $r_n \equiv (-1)^n Y^{2n+1}$ modulo X, so the minimal polynomial we need is $\varphi(r_n) = F^{2n+1} + r_n^2$. Therefore, $G_n := \varphi(r_n)/X = (ZF^{2n} + 2X^{2n}F^nY + X^{4n+1})$ is homogeneous of degree $(4n+1)$.

As shown in [3], Δ_n is irreducible. *Theorem 2* shows $Ker(\Delta_n) \subset k(F, G_n)$. By the *Kernel Criterion*, it follows that $Ker(\Delta_n) = k[F, G_n]$. Therefore Δ_n is of type $(2, 4n+1)$, and is consequently of rank 3. Note that when $n = 1$, we obtain the $(2,5)$ example first discussed in [4].

The set of points fixed by the $\mathbf{G}_a$-action on $\mathbf{A}^3$ induced by Δ_n is precisely the set of points where $\Delta_n X$, $\Delta_n Y$, and $\Delta_n Z$ vanish

simultaneously, and it is easy to check that this set is the line defined by $X = Y = 0$. Every other orbit is a line, i.e., isomorphic to $\mathbf{G}_a \cong \mathbf{A}^1$. Let $\pi : \mathbf{A}^3 \to \mathbf{A}^2$ be the morphism induced by the inclusion $A \hookrightarrow B$ $(A = k[F, G_n])$. Then the fiber over the point $(a, b) \in \mathbf{A}^2$ is defined by the ideal $(f - a, g - b)$ in B, and each fiber is a union of orbits. The fiber over the origin $(0, 0)$ is the line of fixed points. If neither a nor b is 0, then the fiber over (a, b) is a single (coordinate) line in $\mathbf{A}^3$. The most interesting fibers lie over points $(0, b)$ and $(a, 0)$ for $a \neq 0$ and $b \neq 0$. Over $(0, b)$, $b \neq 0$, the fiber consists of $4n + 1$ (coordinate) lines lying on the surface defined by F. And over $(a, 0)$, $a \neq 0$, the fiber consists of two (coordinate) lines lying on the surface defined by G_n.

3.2 Examples of Fibonacci Type

We again use D as above, and fix $r = X^3 + FY$ $(F = XZ - Y^2)$. Inductively, define functions

$$
\begin{aligned}
H_1 &= X \\
H_2 &= F \\
H_{n+1} &= \frac{1}{H_{n-1}}(H_n^3 + r^{a_n}) \qquad (a_n = \deg H_n)\ .
\end{aligned}
$$

As shown in [3], each H_n *is* a polynomial. Observe that $a_{n+1} = 3a_n - a_{n-1}$, giving every other element of the Fibonacci sequence. (These degrees increase rather quickly, e.g., a_{23} already exceeds one billion.) Let $\delta_n = D_{(H_n, H_{n+1})}$ for $n \geq 1$. It turns out that, for each $n \geq 2$, the derivation δ_n is irreducible and locally nilpotent of rank three and type (a_n, a_{n+1}), having kernel $k[H_n, H_{n+1}]$. To prove this, it is shown that $\delta_n = \Delta(H_n, H_{n-1}, r)$ for each such n.

4 The Graph of Kernels

We are actually interested in subrings A of B which occur as the kernel of some locally nilpotent derivation, rather than in any specific derivation of which A is the kernel. With this in mind, we will, in this section, rephrase some of our terms, results and questions in the language of graphs.

Define Γ to be the graph such that vert(Γ) is the set of all kernels of non-zero locally nilpotent k-derivations of B, and such that two vertices A and A' are connected by an edge iff there exist derivations D, D' of B with $Ker(D) = A$, $Ker(D') = A'$, and D' is obtained from D by means of a local slice construction. Let A_0 denote the vertex $k[X, Y]$, corresponding to the partial derivative $(\partial/\partial Z)$.

The group $GA_3(k)$ of k-automorphisms of B acts on Γ by conjugation, and we let $\mathcal{G}$ denote the quotient graph. (Observe that, by Rentschler's Theorem [6], the corresponding quotient graph in dimension 2 consists of a *single* vertex, namely, that corresponding to the partial derivative.) Let $\mathcal{G}_0$ denote the connected component of

A_0 in $\mathcal{G}$. The following terminology will be used. (We do not distinguish between $A \in \text{vert}(\Gamma)$ and its equivalence class in $\text{vert}(\mathcal{G})$.)

1. The *rank* of a vertex A is the rank of any derivation D on B with $Ker(D) = A$.
2. A vertex A is *homogeneous* if there exists a homogeneous derivation D of B with $Ker(D) = A$. The *type* of a homogeneous vertex is the type of the corresponding derivation. (It is possible that more than one vertex could be associated with a given type.)
3. A vertex A is *free* if there exists a locally nilpotent derivation D on B with $Ker(D) = A$ and $(im\, D) = (1)$.

Note that A_0 is the unique vertex of $\mathcal{G}$ of rank one. Moreover, the results of [3] show the following.

Proposition 4 *$\mathcal{G}_0$ contains every rank-two vertex of $\mathcal{G}$.*

We close with some questions.

Question 1 *Is $\mathcal{G}$ connected ?*

Question 2 *Does every homogeneous vertex of $\mathcal{G}$ lie in $\mathcal{G}_0$?*

Question 3 *Does $\mathcal{G}_0$ contain a free vertex other than A_0 ?*

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