A TILT AT TILFS

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This paper is dedicated to Gary H. Meisters

Abstract. In this talk I will endeavour to give an overview of some aspects of the theory of Translation Invariant Linear Forms (TILFs) and associated Hilbert spaces of functions. In particular, I will discuss some of the early ideas and results of Gary Meisters in this area, and try to explain how these ideas have led to applications in various areas of harmonic analysis.

1 Introduction

Let $\mathbb{T}$ denote the circle group, $\mathbb{Z}$ the group of integers and $\mathbb{R}^n$ the n-dimensional Euclidean group. Let G denote any one of these groups (or any locally compact abelian group). Let $M(G)$ denote the measures on G of finite variation, and if $x \in G$ let δ_x denote the Dirac measure at x. Let X denote a vector space of functions or distributions on G such that if $\mu \in M(G)$ and $f \in X$, the convolution $\mu * f$ is defined and is such that $\mu * f \in X$. Let X' denote the algebraic dual of X and let $S \subseteq M(G)$. Then, a linear form $L \in X'$ is called *S-invariant*, or a SILF, if $L(\mu * f) = L(f)$ for all $\mu \in S$ and all $f \in X$.

If $x \in G$ and $f \in L^2(G)$, $\delta_x * f$ is the function $t \longmapsto f(t-x)$ and is called the *translation of f by x*. Then, when $S = \{\delta_x : x \in G\}$, an S-invariant form is called a *translation invariant linear form*, or a TILF, because it takes the same value on all translates of any given function. Thus, a TILF is a SILF!

In the case where $X = L^\infty(G)$, the positive and normalised TILFs are called invariant means (or Banach limits in the case when $G = \mathbb{Z}$), and are related to the theory of amenable groups.

Gary Meisters has obtained many interesting results about invariant linear forms in different contexts. One of these is a result of [8], which says that if $C_c^\infty(\mathbb{R})$ denotes all the complex-valued C^∞-functions on $\mathbb{R}$ which have compact support, then the only TILFs on $C_c^\infty(\mathbb{R})$ are the constant multiples of the TILF which is integration with respect to the usual Lebesgue measure. So, it follows that there are no discontinuous TILFs on $C_c^\infty(\mathbb{R})$. Gary also proved in [8] that if $\mathcal{E}(\mathbb{R})$ denotes the C^∞-functions of arbitrary support in $\mathbb{R}$, and if $\mathcal{S}(\mathbb{R})$ denotes the C^∞-functions which are rapidly decreasing and whose derivatives of all orders are rapidly decreasing, then neither $\mathcal{E}(\mathbb{R})$ nor $\mathcal{S}(\mathbb{R})$ have discontinuous TILFs.

The results in [8] are striking and elegant, and further information on them may be found in Jean Dieudonné's book [2, pp.208–209] and in [7, 11, 12]. However, in this paper I am mainly concerned to explain some developments which have arisen from work of Gary and Wolfgang Schmidt on the $L^2(G)$ case when G is a compact connected abelian group. It is not the purpose to present detailed proofs (which may or will be found elsewhere), but rather to look at some ideas and describe some of their applications.

2 The basic problems

Let X be a subspace of $L^2(G)$ and let S be a set of measures on G. Then the basic problem initially can be considered to be: *identify all S-invariant linear forms on X*. If we can't do this, at least let us try and say a few interesting things about them – for example, 0 is always an S-invariant form! Before proceeding, I would like to consider a refinement of this basic problem.

Let $\mathcal{D}(X, S)$ denote the vector subspace of X spanned by all vectors of the form $f - \mu * f$, for some $\mu \in S$ and $f \in X$. Thus, for $f \in X$, $f \in \mathcal{D}(X, S)$ if and only if there are $n \in \mathbb{N}$, $g_1, g_2, \dots, g_n \in X$ and $\mu_1, \mu_2, \dots, \mu_n \in S$ such that $f = \sum_{j=1}^n (g_j - \mu_j * g_j)$. When $S = \{\delta_x : x \in G\}$, $\mathcal{D}(X, S)$ is denoted by $\mathcal{D}(X)$. In the case where $G = \mathbb{R}$ and $S = \{\delta_x : x \in \mathbb{R}\}$, a function $f - \delta_x * f$ is a *first order difference*, as used to approximate derivatives of functions. For this reason, a general space $\mathcal{D}(X, S)$ is called a *difference space*.

The significance of the space $\mathcal{D}(X, S)$ lies in the immediate fact that, if $L \in X'$, L is S-invariant if and only if $L(\mathcal{D}(X, S)) = \{0\}$. This provides a characterization of the S-invariant forms, and provides further information about them. In fact, it was observed by Gary Meisters in 1973 [10] that there is a non-zero S-invariant form on X if and only if $\mathcal{D}(X, S) \neq X$, and that $\mathcal{D}(X, S)$ is dense in X if and only if the only continuous S-invariant form on X is 0. In view of these observations of Gary's, it is reasonable to regard the following problem as a refinement of the problem of identifying the S-invariant forms. The problem is: *characterize the space $\mathcal{D}(X, S)$ as a subspace of $L^2(G)$*.

3 The circle group case

The following result is due to Gary and Wolfgang Schmidt, and dates from 1972 [9]. As well as being a beautiful result in its own right, it can be regarded as the protoype for later results in a general L^2 context. Note that for a locally compact abelian group G, μ_G denotes a Haar measure on G, normalized in the case when G is compact.

Theorem 1 *If G is a compact and connected abelian group, and in particular if G is the circle group $\mathbb{T}$, then*

$$\mathcal{D}(L^2(G)) = \Big\{ f : f \in L^2(G) \text{ and } \int_G f \; d\mu_G = 0 \Big\}.$$

Thus, in this case, $\mathcal{D}(L^2(G))$ has codimension 1 in $L^2(G)$, and every TILF on $L^2(G)$ is continuous and is a multiple of the Haar measure on G. I would like to present an idea of the proof of this result, especially as the proof illustrates the general approach which later proved to be significant in the non-compact case.

Idea of proof. When G is compact, integration using the Haar measure is a TILF on $L^2(G)$, so that $\int_G f \; d\mu_G = 0$ for all $f \in \mathcal{D}(L^2(G))$. It is harder to prove that if $\int_G f \; d\mu_G = 0$, then $f \in \mathcal{D}(L^2(G))$. However, to this end, let $\widehat{\mu}$ denote the Fourier transform of the function or measure μ. Let $\widehat{G}$ be the dual group of G ($= \mathbb{Z}$, if $G = \mathbb{T}$). Then in [9], a Fourier transform argument is used to show that if $f \in L^2(G)$, $f \in \mathcal{D}(L^2(G))$ if and only if there are $x_1, x_2, \ldots, x_n \in G$ such that

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1-\gamma(x_j)|^2} d\mu_{\widehat{G}}(\gamma) < \infty. \tag{1}$$

In fact, this characterization of $\mathcal{D}(L^2(G))$ is valid for non-compact G as well.

Now when (1) holds, there are $f_1, \ldots, f_n$ such that $f = \sum_{j=1}^n (f_j - \delta_{x_j} * f_j)$. However, the trouble is that for a given f, it is hard to tell whether there are $x_1, x_2, \ldots, x_n \in G$ such that (1) holds. We need a characterization which depends more directly upon f itself. To this end, let $\int_G f \; d\mu_G = 0$ and note that this means that $\widehat{f}(\widehat{e}) = 0$, where $\widehat{e}$ is the identity of $\widehat{G}$.

First, make some preliminary observations. Let $\gamma \in \widehat{G}$, $\gamma \neq \widehat{e}$. Then, the function $(x_1, \ldots, x_n) \longmapsto (\gamma(x_1), \ldots, \gamma(x_n))$ is continuous on the connected group G^n, so its range must be connected and in fact equals $\mathbb{T}^n$. So, if h is any continuous non-negative function on $\mathbb{T}^n$,

$$\int_{G^n} h(\gamma(x_1), \ldots, \gamma(x_n)) d\mu_G(x_1) \ldots d\mu_G(x_n)$$
$$= (2\pi)^{-n} \int_{[-\pi,\pi]^n} h(e^{i\theta_1}, \ldots, e^{i\theta_n}) \; d\theta_1 \ldots d\theta_n.$$

Now, we use this and the assumed fact that $\widehat{f}(\widehat{e}) = 0$ to obtain

$$\begin{aligned}
&\int_{G^n}\left(\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1-\gamma(x_j)|^2} d\mu_{\widehat{G}}(\gamma)\right) d\mu_G(x_1)\ldots d\mu_G(x_n)\\
= &\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 \left(\int_{G^n} \frac{d\mu_G(x_1)d\mu_G(x_2)\ldots d\mu_G(x_n)}{\sum_{j=1}^n |1-\gamma(x_j)|^2}\right) d\mu_{\widehat{G}}(\gamma)\\
= &(2\pi)^{-n}\left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\mu_{\widehat{G}}(\gamma)\right)\left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2\ldots d\theta_n}{\sum_{j=1}^n |1-e^{i\theta_j}|^2}\right)\\
= &(2\pi)^{-n}\left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\mu_{\widehat{G}}(\gamma)\right)\left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2\ldots d\theta_n}{4\sum_{j=1}^n \sin^2\theta_j/2}\right)\\
< &\ \infty,
\end{aligned}$$

as long as $n \geq 3$.

Thus, if we take $n = 3$, for almost all $(x_1, x_2, x_3) \in G^3$, (1) holds and it follows that any $f \in \mathcal{D}(L^2(G))$ is the sum of 3 first order differences.

This argument of Meisters and Schmidt showed there is a connection between estimating certain n-dimensional integrals and the characterization of spaces such as $\mathcal{D}(L^2(G))$. Note that in [9] it is shown that there is a function in $\mathcal{D}(L^2(\mathbb{T}))$ which is *never* the sum of 2 first order differences, so that the choice of 3 is best possible. Note also that the determination of sharp bounds on the required number of differences is related to problems of Diophantine approximation [9, 12].

Meisters and Schmidt in fact showed that on any compact group with a finite number of components, any TILF on $L^2(G)$ is a multiple of Haar measure and so continuous. In 1973 Meisters [10] showed that the L^2-space of the Cantor group has discontinuous TILFs. Further results of Meisters and Bagget [11] and Johnson [3], produced a characterisation of the compact abelian groups G such that every TILF on $L^2(G)$ is continuous. Then Bourgain [1] showed that for $1 < p < \infty$, every TILF on $L^p(\mathbb{T})$ is continuous, and this was extended to other groups by Wai Lok Lo [5] who also gave an estimate of the required number of differences for higher order difference spaces of $L^p(\mathbb{T})$.

The result of Meisters and Schmidt suggests a reformulation of the basic problem for $L^2(G)$. For, in the circle group case let μ be the measure on $\widehat{\mathbb{T}} = \mathbb{Z}$ given by

$$\mu(\{x\}) = \begin{cases} 1, & \text{if } x \in \mathbb{Z} \text{ and } x \neq 0\ ; \\ \infty, & \text{if } x = 0. \end{cases}$$

Then observe that for $f \in L^2(\mathbb{T})$, $\int_{\mathbb{Z}} |\widehat{f}|^2 d\mu < \infty$ if and only if $\widehat{f}(0) = 0$. So, for the circle group, their result can be stated as: *the Fourier transform maps* $\mathcal{D}(L^2(\mathbb{T}))$ *bijectively onto* $L^2(\widehat{\mathbb{T}}, \mu)$. Thus, on a general locally compact abelian group G, the basic problem for $L^2(G)$ may be considered to be: *describe a measure* μ *on* $\widehat{G}$ *such that the Fourier transform maps a difference space* $\mathcal{D}(L^2(G), S)$ *bijectively onto* $L^2(\widehat{G}, \mu)$.

4 The case of the real line

4.1 The first order difference space

In 1973, Gary Meisters [10] proved the following:

$$f \in \mathcal{D}(L^2(\mathbb{R})) \Longrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty. \tag{2}$$

It follows from this that $\mathcal{D}(L^2(\mathbb{R}))$ is a proper subspace of $L^2(\mathbb{R})$, and it is also dense in $L^2(\mathbb{R})$. So, as found in [10], *there are non-zero TILFs on* $L^2(\mathbb{R})$ *and every such TILF is discontinuous.* Further results for non-compact groups were obtained by Sadahiro Saeki [18], Gordon Woodward [21] and myself [14], all concerning the existence and even the profusion of discontinuous TILFs for non-compact cases such as $\mathbb{R}$ and, more generally, for non-compact amenable groups. On the other hand, G. Willis [20] showed that for a non-amenable group G, the only TILF on $L^p(G)$ (for $1 < p \leq \infty$) is 0.

Now (2) shows that Fourier transforms of functions in $\mathcal{D}(L^2(\mathbb{R}))$ have a certain precise behaviour *near the origin.* In fact, it was proved in [13] that the functions in $L^2(\mathbb{R})$ which are in $\mathcal{D}(L^2(\mathbb{R}))$ are *characterized* by the behaviour expressed in (2). So this result, together with Gary's earlier result in [10], gives the following.

Theorem 2 *Let* $f \in L^2(\mathbb{R})$. *Then*

$$f \in \mathcal{D}(L^2(\mathbb{R})) \Longleftrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty.$$

The space $\mathcal{D}(L^2(\mathbb{R}))$ *is Hilbert, with the inner product given by*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2})dx,$$

for all $f, g \in \mathcal{D}(L^2(\mathbb{R}))$. *The Fourier transform maps* $\mathcal{D}(L^2(\mathbb{R}))$ *isometrically onto* $L^2(\mathbb{R}, (1 + |x|^{-2})dx)$.

Now the first order Sobolev space is denoted by $H^1(\mathbb{R})$ and consists of the functions in $L^2(\mathbb{R})$ whose derivatives are in $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, $f \in H^1(\mathbb{R})$ if and only if $\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^2 dx < \infty$, so functions in $H^1(\mathbb{R})$ are characterized by the behaviour of their Fourier transforms *at infinity.* The space $H^1(\mathbb{R})$ is Hilbert with inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^2)dx.$$

The spaces $\mathcal{D}(L^2(\mathbb{R}))$ and $H^1(\mathbb{R})$ are "complementary" in that the functions in the former are characterized by the behaviour of their Fourier transforms near the origin, while the functions in the latter are characterized by the behaviour of their Fourier transforms at infinity (see [15] for more aspects of this).

Now let D denote differentiation. Then, for $f \in H^1(\mathbb{R})$, $D(f)\hat{}(x) = ix\widehat{f}(x)$. It follows from this that *differentiation is a Hilbert space isometry from* $H^1(\mathbb{R})$ *onto* $\mathcal{D}(L^2(\mathbb{R}))$, so the latter space is the range of D in a natural sense. Since an element L of $L^2(\mathbb{R})'$ is a TILF if and only if $L(\mathcal{D}(L^2(\mathbb{R}))) = 0$, it follows by a Hamel basis argument that

$$D(H^1(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is a TILF} \Big\},$$

which establishes a connection between TILFs and differentiation.

4.2 Higher order and fractional difference spaces

Now let $s > 0$, let $m \in \mathbb{N}$, and let α be a 2π-periodic function on $\mathbb{R}$ which has an absolutely convergent Fourier series. Assume that:

(i) for some $\delta > 0$,

$$\delta |x|^s \leq |\alpha(x)| \leq \delta^{-1}|x|^s, \text{ for all } x \text{ in } [-\delta, \delta];$$

(ii)

$$\int_{[-\pi,\pi]^m} \frac{dx_1 \ldots dx_m}{\sum_{j=1}^m |\alpha(x_j)|^2} < \infty.$$

Let $S_\alpha = \{\delta_0 - \sum_{j=-\infty}^{\infty} \widehat{\alpha}(j)\delta_{-jy} : y \in \mathbb{R}\}$. Then it was proved in [13] that $\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ *consists precisely of all functions* $f \in L^2(\mathbb{R})$ *such that*

$$\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty.$$

Thus, as $\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ depends upon s but is independent of α, it may be denoted by $\mathcal{D}_s(L^2(\mathbb{R}))$. The following result is then analogous to Theorem 2 and extends it.

Theorem 3 *Let* $f \in L^2(\mathbb{R})$. *Then*

$$f \in \mathcal{D}_s(L^2(\mathbb{R})) \Longleftrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty.$$

The space $\mathcal{D}_s(L^2(\mathbb{R}))$ *is Hilbert, with the inner product given by*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2s})dx, \text{ for } f, g \in \mathcal{D}_s(L^2(\mathbb{R})).$$

For $s \in \mathbb{N}$, let $H^s(\mathbb{R})$ denote the Sobolev space consisting of the functions in $L^2(\mathbb{R})$ all of whose derivatives of order at most n are in $L^2(\mathbb{R})$. Then $H^s(\mathbb{R})$ is Hilbert in the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{2s})dx, \text{ for } f, g \in \mathcal{D}_s(L^2(\mathbb{R})),$$

and it follows that D^s is an isometry from the Sobolev space $H^s(\mathbb{R})$ onto $\mathcal{D}_s(L^2(\mathbb{R}))$.

Now, let us make the definition that an S_α-invariant linear form, with S_α as above, is called an s-ILF (or sILF). Thus, for $s \in \mathbb{N}$,

$$D^s(H^s(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is a } s\text{ILF} \Big\}.$$

Example 1 A function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if there are $x_1, \ldots, x_5 \in \mathbb{R}$, and $f_1, \ldots, f_5 \in L^2(\mathbb{R})$ such that $f = \sum_{j=1}^{5} \Big(f_j - 2^{-1}(\delta_{x_j} + \delta_{-x_j}) * f_j \Big)$. Also, a function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if $L(f) = 0$ for every $\{2^{-1}(\delta_x + \delta_{-x}) : x \in \mathbb{R}\}$-invariant form on $L^2(\mathbb{R})$.

5 Partial differential operators

The preceding discussion has established a connection between invariant forms, differentiation and difference subspaces, all in the context of one variable. The question arises as to what happens when several real variables are considered. In order to give an idea of what happens in this context, let V be a vector subspace of $\mathbb{R}^n$, let $|x|$ be the usual norm of $x \in \mathbb{R}^n$, and let $e_1, \ldots, e_r$ be an orthonormal basis for V. The *V-Laplacian* Δ_V is given by $\Delta_V = \sum_{j=1}^{r} D_{e_j}^2$, where D_{e_j} is differentiation in direction e_j. If P_V is the orthogonal projection onto V,

$$\Delta_V(f)^\wedge(x) = -|P_V(x)|^2 \widehat{f}(x).$$

Now let $V_1, V_2, \ldots, V_q$ be non-zero vector subspaces of $\mathbb{R}^n$, and let $s_1, s_2, \ldots, s_q$ be q strictly positive real numbers. Let

$$\begin{aligned} \Upsilon &= \prod_{j=1}^{q} |P_{V_j}|^{s_j}, \\ \Psi &= \sum_{A \subseteq \{1,\ldots,q\}} \prod_{j \in A} |P_{V_j}|^{-s_j}, \\ \Theta &= \sum_{A \subseteq \{1,\ldots,q\}} \prod_{j \in A} |P_{V_j}|^{s_j}. \end{aligned}$$

Let

$$W(L^2(\mathbb{R}^n), \Psi) = \Big\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{f}(x)|^2 \Psi^2(x) dx < \infty \Big\},$$

and similarly define $W(L^2(\mathbb{R}^n), \Theta)$. Then, using the fact that $\Upsilon\Psi = \Theta$, the operator $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ may be defined and is an isometry from $W(L^2(\mathbb{R}^n), \Theta)$ onto $W(L^2(\mathbb{R}^n), \Psi)$. This definition is given by the requirement that

$$\Big(\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}(f) \Big)^\wedge = \Big(\prod_{j=1}^{q} |P_{V_j}|^{2s_j} \Big) \widehat{f},$$

for all $f \in W(L^2(\mathbb{R}^n), \Theta)$. The statement above that $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ is an isometry results from routine manipulations of the definitions; but the point is that the space $W(L^2(\mathbb{R}^n), \Psi)$, the range of the operator, may be described alternatively as a "generalized" difference space which is analogous to the space $\mathcal{D}(L^2(\mathbb{R}))$ as it appears in Theorem 2. A corresponding description of the range is valid for the similarly defined operators $D_{u_1} D_{u_2} \dots D_{u_r}$, for independent vectors $u_1, \dots, u_r \in \mathbb{R}^n$. Before proceeding to technicalities, let us consider a special case.

Example 2 The Wave Operator is

$$\mathcal{W} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = D_{u_1} D_{u_2},$$

where $u_1 = (1,-1), u_2 = (1,1)$. The domain of $\mathcal{W}$ is the Sobolev-type space consisting of all $f \in L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} |\widehat{f}(x,y)|^2 \Big\{1 + |x-y| + |x+y| + |x-y| \cdot |x+y|\Big\}^2 dxdy < \infty.$$

The range of $\mathcal{W}$ consists of those functions in $L^2(\mathbb{R}^2)$ which are the sum of 9 functions, each of which is of the form $(x,y) \longmapsto g(x,y) - g(x+a, y+a) - g(x+b, y-b) + g(x+a+b, y+a-b)$, for some $a, b \in \mathbb{R}$ and some $g \in L^2(\mathbb{R})$. The range of $\mathcal{W}$ is the intersection of the kernels of all the linear forms which are $\{\delta_{(a,a)} + \delta_{(b,-b)} - \delta_{(a+b,a-b)} : a, b \in \mathbb{R}\}$-invariant.

Now for the more general case. Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be q continuous, complex valued 2π-periodic functions on $\mathbb{R}^n$ such that for some $\delta_1, \delta_2 > 0$,

$$\delta_1 |x|^{s_j} \le |\alpha_j(x)| \le \delta_2 |x|^{s_j},$$

for all $x \in [-\pi, \pi]$ and all $j = 1, 2, \dots, q$. For each $j = 1, 2, \dots, q$, let $m_j \in \mathbb{N}$ with $m_j > 2s_j$, and let $J_1, J_2, \dots, J_q$ be q disjoint subintervals of $\mathbb{N}$ such that each J_j has m_j elements. Now consider the set of all functions f in $L^2(\mathbb{R}^n)$ such that f is equal to a sum of the form

$$\sum_{(k_1,\dots,k_q) \in \prod_{j=1}^{q} J_j} \left(\prod_{j=1}^{q} \sum_{\ell=-\infty}^{\infty} \widehat{\alpha}_j(\ell) \delta_{-\ell y_{k_j}} \right) * \ h_{k_1 k_2 \dots k_q},$$

where for each $k \in \{1, 2, \dots, q\}$, $y_k \in V_k$; and for each $(k_1, \dots, k_q) \in \prod_{j=1}^{q} J_j$, $h_{k_1 k_2 \dots k_q} \in L^2(\mathbb{R}^n)$. This set of functions is a subset of $L^2(\mathbb{R}^n)$ which depends upon $V_1, \dots, V_q$, $s_1, \dots, s_q$ but is *independent* of $\alpha_1, \dots, \alpha_q$ and $m_1, \dots, m_q$. Accordingly, this set of functions is denoted by $\mathcal{D}_{s_1, s_2, \dots, s_q}(L^2(\mathbb{R}^n), V_1, \dots, V_q)$.

Theorem 4 *$\mathcal{D}_{s_1, s_2, \dots, s_q}(L^2(\mathbb{R}^n), V_1, \dots, V_q)$ is a vector subspace of $L^2(\mathbb{R}^n)$, and it is a Hilbert space in the inner product $[\cdot, \cdot]$ given by*

$$[f, g] = \int_{\mathbb{R}^n} \left(\sum_{A \subseteq \{1,2,\dots,q\}} \left[\prod_{j \in A} |P_{V_j}|^{-s_j} \right] \right) \widehat{f}(x) \overline{\widehat{g}(x)}\ dx.$$

In particular, in $\mathbb{R}^2$*, if* V_1 *is the subspace spanned by* $(-1,1)$ *and* V_2 *is the one spanned by* $(1,1)$*, the range of* $\mathcal{W}$*, as in the above example, is the space* $\mathcal{D}_{1,1}(L^2(\mathbb{R}^n), V_1, V_2)$.

More generally, if Θ, Ψ are the functions as before, Theorem 4 shows that $W(L^2(\mathbb{R}^n), \Psi) = \mathcal{D}_{s_1,s_2,\ldots,s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$. Thus, we have

Theorem 5 $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ *is an isometry from the space* $W(L^2(\mathbb{R}^n), \Theta)$ *onto the space* $\mathcal{D}_{s_1,s_2,\ldots,s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$.

This result identifies the range of an operator $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ as a "generalised difference space" when the domain of the operator is taken as the "Sobolev-type" space $W(L^2(\mathbb{R}^n), \Theta)$.

6 Multiplier operators

Partial differential operators are special cases of (unbounded) multiplier operators with mutipliers of the form $\prod_{j=1}^{r} |P_{V_j}|^{s_j}$ or $\prod_{j=1}^{r} \langle \cdot, e_j \rangle$, in the present context. A multiplier operator T on a space X is one for which there is a function φ such that $T(f)^\wedge = \varphi \widehat{f}$, for all $f \in X$. Recent work with Susumu Okada has shown that the ranges of a large class of multiplier operators on locally compact abelian groups may be described by means of "generalized difference spaces" on these groups. For example, the following result is proved in [17].

Theorem 6 *If* T *is a bounded multiplier operator on* $L^2(G)$ *with multiplier* φ*, there is a family of pseudomeasures* $S = \{\mu_a : a \in \mathbb{R}\}$ *on* G *such that:*

1. $\mu_a * \mu_b = \mu_{a+b}$ *for all* $a, b \in \mathbb{R}$*;*
2. *the range of* T *is the difference space* $\mathcal{D}(L^2(G), S)$ *and this space is a Hilbert space in the inner product* $\langle \cdot, \cdot \rangle$ *given by*

$$\langle f, g \rangle = \int_{\widehat{G}} \widehat{f}\, \overline{\widehat{g}}\, (1 + |\varphi|^{-2})\, d\mu_{\widehat{G}},$$

   for all f, g *in the range of* T*; and*

3. *for each* f *in the range of* T*, there are* $a_1, a_2, a_3 \in \mathbb{R}$ *and* $f_1, f_2, f_3 \in L^2(G)$ *such that* $f = \sum_{j=1}^{3} (f_j - \mu_{a_j} * f_j)$.

Results with S. Okada have also been obtained which extend this sort of result to unbounded multiplier operators, the partial differential operators of Section 5 being special cases.

7 Further applications

7.1 The Riesz potential operators

Let $n, s \in \mathbb{N}$ with $0 < s < n/2$. The Riesz potential operator I_s of order s is given by

$$I_s(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy,$$

for $x \in \mathbb{R}^n$. Then, as discussed in Chapter 7 of [19],

$$I_s(f)^\wedge(x) = \widehat{f}(x)/|x|^{n-s}.$$

The Sobolev space $W^s(L^2(\mathbb{R}^n))$ of order s on $\mathbb{R}^n$ consists of all functions $f \in L^2(\mathbb{R}^n)$ such that

$$||f|| = \left(\int_{\mathbb{R}^n} \left(1 + |x|^{2s}\right) |\widehat{f}(x)|^2 dx\right)^{1/2} < \infty,$$

and it is Hilbert in this norm $||\cdot||$. Note that $W^s(L^2(\mathbb{R})) = H^s(\mathbb{R})$, for $s \in \mathbb{N}$. The Laplace operator Δ is given by $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$.

Theorem 7 *[13, 14]. The operator $|\Delta|^{s/2}$ is an isometry from $W^s(L^2(\mathbb{R}^n))$ onto the difference space $\mathcal{D}_s(L^2(\mathbb{R}^n))$, and its inverse is the Riesz potential operator of order s. Also, $\mathcal{D}_s(L^2(\mathbb{R}^n))$ consists of the functions f in $L^2(\mathbb{R}^n)$ such that $I_s(f) \in L^2(\mathbb{R}^n)$.*

7.2 The Hilbert transform and related operators

The Hilbert transform on $L^2(\mathbb{R})$ arises from convolution by the kernel $x \longmapsto 1/\pi x$. Now let s be an even non-negative integer, and consider the function

$$K_{s,y} : x \longmapsto \frac{1}{\pi x \prod_{k=1}^{s/2}(x^2 - k^2y^2)}.$$

Owing to the identity

$$\sum_{k=0}^{s} \binom{s}{k} \frac{(-1)^k}{x - ky} = \frac{(-1)^s s! y^s}{\prod_{k=0}^{s}(x - ky)},$$

convolution by $K_{s,y}$ defines a bounded operator $H_{s,y}$ on $L^2(\mathbb{R})$ in the same way as the Hilbert transform. In fact the Hilbert transform is the case $s = 0$.

Theorem 8 *[14]. Let $y \in \mathbb{R}, y \neq 0$. The operator $H_{2,y}$ on $L^2(\mathbb{R})$ is given by convolution by the kernel $x \longmapsto 1/\pi x(x^2 - y^2)$. This operator has multiplier $x \longmapsto -2iy^{-2} sign(x) \sin^2(xy/2)$. The range of this operator consists of all functions in $L^2(\mathbb{R})$ which can be expressed in the form $g - 2^{-1}(\delta_y + \delta_{-y}) * g$ for some $g \in L^2(\mathbb{R})$. That is, the range is the intersection of the kernels of all the $\{\delta_y + \delta_{-y})/2\}$-invariant linear forms on $L^2(\mathbb{R})$.*

Whereas this result describes the range of $H_{2,y}$ in terms of certain second order differences, convolution by the kernel $x \longmapsto 1/\pi(x^2 - y^2)$ has a range which can be expressed in terms of *first* order differences.

7.3 Wavelets

Let $\mathbb{R}^*$ denote the non-zero real numbers. If $h \in L^2(\mathbb{R})$, we define a function U_h from $L^2(\mathbb{R})$ into the functions on $\mathbb{R} \times \mathbb{R}^*$ by

$$U_h(f)(a,b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\ \overline{h\left(\frac{x-b}{a}\right)}\ dx,$$

for all $f \in L^2(\mathbb{R})$ and all $a \in \mathbb{R}^*, b \in \mathbb{R}$. The function U_h is linear and is called the *wavelet transform* with *wavelet* h. A standard identity in the theory of the wavelet transform, which is analogous to the Plancherel Theorem, is

$$\int_{-\infty}^{\infty} \int_{\infty}^{\infty} \frac{|U_h(f)(a,b)|^2}{|a|^2}\ dadb = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|} dx \right) \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right).$$

This singles out the wavelets h which have the property that

$$\int_{-\infty}^{\infty} |\widehat{h}(x)|^2 |x|^{-1} dx < \infty,$$

and such wavelets are called *admissible*, in which case, the wavelet transform maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^* \times \mathbb{R}, |a|^{-2} dadb)$. It is clear from what has been said earlier that h *is admissible* if and only if $h \in \mathcal{D}_{1/2}(L^2(\mathbb{R}))$. Equivalently, h is admissible if and only if there is $g \in W^{1/2}(L^2(\mathbb{R}))$ such that $|D|^{1/2}(g) = h$. Alternatively, if we think of a TILF as being a "1-invariant linear form", h is an admissible wavelet if and only if for any 1/2-ILF, L say, $L(h) = 0$. Further aspects of wavelets and difference spaces may be found in [14, 15, 16].

8 Conclusion

The ideas in Gary Meisters' early papers on TILFs have proved to be of central and essential importance for subsequent work. In particular, the connection between TILFs and the estimation of certain specific integrals in $\mathbb{R}^n$, developed by Gary with Wolfgang Schmidt [9], has proved to be the cornerstone in developing the theory and applications of SILFs and TILFs in the non-compact case. Details of many of the ideas presented in this paper for the non-compact case, together with proofs, can be found in detail in [13, 14]. Since the publication of [14], some other work in this area may be found in [4, 5, 6, 15, 16, 17]. The expository paper [15] discusses the case of the real line, including proofs of the fundamental results for this case. It is fair to say that without Gary's input into this area of harmonic analysis, I would certainly not have carried out this work, so I owe him a very great debt; and not only because of this, but also owing to his warm and generous resonse to this work.

It may seem a long way from the concept of a TILF to the characterization of the ranges of differential and multiplier operators, to which the concept has led. However, as I tried to explain in [15], if one looks at the fundamental result of Gary and Wolfgang Schmidt in [9] from the appropriate viewpoint, it may be regarded as characterizing the range of d/dx on the first order Sobolev space of $L^2([-\pi,\pi])$

as the space consisting precisely of those functions expressible in a certain way as a sum of first order differences. So, their result can be regarded as establishing a precise relationship between the usual or "continuous" calculus on $[-\pi, \pi]$ and the "discrete" calculus which is based upon finite differences. That is, from this viewpoint, their result can be looked upon as unifying or reconciling "opposite" concepts, the continuous and the discrete. Still another way of thinking of their result is by regarding it as characterizing the "admissible" wavelets on the circle group, well in advance of the theory of wavelets!

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