# Essential laminations in Seifert-fibered spaces: boundary behavior

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ABSTRACT. We show that an essential lamination in a Seifert-fibered space $M$ rarely meets the boundary of $M$ in a Reeb-foliated annulus.

## §0
## INTRODUCTION

In [Br1], we showed that every essential lamination $\mathcal{L}$ in a Seifert-fibered space $M$ contains a sublamination $\mathcal{L}_0$ which can be isotoped to be either vertical or horizontal in $M$, i.e., each leaf is either saturated by the circle fibers of $M$, or each leaf is everywhere transverse to the circle fibers of $M$. We also described, in many cases, how the other leaves of $\mathcal{L}$ behaved; in most cases, they could be isotoped to be horizontal in $M$. In [Br2], we completed this description, in the case when $M$ is closed, by showing that, except for the leaves in 'Reeb sublaminations', $\mathcal{L}$ can be isotoped so that each leaf is either vertical or horizontal.

A Reeb sublamination is a generalization of the concept of 'cylindrical component' found in foliation theory, which is in turn related to the concept of a Reeb annulus. A Reeb annulus is an annulus $A=S^1 \times \mathbb{I}$, which is foliated by lines which approximate the $\mathbb{I}$-fibers of the annulus, except near the ends, where they spiral in the same direction toward the two boundary circles (which are also leaves). A cylindrical component $C$ (called a component of type II in [Ro]) is a Reeb annulus crossed with $S^1$, which is foliated by (leaf)$\times S^1$. The interior leaves are therefore open annuli which spiral in the same dirction towards the two boundary tori (which are also leaves). A Reeb sublamination is a sublamination, which contains at least one of the non-compact leaves, of this foliation, or of the related foliations of the orientable and non-orientable $\mathbb{I}$-bundles over the Klein bottle. and the non-orientable $\mathbb{I}$-bundle over the torus, which a cylindrical component double or four-fold covers.

In this paper we prove much the same theorem as described above, in the case that $M$ has non-empty boundary. To do so we extend the notion of cylindrical component to include a Reeb annulus crossed with $\mathbb{I}$ (and its relatives), and extend the notion of Reeb sublamination accordingly. The boundary of the cylindrical component now includes an $\mathbb{I}$-saturated part, which consists of a pair of annuli, each foliated as a Reeb annulus. The leaves of the cylindrical component meeting

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the interior are now infinite rectangles, meeting each of the Reeb annuli in a leaf of its foliation (see Figure 4 below).

**Theorem.** *Let $M$ be an orientable, connected, compact Seifert-fibered space with non-empty boundary, and $\mathcal{L}$ an essential lamination in $M$, which is either transverse to, or contains as a leaf, each boundary component of $M$. Then, possibly after splitting $\mathcal{L}$ open along a finite number of leaves, either $\mathcal{L}$ can be isotoped so that each leaf is either vertical or horizontal, or it has finitely many Reeb sublaminations with horizontal boundary. In particular, if $\mathcal{L}$ has a Reeb sublamination, then $M$ contains a horizontal annulus.*

An appropriate statement for non-orientable $M$ can be obtained by applying the above theorem to its orientation double covering.

Essential laminations in Seifert-fibered spaces therefore fall into four distinct classes. There are the horizontal laminations, the vertical laminations, the mixed horizontal/vertical laminations, and the laminations with 'horizontal' Reeb sublaminations. A horizontal lamination can be 'filled in' with additional horizontal leaves in its complement, to 'complete' it to a horizontal foliation. Such foliations have been extensively studied (see [EHN],[JN],[Na]) to the point where we can now determine (in terms of their Seifert invariants) exactly which Seifert-fibered spaces admit horizontal foliations. This fact serves as the basis for nearly all of the known non-existence results ([Br1],[Cl],[BNR]) for essential laminations and foliations. A Seifert-fibered space $M$ can be thought of as a circle bundle over a 2-dimensional orbifold (the space obtained by crushing each circle fiber to a point is topologically a surface, but geometrically an orbifold), and a vertical lamination in $M$ is simply the full preimage, under the projection to the base orbifold $B$, of a 1-dimensional lamination in $B$. Every leaf can therefore be foliated by circles, and therefore consists of tori, compact annuli, open annuli, half-open annuli, and their non-orientable analogues. If an essential lamination $\mathcal{L}$ has a vertical šublamination $\mathcal{L}_0$, then the horizontal leaves of $\mathcal{L}\backslash\mathcal{L}_0$ can be thought of as coming from a horizontal lamination of $M|\mathcal{L}_0$, the manifold obtained by splitting $M$ open along the leaves of $\mathcal{L}_0$. This is a (usually non-compact) 3-manifold with boundary. The leaves of $\mathcal{L}$ are obtained from the leaves of this horizontal lamination by having them spiral towards the leaves of $\mathcal{L}_0$ as they approach the boundary of $M|\mathcal{L}_0$, as in the definition of a Reeb annulus. We don't place any restriction on the direction in which they spiral, however. Finally, when $\mathcal{L}$ contains a Reeb sublamination, the leaves of $\mathcal{L}$ inside of the cylindrical components have already been described, while (by simply ignoring the leaves inside the components) the leaves outside of the cylindrical components, together with the boundaries of the components, still form an essential lamination, and can therefore be isotoped to be horizontal. Such a lamination therefore looks like a collection of Reeb sublaminations sandwiched between horizontal laminations.

Just as with horizontal tori in closed Seifert-fibered spaces, horizontal annuli and Möbius bands are rare; there are, in fact, only three orientable Seifert-fibered spaces which can contain them. They are the space with base $D^2$ and two multiple fibers of multiplicity 2, the trivial $S^1$-bundle over the annulus, and the non-trivial $S^1$-bundle over the Möbius band. The above theorem therefore says that essential laminations in most Seifert-fibered spaces cannot have 'horizontal' Reeb annuli in their boundary. As such, this result is much in the spirit of a paper of Gabai [Ga], where it is shown that an essential lamination in the exterior of a knot in $S^3$ must meet the boundary torus either in a suspension (i.e., no Reeb annuli) or in Reeb annuli whose compact loops describe a curve in the torus which meets the meridian of the knot at most once. If the lamination is in fact a foliation, the Reeb annuli must be meridional.

Torus knots are the only knots in $S^3$ with Seifert-fibered exterior, and their exteriors cannot contain a vertical essential lamination, other than the obvious vertical annulus separating the two multiple fibers, or, for (2,q)-torus knots, the vertical Möbius band that this annulus 'double covers'; the proof of this is entirely similar to that of Proposition 3 in [Br1]. These annuli also happen to be the cabling annuli for these knots. Since none of these knot exteriors contain horizontal annuli (although the (2,2)-torus link exterior does), we can conclude that no essential lamination in a torus knot exterior can have Reeb annuli in its boundary, other than vertical ones. Our main theorem therefore implies:

**Corollary.** *Every essential lamination in the exterior of a torus knot either contains the cabling annulus as a leaf or (for (2,q)-torus knots) the Möbius band it double covers, or is isotopic to a horizontal lamination.*

Naimi [Na] has completed the classification of those slopes in the boundary of a torus knot exterior that can be realized by horizontal laminations. In particular, a horizontal lamination in the $(p,q)$-torus knot exterior must meet the boundary torus in curves of slope $r\in(-\infty,q-2]$, and all such slopes are realized. If $r\in(-\infty,q-2)$, then the essential lamination can be chosen to meet the boundary in parallel loops of slope $r$ (when $r$ is rational). If $r=q-2$, then the boundary lamination must contain non-compact leaves. The above corollary allows us to drop the word 'horizontal' from this result.

**Corollary.** *Every essential lamination in a $(p,q)$-torus knot exterior either contains the cabling annulus or Möbius band as a leaf or is horizontal and meets the boundary torus in a suspension, whose curves have slope $r\in(-\infty,q-2]$. Further, all such slopes are realized.*

These facts, in turn, are among the ingredients in the proof [BNR] that the incompressible torus in the manifold $M$ obtained by 37/2 surgery on the (-2,3,7) pretzel knot, first identified by Hatcher and Oertel [HO], is a leaf of every essential lamination contained in $M$. This same approach can also be applied to incompressible tori in many other graph manifolds [BNR].

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## §1
## Vertical and horizontal sublaminations

We refer the reader to [Or] for background information on Seifert-fibered spaces, and to [G-O] for basic information on essential laminations. For the more technical portions of what follows, a familiarity with the techniques of [Br1] will be helpful.

Let $M$ be an orientable, compact Seifert-fibered space with non-empty boundary, and let p:$M\to F$ be the associated quotient map, crushing every circle fiber to a point. The quotient space $F$ is a 2-dimensional orbifold, whose cone points correspond to the multiple fibers of $M$. As in [Br1], we can describe a decomposition of $M$ into solid tori, by cutting $F$ along disjoint properly embedded arcs (whose union we call $\alpha$) into a collection of disks (whose union we call $D$), each containing at most one cone point. The inverse image of the arcs $\alpha$ is a collection of annuli, which we denote $A$. These annuli split $M$ into the set of inverse images of the disks $D$, which are therefore 3-manifolds which are Seifert-fibered over the disk, with at most one multiple fiber, and so are solid tori. Each has a (usually non-trivial) Seifert-fibering induced from $M$. We will denote these solid tori $M_1,\dots,M_n$. For simplicity, we shall actually choose two parallel arcs for each arc in the original collection $\alpha$, so that every solid torus we obtain after cutting open along the annuli, when thought of as lying in $M$, is embedded. That is, no solid torus $M_i$ abuts the same component of $A$ from both sides.

Let $\mathcal{L}$ be an essential lamination in $M$. The proof that $\mathcal{L}$ contains a sublamination $\mathcal{L}_0$ which can be made either vertical or horizontal w.r.t. the Seifert-fibering of $M$ is entirely similar to the argument given in [Br1] for closed $M$. Some additional care, however, must be taken when working near the boundary of $M$. For completeness, we give the proof here.

**Theorem 1.** *Every essential lamination $\mathcal{L}$ in a Seifert-fibered space $M$ with non-empty boundary can be isotoped, possibly after splitting $\mathcal{L}$ open along a finite number of leaves, so that it contains a sublamination which is either vertical or horizontal in $M$. If $\mathcal{L}$ contains a vertical sublamination, then $\mathcal{L}$ can be isotoped so that each leaf of $\mathcal{L}$ is either vertical or horizontal in $M$.*

**Proof:** As in [Br1], the argument focusses on the intersection $\mathcal{L}\cap S$, where $S$ is a finite collection of regular fibers of the fibering of $M$, which we call the *sentinel fibers*. In this instance $S = A\cap\partial M = \partial A$.

By splitting $\mathcal{L}$ along a finite number of leaves, if necessary, we can assume that $\mathcal{L}$ is carried by a branched surface $B$, and so (by making $B$ transverse to $A$) $\mathcal{L}$ can be made transverse to the annuli $A$. Note that splitting does not qualitatively change the boundary behavior of $\mathcal{L}$. $\partial\mathcal{L}\subseteq\partial M$ is a one-dimensional lamination in a union of tori, and so we can, by isotopy, pull $\mathcal{L}$ 'taut' w.r.t. $\partial A$. By this we mean that $\partial A$ misses any vertical loops of $\partial\mathcal{L}$, meets any horizontal loops of $\partial\mathcal{L}$ tautly, as well as any Kronecker-type (i.e., non-Reeb) leaves lying between horizontal loops of $\partial\mathcal{L}$, and meets non-compact leaves lying between vertical loops tautly. Finally, we can assume that $\partial A$ meets all Reeb-type leaves lying between horizontal loops as tautly as possible, i.e., the direction of intersection with each component of $\partial A$, when we orient the Reeb leaf, changes exactly once. This isotopy can be carried out *conservatively*, i.e., without introducing new points of intersection with $S$, and without moving any of the points that it doesn't erase.

We now build an infinite sequence of conservative isotopies of $\mathcal{L}$, by running cyclically through the solid tori $M_i$, and, at each stage, 'cleaning up' $\mathcal{L}_i = \mathcal{L}\cap M_i$. $\mathcal{L}\cap\partial M_i$ is a 1-dimensional lamination $\lambda_i$ in the torus $\partial M_i$, and so, since it cannot contain any monogons - otherwise $\mathcal{L}$ would admit an end-compressing disk, a contradiction - it consists of an incompressible lamination in $\partial M_i$, together with a collection $\mathcal{T}$ of trivial circles, which are open and closed in $\lambda_i$. By surgering $\mathcal{L}$ along disks in $M_i$, we can assume that all of the trivial circles bound disks in $\mathcal{L}_i$; because $\mathcal{L}$ is essential, this surgery can be achieved by a (conservative) isotopy. We can now isotope these disks out of $M_i$, to eliminate $\mathcal{T}$ from $\lambda_i$. If some of these trivial circles intersect $S$, this must be done in several steps: first, by a (conservative) isotopy of $\partial\mathcal{L}$, the bounding disks can be pushed off of $M_i\cap\partial M$ (see Figure 1), after which their boundaries lie entirely on $A \cap \partial M_i$, and the disks can then be pushed out of $M_i$.

Figure 1

Any meridional loop $\gamma$ of $\lambda_i$ now bounds a disk leaf of $\mathcal{L}_i$; this is because it bounds a disk $D$ in the leaf of $\mathcal{L}$ containing it. Since $\gamma$ intersects $\partial M$, this disk lies on the $M_i$-side of $\gamma$. $D$ must therefore be entirely contained in $M_i$, since otherwise $D \cap \lambda_i \subseteq D$ contains leaves in the interior of $D$. If any of these leaves are non-compact, then $D \cap \lambda_i$ either contains a monogon or holonomy around a trivial loop, both contradicting the essentiality of $\mathcal{L}$. But any compact loops are either non-null-homotopic in ($M_i$, hence) $M$, or are meridional, hence intersect $\partial M$, also both contradictions, since in the first case $D$ provides a null-homotopy for the loop, and in the second case $D \cap \partial M \subseteq \partial D$. So $D \cap \lambda_i = D \cap \partial M_i = \partial D$.

By Lemma 2.1 of [Br1], the leaves of $\mathcal{L}_i$ are $\pi_1$-injective in $M_i$. By Theorem 3.1 of [Br1], $\mathcal{L}_i$ either consists of meridional disks, or it contains a sublamination, consisting of annuli and possibly one Möbius band, which is vertical w.r.t. a possibly different model Seifert-fibering of $M_i$; all other leaves can be made horizontal w.r.t. this fibering.

If $\mathcal{L}_i$ has annular leaves and no Möbius band leaf, then by a further isotopy of $\mathcal{L}$, we can pull $\lambda_i$ taut w.r.t. $S$, except possibly for Reeb leaves lying between non-vertical loops of $\lambda_i$. If $\lambda_i$ contains any vertical loops, we can then isotope $\mathcal{L}_i$ so that each leaf is vertical or horizontal w.r.t the Seifert-fibering of $M_i$ induced from $M$. If $\lambda_i$ contains any horizontal, non-meridional loops, we can eliminate their intersection with $S$ by a conservative isotopy of $\mathcal{L}$, since the leaves of $\mathcal{L}_i$ containing them are boundary-parallel annuli (see Figure 2). Note that since $S \subseteq \partial M$, what is pictured is a $\partial$-compression of $\mathcal{L}$, not an isotopy. But since $\mathcal{L}$ is $\partial$-incompressible, this compression results in a lamination isotopic to $\mathcal{L}$, together with a collection of $\partial$-parallel disks (which we throw away). Once all of the annular leaves have been eliminated, we are left with a collection of meridional disks, by Theorem 3.1 of [Br1] (since the leaves are horizontal w.r.t. some Seifert-fibering on $M_i$), which we can make horizontal w.r.t the Seifert-fibering of $M_i$ induced from $M$, by a conservative isotopy.

Figure 2

This gives us a conservative isotopy $I$ of $\mathcal{L}$, so that $I(\mathcal{L}) \cap M_i$ either contains a Möbius band leaf, or a vertical sublamination w.r.t the Seifert-fibering of $M_i$ induced from $M$, or a collection of horizontal meridional disks w.r.t. the same Seifert-fibering. The first two cases lead us immediately to a vertical sublamination of $\mathcal{L}$; the third leads us to repeat the process, building our infinite string of isotopies. We deal with the first two cases first.

A Möbius band leaf of $\mathcal{L}_i$ must contain a loop isotopic to the core of $M_i$, which is a fiber of the Seifert-fibering of $M$. So $\mathcal{L}$ is isotopic to a lamination which contains an (interior) fiber of $M$. This is obviously also true if some $\mathcal{L} \cap \partial M_i$ contains a vertical fiber, since the annulus leaf of $\mathcal{L}_i$ containing it contains interior fibers. Splitting $\mathcal{L}$ along the leaf containing the Möbius band or annulus, we then have an essential lamination $\mathcal{L}$ missing a fiber of $M$. If we drill out a small neighborhood of this fiber, we get a new Seifert-fibered space $M'$, containing $\mathcal{L}$. $\mathcal{L}$ is essential in $M'$, and now misses one of the boundary components $T'$ of $M'$.

**Proposition 2.** *Every essential lamination $\mathcal{L}$ as above can be isotoped to contain a vertical sublamination $\mathcal{L}_0$ ; all other leaves can be made horizontal in $M'$.*

**Proof:** If we choose a new splitting of $M'$, using annuli all of which meet $T'$, then for every resulting solid torus $M_i'$, $\mathcal{L}\cap M_i'=\mathcal{L}_i'$ misses a vertical annulus in $\partial M_i'$. Therefore, $\lambda_i'=\mathcal{L}\cap\partial M_i'$ either contains vertical loops or consists of trivial disks.

We will now build a vertical sublamination of $\mathcal{L}$. Starting with $M_1$, either we find vertical loops or, after surgering and then pushing trivial disks out, $\mathcal{L}\cap M_1 = \emptyset$. Note that, in the second case, no further conservative isotopy will push anything back into $M_1$, since all such pushes require $\mathcal{L}\cap\partial M_1\neq \emptyset$. Continuing cyclically through our solid tori, we must find some $i$ so that, after isotopy, $\mathcal{L}\cap\partial M_i$ contains vertical loops. Otherwise, after passing through our list of solid tori once, we will have $\mathcal{L}\cap M_i = \emptyset$ for all $i$, hence $\mathcal{L} = \emptyset$, a contradiction. So we may assume that, after isotopy, $\mathcal{L}\cap\partial M_i$ contains a vertical loop, for some $i$. What we will see is that if we now run cyclically through our solid tori once more, then when we are done, the isotoped lamination will have a vertical sublamination.

The key point is that our isotopies will never move a vertical loop $\gamma$ of $\mathcal{L}\cap\partial M_i$. This is because our isotopies only deal with straightening curves ($\gamma$ is already straight), dealing with intersections of the $\lambda_i$ with $S$ ($\gamma$ has none), and throwing away trivial pieces of $\mathcal{L}$ after surgery ($\gamma$ is essential in $M$, so will not be contained in any).

In fact, even more is true. Once we have made $\mathcal{L}$ meet some $\partial M_i'$ in vertical loops, and no trivial loops, we know by Theorem 3.1 of [Br1] that $\mathcal{L}_i'$ contains a vertical sublamination, and all other leaves of $\mathcal{L}_i'$ can be made horizontal (see Figure 3). But now if we move on to begin straightening $\mathcal{L}$ in other solid tori, we have the following important fact:

**Lemma 3.** *$\mathcal{L}_i'$ remains fixed under all further conservative isotopies of $\mathcal{L}$. In particular, no $\partial$-compression of $\mathcal{L}$ will abut the components of $S$ contained in $\partial M_i'$.*

**Proof:** $\partial\mathcal{L}_i'$ consists of vertical loops and non-compact leaves, neither of which can be contained in a compact piece of a leaf $L$ of $\mathcal{L}$ (in particular, in a disk in $L$), so no ordinary surgery of $\mathcal{L}$ can affect $\partial\mathcal{L}_i$. A $\partial$-compression, on the other hand, would have to join together two non-compact leaves of $\partial\mathcal{L}_i'$; they are the only leaves which intersect $S$. The $\partial$-compressing disk, together with half of the infinite rectangle between the two leaves, would yield an end-compressing disk for $\mathcal{L}$, a contradiction; see Figure 3. █

Figure 3

Consequently, as we work cyclically through the solid tori, we either completely clear $\mathcal{L}$ out of $M_i'$, or leave a lamination, all of whose leaves are vertical or horizontal, behind. In so doing, we neither push anything back into solid tori we have cleared out, nor disturb any of the horizontal/vertical laminations that we have previously built. So once we return to our starting point, $\mathcal{L}$ has been isotoped so that in every solid torus $M_i'$, $\mathcal{L}\cap M_i'$ is a lamination all of whose leaves are either vertical or horizontal. The union of the vertical pieces form a vertical sublamination $\mathcal{L}_0$ of $\mathcal{L}$; all leaves of $\mathcal{L}\backslash\mathcal{L}_0$ are made up of horizontal pieces in the solid tori, so are horizontal in $M'$.

Since the Seifert-fibering of $M'$ was induced from the one on $M$ by inclusion, $\mathcal{L}_0$ is a vertical sublamination of $\mathcal{L}$ in $M$. Since the leaf the we split $\mathcal{L}$ open along is now vertical (it met the annuli $A$ in vertical loops lying near $T'$), when we collapse it back, we get a vertical sublamination of our original lamination $\mathcal{L}$; all other leaves are (identical, hence) horizontal. █

Note that this in particular implies that $\partial\mathcal{L}$ contains no horizontal loops with Reeb-type leaves in between. Such leaves could not be isotoped to lie either vertically or horizontally in $M$. This could have been seen earlier in the proof, since every component of $A$ abuts $T'$, so contains a vertical loop of $\mathcal{L}\cap A$. Any Reeb leaf in some component T of $\partial M$ could have been joined to a pair of arcs in an annulus of $A$ abutting T, as in the lemma, to give an end-compressing disk.

The proof of Thoerem 1 now finishes exactly as in [Br1]. We work cyclically through the solid tori $M_1,\dots,M_n$, using conservative isotopies to make $\mathcal{L}$ meet the $M_i$ tautly. If at any point we encounter a Möbius band leaf or a vertical annulus, we stop and apply the above argument to find a vertical sublamination of $\mathcal{L}$. Otherwise, we continue, building an infinite sequence of isotopies $I_r$ which make $\mathcal{L}$ meet $M_i$ in horizontal, meridional disks, for $r \equiv i (\mathrm{mod}\ n)$. Then, by focussing on the stable points $\bigcap I_n(\mathcal{L})\cap S$ of $\mathcal{L}$ in the sentinel fibers $S$, we can see, as in [Br1], that a stable, horizontal lamination $\mathcal{L}_0$ grows out of them, built out of the pieces of $\mathcal{L}$ which eventually stabilize under the isotopies $I_r$. A final argument, as in [Br1], shows that the pieces of this horizontal lamiantion actually all stabilize in finite time, so $\mathcal{L}_0$ is in fact a sublamination of $\mathcal{L}$. The proofs are identical to those given in [Br1], so we will not repeat them here. █

## §2
## Finding horizontal annuli

Theorem 1 provides a complete description of an essential lamination $\mathcal{L}$ when $\mathcal{L}$ contains a vertical sublamination. All leaves of $\mathcal{L}$ can then be made either vertical or horizontal w.r.t. the Seifert-fibering of $M$. This need not be true when $\mathcal{L}$ contains a horizontal lamination, however. [Br2] explored this phenomenon when $M$ was closed, and showed that cylindrical components - parallel horizontal tori, with Reeb-type annuli lying between - essentially gave the only counterexamples. When $\partial M{\neq}\emptyset$, there is a similar phenomenon; the lamination can contain two parallel horizontal annuli, with Reeb-type leaves lying in between (see Figure 4). The Reeb leaves cannot be isotoped to be vertical or horizontal in $M$. This is what we have called a 'Reeb sublamination'.

Figure 4

The purpose of this section is to show that such annuli are essentially the only possible counterexamples. Since horizontal annuli (and horizontal tori, for that matter) are scarce in Seifert-fibered spaces, this implies that every essential lamination in most Seifert-fibered spaces can be isotoped so that every leaf is horizontal or vertical. This result therefore parallels the main result of [Br2]. In contrast with the previous section, the fact that we have non-empty boundary actually simplifies the argument, instead of complicating it; the boundary gives us 'edges' to start arguing from, instead of having to start from the 'middle' of the manifold, as in [Br2]. We begin with the analogue of Proposition 6 of [Br1].

**Proposition 4.** *Let $\mathcal{L}$ be an essential lamination in the Seifert-fibered space $M$, with $\partial M \neq \emptyset$. If $\mathcal{L}$ contains a horizontal sublamination $\mathcal{L}_0$ which contains no compact leaves, then $\mathcal{L}$ is isotopic to a horizontal lamination. In particular, the 1-dimensional lamination $\mathcal{L} \cap \partial M$ is isotopic (in $\partial M$) to a lamination everywhere transverse to the circle fibers of $M$.*

**Proof:** If we impose a Riemannian metric on $M$, then the (acute) angle which the leaves of $\mathcal{L}$ make with the circle fibers of $M$ is (basically by definition) a continuous function $\theta{:}\mathcal{L} \to [0,\pi/2]$; since $M$ is normal and $\mathcal{L}$ is closed, we can extend $\theta$ to a continuous function $\theta{:}M \to [0,\pi/2]$. Since $\mathcal{L}_0$ is horizontal, $\theta$ never takes on the value 0 on $\mathcal{L}_0$, and so, since $\mathcal{L}_0$ is compact, $\theta$ is bounded away from 0 (by $\epsilon$, say) on $\mathcal{L}_0$. Consequently, there is an open neighborhood $\mathcal{U}$ of $\mathcal{L}_0$ in $\mathcal{L}$ (for example, $\theta^{-1}((\epsilon/2,\pi/2))$) where $\theta$ is non-zero. Now choose a component $N$ of $M|\mathcal{L}_0$, the manifold $M$ split open along the lamination $\mathcal{L}_0$; it is a (non-compact) manifold with boundary. Since $\mathcal{L}_0$ is horizontal, it cuts the circle fibers of $M$ into intervals, which foliate $N$, making $N$ an $\mathbb{I}$-bundle over some (by hypothesis) non-compact base $B$. Note that every such component $N$ must meet $\partial M$, since every leaf of $\mathcal{L}_0$ must meet $\partial M$. Some of the vertical boundary components of $N$ could be non-compact, if the boundary leaves in $\mathcal{L}_0$ meet $\partial M$ in non-compact leaves.

**Claim:** $N_0 = N \setminus \mathcal{U}$ is compact.

This is because if we pick points $x_i$ in $N_0$ with no convergent subsequence in $N_0$, i.e., whose images in the base $B$ tend to infinity, the $\mathbb{I}$-fibers containing them must become arbitrarily short. For otherwise, a subsequence converges in $M$ (since $M$ is compact) to a point $x$. Since this convergence cannot be taking place in $N_0$ (hence not in $N$, since $N_0$ is closed in $N$), $x$ must lie in $\overline{N} \setminus N \subseteq \mathcal{L}$. But then in a product neighborhood $I^2 \times I$ of $x$, the subsequence cannot eventually lie on the same vertical level (since then the sequence would converge in $N$), and so the heights of the levels containing the subsequence must go to 0.

But since $\mathcal{L}_0$ is compact and $\mathcal{U}$ is open, there is an $\epsilon > 0$ so that the $\epsilon$ neighborhood of every point of $\mathcal{L}_0$ is contained in $\mathcal{U}$; $\epsilon$ is simply the Lebesgue number for the open cover $\mathcal{U}$ of $\mathcal{L}_0$. But since the endpoints of every $I$-fiber in $N$ lies in $\mathcal{L}_0$, this implies that every sufficiently short fiber of $N$ lies in $\mathcal{U}$, and therefore our subsequence eventually lies in $\mathcal{U}$, hence not in $N_0$, a contradiction.

Consequently, its projection $B_0$ of $N_0$ to the base $B$ of $N$ is compact, so we can choose simple loops and arcs $\gamma_i$ missing $B_0$ so that the component $B_1$ of $B|(\gamma_1 \cup \cdots \cup \gamma_k)$ containing $B_0$ is compact, and connected. Taking inverse images, we get a collection $R$ of 'vertical' annuli and rectangles in $N$ so that the component $N_1$ of $N|R$ containing $N_0$ is a connected $\mathbb{I}$-bundle over the compact base $B_1$. By deleting components of $R$, if necessary, we can include any other compact components of $N|R$ in $N_1$, so we may assume that every component of $N \setminus (N_1 \setminus R)$ is non-compact, i.e., is an $\mathbb{I}$-bundle over a non-compact base. $\partial N_1$ splits naturally into two pieces, the 'vertical' boundary, $\partial_v N_1 = R$, saturated by $\mathbb{I}$-fibers, and the 'horizontal' boundary $\partial_h N_1 = N_1 \cap \mathcal{L}_0$, the associated $\partial\mathbb{I}$-bundle.

Outside of $N_1$, every point of $\mathcal{L} \cap N$ is contained in $\mathcal{U}$, so $\mathcal{L}$ is transverse to the circle fibers of $M$ at these points. In particular, $\mathcal{L} \cap \partial_v N_1$ is a horizontal lamination in the vertical boundary of $N_1$. In addition, $\mathcal{L}$ meets $N_1 \cap \partial M$ in a horizontal lamination, since otherwise a turnaround arc can be joined to a half-infinite 'vertical' rectangle in $N \setminus N_1$ to give an end-compressing disk for $\mathcal{L}$, a contradiction (see Figure 5a).

Figure 5

Therefore, $\mathcal{L}$ meets $\partial N_1$ in a horizontal lamination. Since $N_1$ is an $\mathbb{I}$-bundle over a compact base, it can be cut open along vertical rectangles to give an $\mathbb{I}$-bundle over a disk, i.e., a 3-ball. Working inductively, as in Proposition 6 of [Br1], we can see that $\mathcal{L}$ meets each of these rectangles in horizontal arcs. For otherwise we can once again join a turnaround arc to a half-infinite rectangle to give an end-compressing disk for $\mathcal{L}$ (Figure 5b); or, if the arc hits $\partial M$, it gives us a $\partial$-compressing disk for $\mathcal{L}$ (Figure 5c), which is also a contradiction.

We can therefore absorb neighborhoods of these rectangles into $N \backslash N_1$, maintaining the property that $\mathcal{L}$ be horizontal there. In the end, we are left with the 3-ball, which is fibered over a disk, and $\mathcal{L}$ is horizontal along its vertical boundary. $\mathcal{L}$ must therefore meet the boundary in loops; otherwise $\mathcal{L}$ has non-trivial holonomy around a homotopically trivial loop, which is impossible. These loops must bound disks in the 3-ball, which can therefore be made horizontal. This finishes pulling $\mathcal{L}$ horizontal in $N$. Doing this for all components of $M|\mathcal{L}_0$ completes the proof. ∎

The above proposition also shows that the leaves in a complementary component $N$ of the horizontal lamination $\mathcal{L}_0$ found in Theorem 1 can be made horizntal, if one (hence both) of the boundary leaves $\mathcal{L}_0 \cap N = \partial_h N$ is non-compact. Therefore, to determine what leaves of $\mathcal{L}$ can be made horizontal, it will be sufficient to focus our attention on those leaves which can live in an $\mathbb{I}$-bundle component $N$ lying between horizontal, compact leaves. By splitting the horizontal leaves, if necessary, we may assume that $\partial_h N$ is embedded in $M$. (Note that this turns a horizontal Möbius band into a horizontal annulus.) These horizontal leaves must meet $\partial M$, since they meet every circle fiber of $M$. $N$ is therefore a handlebody. We can therefore conclude that every leaf of $\mathcal{L} \cap N$ meets $\partial M$ : otherwise, the set of leaves which didn't meet $\partial M$ would be an (essential) sublamination of $\mathcal{L}$, living in a handlebody, which is impossible.

**Proposition 5.** *If $\mathcal{L}$ contains a horizontal sublamination $\mathcal{L}_0$, and a leaf $L$ of $\mathcal{L}$ cannot be isotoped to lie horizontally in $M$, then $L$ is a standardly embedded planar leaf lying either in a trivial $\mathbb{I}$-bundle component of $M|\mathcal{L}_0$ bounded by two annulus leaves of $\mathcal{L}_0$, or lying in a non-trivial $\mathbb{I}$-bundle component of $M|\mathcal{L}_0$ bounded by a single annulus leaf.*

**Proof:** $\mathcal{L}_0$ consists of the 'eventually stable' pieces of $\mathcal{L}$ under the sequence of isotopies that we constructed. It is easy to see that any horizontal leaf of $\partial\mathcal{L} \subseteq \partial M$ is stable; no surgery disk could meet $\partial M$, and a $\partial$-compression meeting such a leaf would either constitute a 'real' $\partial$-compression for $\mathcal{L}$ (Figure 6a) or provide an end-compressing disk for $\mathcal{L}$ (Figure 6b). Its intersection with the sentinel fibers $S$ is therefore stable, so the leaf of $\mathcal{L}$ containing them will be eventually stable, hence contained in $\mathcal{L}_0$ . Consequently, every leaf of $\mathcal{L} \cap N$, except for the horizontal boundary leaves, meets $\partial_v N$ in Reeb-type leaves.

Figure 6

We will now show, first, that the Reeb-type behavior on any two components of $\partial_v N$ must be 'coherently oriented', which will lead us quickly to the fact that $\partial_v N$ has at most two components. Two $\partial_v$-components leads us to the first possibility, by an argument reminiscent of Novikov's [No] construction of Reeb components; one $\partial_v$-component will lead us to the second possiblity.

By coherently-oriented we mean that if the two $\partial_v$-components $A_1$, $A_2$ are joined by a vertical rectangle $R$ in $N$ (see Figure 7a), then the Reeb leaves in $A_i$ open up in the same direction normal to $R$. If not, then by the usual isotopies we can make $\mathcal{L}$ meet $R$ in horizontal arcs. The innermost turnaround leaves of $\mathcal{L} \cap A_i$ must be joined by these arcs, otherwise we can find an end-compressing disk for $\mathcal{L}$; see Figure 7b. If these turnaround arcs are not coherently-oriented, then we can (Figure 7c) find a loop in a leaf of $\mathcal{L}$ which is null homotopic in $M$, hence bounds a disk in its leaf, yet meets a proper arc in the leaf exactly once. This is impossible; proper arcs meet, transversely, the boundaries of disks an even number of times.

Figure 7 Figure 8

But now if $N$ has three or more $\partial_v$-components, their Reeb leaves must be pairwise coherently-oriented. Figure 8 shows that this is impossible; given two of the $\partial_v$-components, there is no way to orient the Reeb leaves in the third coherently with the other two. This figure tacitly assumes that the circle fibers of $M$ can be given an orientation, so that none of the vertical rectangles have a half-twist (relative to one another). But we can orient the fibers of M, after passing to a double cover of $M$, if necessary. $\mathcal{L}$ then lifts to an essential lamination, $\mathcal{L}_0$ lifts to a horizontal sublamination, and $N$ lifts to a (trivial) $\mathbb{I}$-bundle, each component having three or more $\partial_v$-components, a contradiction.

Consequently, $N$ has either one or two $\partial_v$-components. If it has two, then, joining the components by a vertical rectangle $R$, the innermost Reeb-type arcs of $(\partial_v N)|R$ on each $\partial_v$-component are joined by horizontal arcs in $R$, as above. These pieces and arcs form a loop in a leaf of $\mathcal{L}$, which is null-homotopic in $N$, hence bounds a disk in its leaf; see Figure 9; we will call such a disk a Reeb disk. Reeb Stability implies that this disk lifts to Reeb disks in all nearby leaves.

Figure 9 Figure 10

But then all of the Reeb arcs are contained in Reeb disks. For otherwise, starting from the outermost Reeb arc in one of the components $F$ of $(\partial_v N)|R$, there is a first arc $\alpha$ which is not contained in a Reeb disk, by the (relative) openness of Reeb arcs. If this arc is limited upon by arcs in Reeb disks, it is easy to see that the Reeb disks are limiting upon another Reeb arc on the other component of $\partial_v N$ (Figure 10). The horizontal arcs in $R$ must join these two Reeb arcs together, because otherwise we find non-trivial holonomy around a null-homotopic loop. These arcs combine to give us the boundary of another Reeb disk, a contradiction. If, on the other hand, $\alpha$ is not limited upon by Reeb disks, then it is in particular isolated in $\partial\mathcal{L}$ from the 'outermost' side. So we push all of the Reeb disks in front of $\alpha$across $R$; now $\alpha$is an outermost Reeb arc in $F|R$. But now the argument above implies that $\alpha$ is contained in a Reeb disk $D$, so pushing the Reeb disks back across $R$ (which does not move $D$) means that $\alpha$ had already been contained in a Reeb disk, a contradiction.

Consequently, all of the Reeb arcs are contained in (parallel) Reeb disks. If we push all of these disks across $R$, we create new (longer) Reeb arcs, which are all therefore contained in new (larger) Reeb disks. An old Reeb disk cuts a new Reeb disk into two horizontal rectangles, whose boundaries lie near $\partial_h N$. These rectangles project up and down along the $\mathbb{I}$-fibers to $\partial_h N$. This projection identifies the two arcs which lie in $R$ in each of their boundararies together, showing that $\partial_h N$ consists of a pair of (horizontal) annuli. Therefore, $\mathcal{L} \cap \partial_h N$ consists of two parallel horizontal annuli. It is easy to see that the Reeb disks glue to the remaining horizontal rectangles to give a Reeb-type foliation in between the annuli, as desired.

If $\partial_v N$ consists of a single vertical annulus, then (since the core of this annulus is not homotopically trivial in $N$ - the parallel loops $\partial_v N \cap \partial_h N \subseteq \mathcal{L}$ have non-trivial holonomy around them) there is an essential arc in the base of the $\mathbb{I}$-bundle $N$ giving us a similar vertical rectangle $R$ in $N$ to work with. Our previous arguments apply, so, as in Figure 7, the Reeb arcs must meet the ends of $R$ (i.e., the vertical components of $\partial R$) in a coherently oriented fashion. We can therefore once again find Reeb disks containing each of these Reeb arcs. Pushing them all across $R$, we find new Reeb disks, which the old ones cut into horizontal rectangles. In this case, however, they do not each project onto an annulus in $\partial_h N$, because this would imply that $\partial_h N$ had four boundary components, instead of two. These two rectangles therefore glue end-to-end to form a single annulus in $\partial_h N$, giving us our second situation. $N$ is an $\mathbb{I}$-bundle with boundary a torus, having $\partial_h N$ and $\partial_v N$ each a single annulus; it is therefore the non-trivial $\mathbb{I}$-bundle over the Möbius band. The picture of $\mathcal{L} \cap N$ follows as in the previous case. ∎

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