THE CLASSIFICATION OF DEHN SURGERIES ON 2-BRIDGE KNOTS

Mark Brittenham and Ying-Qing Wu

Abstract. We will determine whether a given surgery on a 2-bridge knot is reducible, toroidal, Seifert fibered, or hyperbolic.

In [Th1] Thurston showed that if K is a hyperbolic knot, then all but finitely many surgeries on K are hyperbolic. In particular, for the Figure 8 knot, it was shown that exactly 9 nontrivial surgeries are non-hyperbolic. Let $K_{p/q}$ be a 2-bridge knot associated to the rational number p/q. When $p \equiv \pm 1 \mod q$, K is a torus knot, on which the surgeries are well understood. By [HT], all other 2-bridge knots are hyperbolic, admitting no reducible surgeries. Moreover, $K_{p/q}$ admits a toroidal surgery if and only if $p/q = [r_1, r_2] = 1/(r_1 - 1/r_2)$ for some integers r_1, r_2. See Lemma 8 below for a complete list of all toroidal surgeries. The Geometrization Conjecture [Th2] asserts that if a closed orientable 3-manifold is irreducible and atoroidal, then it is either a hyperbolic manifold, or a Seifert fibered space whose orbifold is a 2-sphere with at most three cone points, called a *small Seifert fibered space*. The conjecture has been proved for two large classes of manifolds: the Haken manifolds [Th2], and those admitting an orientation preserving periodic map with nonempty fixed point set [Th3,Ho,KOS,Zh]. It can be shown that surgery on a 2-bridge knot yields a manifold which admits such a periodic map, so it has a geometric decomposition. Our main result will classify all surgeries on 2-bridge knots according to whether they are reducible, toroidal, Seifert fibered, or hyperbolic manifolds.

1991 *Mathematics Subject Classification.* 57N10, 57M25, 57M50.

Key words and phrases. Dehn surgery, 2-bridge knots, Seifert fibered spaces, hyperbolic manifolds.

We use $[b_1, \ldots, b_n]$ to denote the partial fraction decomposition $1/(b_1 - 1/(b_2 - \ldots - 1/b_n)\ldots)$. Recall that a 2-bridge knot K is a *twist knot* if it is equivalent to some $K_{p/q}$ with $p/q = [b, \pm 2]$ for some integer b. Since $[b, \pm 2] = [b \mp 1, \mp 2]$, we may assume that b is even. Let $K(\gamma)$ be the manifold obtained by γ surgery on K. We always assume that $\gamma \neq \infty$, that is, the surgery is nontrivial. (4) and (5) in the following theorem are well known. They are included here for the sake of completeness.

Theorem 1. *Let K be a 2-bridge knot.*

(1) If $K \neq K_{[b_1,b_2]}$ for any b_1, b_2, then $K(\gamma)$ is hyperbolic for all γ.

(2) If $K = K_{[b_1,b_2]}$ with $|b_1|, |b_2| > 2$, then $K(\gamma)$ is hyperbolic for all but one γ, which yields toroidal manifold. When both b_1 and b_2 are even, $\gamma = 0$. If b_1 is odd and b_2 is even, $\gamma = 2b_2$.

(3) If $K = K_{[2n,\pm 2]}$ and $|n| > 1$, $K(\gamma)$ is hyperbolic for all but five γ: $K(\gamma)$ is toroidal for $\gamma = 0, \mp 4$, and is Seifert fibered for $\gamma = \mp 1, \mp 2, \mp 3$.

(4) If $K = K_{[2,-2]}$ is the Figure 8 knot, $K(\gamma)$ is hyperbolic for all but nine γ: $K(\gamma)$ is toroidal for $\gamma = 0, 4, -4$, and is Seifert fibered for $\gamma = -1, -2, -3, 1, 2, 3$.

(5) If $K = K_{[b]}$ is a $(2, b)$ torus knot, $K(\gamma)$ is Seifert fibered unless $\gamma = 2b$. $K(2b)$ is a reducible manifold.

The reader is referred to [GO] for definitions and basic properties concerning essential branched surfaces and essential laminations, which play a central role in the proof of the theorem. In [De1, De2] Delman constructed essential branched surfaces and laminations in 2-bridge knot complements, which are persistent in the sense that they remain essential after all nontrivial surgeries. Brittenham [Br] showed that if M is a small Seifert fibered space containing an essential branched surface $\mathcal{F}$, then each component of $M - \mathrm{Int}N(\mathcal{F})$ is an I bundle over a compact surface G. This is a very useful criteria in determining which manifolds are small Seifert fibered spaces. Before proving the theorem, we need to review some results of [De1, De2].

For each rational number p/q, there is associated a diagram $D(p/q)$, which is the minimal subdiagram of the Hatcher-Thurston diagram [HT, Figure 4] that contains all minimal paths from 1/0 to p/q. See [HT, Figure 5] and [De1]. $D(p/q)$ can be constructed as follows. Let $p/q = [a_1, \ldots, a_k]$ be a partial fraction decomposition of p/q. To each a_i is associated a "fan" F_{a_i} consisting of a_i simplices, see Figure 1(a) and 1(b) for the fans F_4 and F_{-4}. The edges labeled e_1 are called initial edges, and the ones labeled e_2 are called terminal edges. The diagram $D(p/q)$ can be constructed by gluing the F_{a_i} together in such a way that the terminal edge of F_{a_i} is glued to the initial edge of $F_{a_{i+1}}$. Moreover, if $a_i a_{i+1} < 0$ then F_{a_i} and $F_{a_{i+1}}$ have one edge in common, and if $a_i a_{i+1} > 0$ then they have a 2-simplex in common. See Figure 1(c) for the diagram of $[2, -2, -4, 2]$.

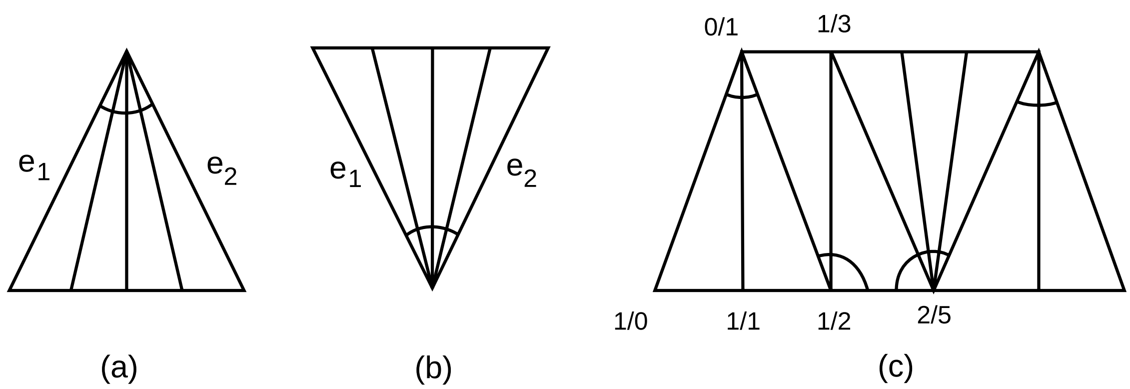

Figure 1

To each vertex v_i of $D(p/q)$ is associated a rational number r_i/s_i. It has one of the three possible parities: odd/odd, odd/even, or even/odd, denoted by o/o, o/e, and e/o, respectively. Note that the three vertices of any simplex in $D(p/q)$ have mutually different parities.

We consider $D(p/q)$ as a graph on a disk D, with all vertices on ∂D, containing ∂D as a subgraph. The boundary of D forms two paths from the vertex 1/0 to the vertex p/q. The one containing the vertex 0/1 is called the *top path*, and the one containing the vertex 1/1 is called the *bottom path*. Edges on the top path are called *top edges*. Similarly for *bottom edges*.

Let Δ_1, Δ_2 be two simplices in $D(p/q)$ with an edge in common. Assume that the two vertices which are not on the common edge are of parity o/o. Then the arcs indicated in Figure 2(a) and (b) are called *channels*. A *path* α in $D(p/q)$ is a union of arcs, each of which is either an edge of $D(p/q)$ or a channel.

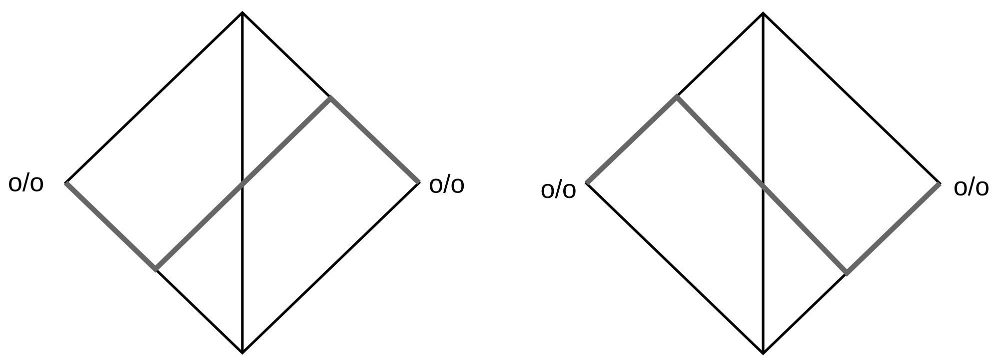

Figure 2

Let α be a path in $D(p/q)$. Let v be a vertex on α. Let e_1, e_2 be the edges of α incident to v. Then there is a fan F_v in $D(p/q)$ with e_1, e_2 as initial and terminal edges. The number of simplices in F_v is called the *corner number* of v in α, denoted by $c(v;\alpha)$ or simply $c(v)$. A path α from $1/0$ to p/q is an *allowable path* if it has at least one channel, and $c(v) \geq 2$ for all v in α.

Now assume that $K = K_{p/q}$ is a 2-bridge knot. Then q is an odd number. Recall that $K_{p/q} = K_{p'/q}$ if $p' \equiv p^{\pm 1}$ mod q, and $K_{-p/q}$ is the mirror image of $K_{p/q}$. We may assume without loss of generality that p is even, and $1 < p < q$. This is because $K_{(q-p)/q}$ is equivalent to the mirror image of $K_{p/q}$, so the result of γ surgery on the first is the same as that of $-\gamma$ surgery on the second. Note that $q-p$ and p have different parity, since q is odd. The following result is due to Delman. See [De1] and [De2, Proposition 3.1].

Lemma 2. *Given an allowable path α of $D(p/q)$, there is an essential branched surface $\mathcal{F}$ in $S^3 - K$ which remains essential after all nontrivial surgeries on K.* □

Lemma 3. *If there is an allowable path α in $D(p/q)$ such that $c(v) > 2$ for some vertex v in α, then $K(\gamma)$ is not a small Seifert fibered space for any γ.*

Proof. It was shown in [Br, Corollary] that if $\mathcal{F}$ is an essential branched surface in a small Seifert fibered space M, then each component of $M - \text{Int}N(\mathcal{F})$ is an I-bundle over a compact surface G, such that the vertical surface $\partial_v N(\mathcal{F})$ (also called cusps) is the I-bundle over ∂G. It has been shown in [De1] that for each vertex v of α there is a component W_v of $S^3 - \text{Int}N(\mathcal{F})$ such that W_v is a solid torus whose meridian disk intersects the cusps $c(v)$ times. In particular, if $c(v) > 2$ then W_v is not an I bundle as above. Since $\mathcal{F}$ is an essential branched surface in $K(\gamma)$, it follows that $K(\gamma)$ is not a small Seifert fibered space. □

Lemma 4. *Suppose p is even, q is odd, and $1 < p < q-1$. If p/q does not have partial fraction decomposition of type $[r_1, r_2]$, then $D(p/q)$ has an allowable path α such that some vertex v on α has $c(v) > 2$.*

Proof. Let $[a_1, \ldots, a_n]$ be the partial fraction decomposition of p/q such that all a_i are even. Then $a_1 \geq 2$. If $a_i = 2$ for all i, then $p/q = (q-1)/q$, contradicting our assumption. Thus either some $a_i < 0$, or some $a_i > 4$. We separate the two cases.

CASE 1. *Some $a_i < 0$.*

Let a_i be the first negative number. Then $a_{i-1} > 0$, so there is a sign change. By [De2] there is a channel α_0 in $F_{a_{i-1}} \cup F_{a_i}$ starting at a bottom edge and ending at a top edge, where F_{a_i} is the fan in $D(p/q)$ corresponding to a_i. Let α_1 be the part of the bottom path of $D(p/q)$ from the vertex $1/0$ to the initial point of α_0, and let α_2 be the part of the top path from the end point of α_0 to the vertex p/q. Then $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ is an allowable path in $D(p/q)$. We need to show that if $c(v) = 2$ for all vertices v on this path, then $p/q = [r_1, r_2]$ for some r_1, r_2.

Consider the vertices on α_1. Since $c(v_i) = 2$ for all v_i, each vertex v_i is incident to exactly one non boundary edge e_i of $D(p/q)$, which must have the other end on a vertex v_i' in the top path. If some of these v_i' are different, then since all faces of $D(p/q)$ are triangles, it is clear that some v_j on α_1 would have at least two non boundary edges, which would be a contradiction. Similarly, each vertex on α_2 has a unique non boundary edge, leading to a common vertex on the bottom path, so the diagram $D(p/q)$ looks exactly as in Figure 3(a). It is the union of two fans F_{r_1} and F_{r_2} with $r_1 > 0$, and $r_2 < 0$. Therefore, $p/q = [r_1, r_2]$.

CASE 2. *Some* $a_i \geq 4$.

In this case there is a channel α_0 with both ends on the bottom path. Construct an allowable path $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ with α_1, α_2 in the bottom path. Similar to Case 1, it can be shown that each vertex on α_i has a unique non boundary edge leading to a common vertex v_i' on the top path, so $D(p/q)$ looks like that in Figure 3(b). In this case $p/q = [r_1, r_2]$, with both $r_i > 0$. □

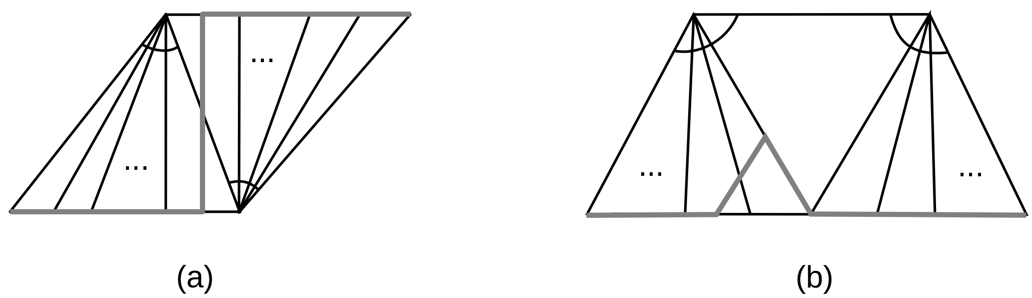

Figure 3

Lemma 5. *Let $K = K_{p/q}$ be a two bridge knot such that $p/q = [r_1, r_2]$, and $|r_i| \geq 3$ for $i = 1, 2$. Then no surgery on K is a small Seifert fibered space.*

Proof. Since K is a knot, q is odd, so at least one of the r_i is an even number. We assume without loss of generality that $r_1 = 2n$ for some integer n, since $K_{[r_1, r_2]}$ is equivalent to $K_{[r_2, r_1]}$, by turning the standard diagram for the first knot upside down.

Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p'/q' = [2, r_2, -2]$. See Figure 4, where $r_2 = -6$. Notice that after $-1/n$ surgery on k_1, the other component k_2 becomes the knot $K = K_{[2n, r_2]}$. Therefore, doing γ surgery on K is the same as doing $-1/n$ surgery on k_1, then doing some γ' surgery on k_2.

By considering the mirror image of K if necessary we may assume that $r_2 < 0$. If r_2 is even, then $[2, r_2, -2]$ is a partial fraction decomposition with even coefficient, and $r_2 \leq -4$. There is an allowable path in $D(p'/q')$ with two channels, as shown in Figure

5(a), where $r_2 = -4$. If r_2 is odd, then $p'/q' = [2, r_2+1, 2]$, in which case $D(p'/q')$ also has an allowable path with two channels. See Figure 5(b) for the case $r_2 = -3$.

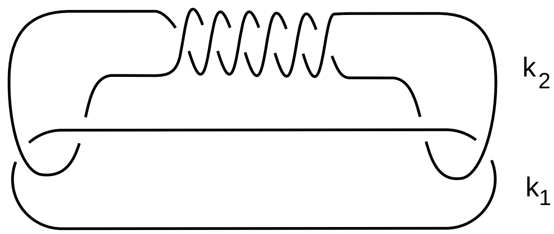

Figure 4

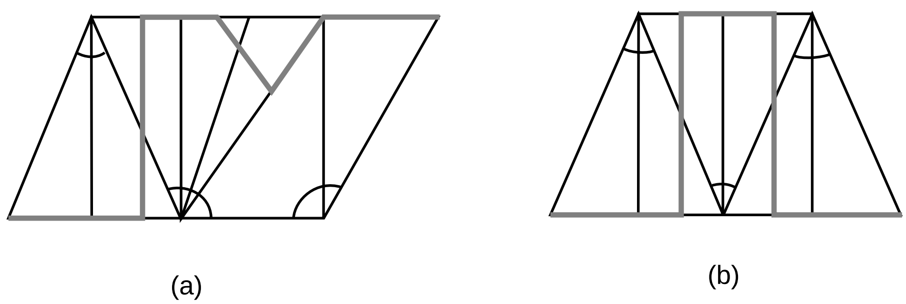

Figure 5

Let $\mathcal{F}$ be the essential branched surface in the link exterior associated to the above allowable path in $D(p'/q')$, as constructed in [De2]. There is one solid torus component V_i in $S^3 - \text{Int}N(\mathcal{F})$ for each k_i, containing k_i as a central curve. From the construction of $\mathcal{F}$ one can see that each channel contributes two cusps, one on each ∂V_i. Actually from [De2, Figure 3.5] we see that the two cusps are around two points of L on a level sphere with same orientation. Since each k_i intersects the sphere at two points with different orientations, those two cusps must be around different components of L. See [Wu] for more details about surgery on 2-bridge links.

As the allowable path above has two channels, each V_i has two meridional cusps. Thus $\mathcal{F}$ remains an essential branched surface after surgery on L. Moreover, since the surgery on k_1 has coefficient $-1/n$, which is non-integral, after surgery V_1 becomes a solid torus whose meridional disk intersects the cusps at least four times. By [Br, Corollary], the surgered manifold is not a small Seifert fibered space. □

Lemma 6. *Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p'/q' = [2, 2, -2]$. Let $L(\gamma_1, \gamma_2)$ be the manifold obtained by γ_i surgery on k_i. If $\gamma_1 = -1/n$ and $\gamma_2 = -1$, -2 or -3, then $L(\gamma_1, \gamma_2)$ is a small Seifert fibered space.*

Proof. By definition $L(\infty, \gamma_2)$ is the manifold obtained from S^3 by γ_2 surgery on k_2. After -1 surgery on k_2, the knot k_1 becomes a trefoil knot in $L(\infty, -1) = S^3$. Since the exterior of a torus knot is a Seifert fibered space with orbifold a disk with two cones, it is easy to see that all but one surgeries yield Seifert fibered spaces, each having an orbifold a disk with at most three cones. For this trefoil, the exceptional surgery has coefficient -6, yielding a reducible manifold. Thus $L(-1/n, -1)$ is a small Seifert fibered space for any n.

After -2 surgery on k_2, k_1 becomes a knot in $\mathbb{R}P^3 = L(\infty, -2)$. The link L is drawn in Figure 6(a), where the curve C is a curve on $\partial N(k_2)$ of slope -2, so it bounds a disk in $L(\infty, -2)$. Thus a band sum of k_1 and C forms a knot k_1' isotopic to k_1 in $L(\infty, -2)$. The link $L' = k_1' \cup k_2$ is shown in Figure 6(b). Using Kirby Calculus one can show that $L(-1/n, -2) = L'(-2 - 1/n, -2)$. The exterior of k_1' in S^3 is a Seifert fibered space with orbifold a disk with two cones, in which k_2 is a singular fiber of index 3. Thus after -2 surgery on k_2, the manifold $L(\infty, -2) - \mathrm{Int}N(k_1')$ is still Seifert fibered, with orbifold a disk with two cones. The fiber slope on $\partial N(k_1')$ is 6. It follows that all but the 6 surgery on k_1' in $L(\infty, -2)$ yield small Seifert fibered manifolds. In particular, $L(-1/n, -2) = L'(-2 - 1/n, -2)$ are small Seifert fibered manifolds for all n.

The proof for $\gamma_2 = -3$ is similar. One can show that the band sum of k_1 and the curve C of slope -3 on $\partial N(k_2)$ is isotopic to the curve k_1' shown in Figure 6(c), which is a $(3, -2)$ torus knot. By the same argument as above one can show that $L(-1/n, -3) = L(-3 - 1/n, -3)$ are small Seifert fibered manifolds for all n. □

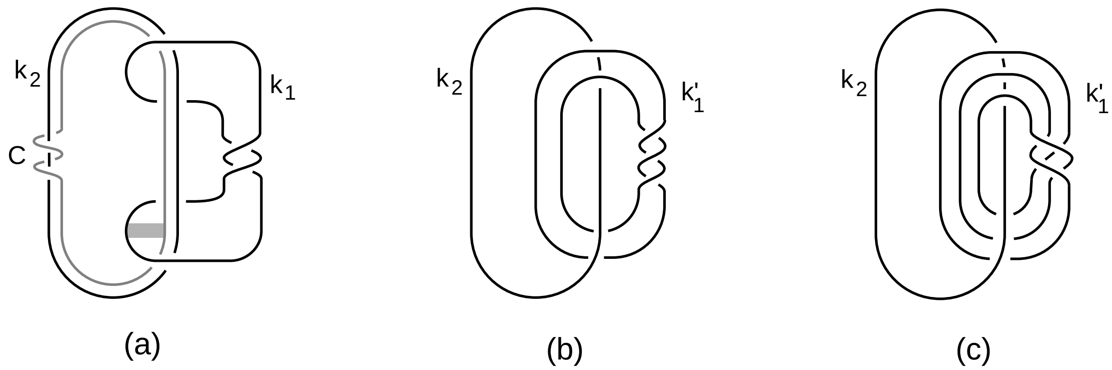

Figure 6

Corollary 7. *Let K be a non-torus 2-bridge knot.*

(1) K admits a small Seifert fibered surgery if and only if it is a twist knot;

(2) If $K = K_{p/q}$ is a twist knot with $p/q = [2n, \pm 2]$, then $K(\gamma)$ is a small Seifert fibered space for $\gamma = \mp 1$, ∓ 2 and ∓ 3.

(3) $K(\gamma)$ is not Seifert fibered unless γ is an integer.

Proof. As noticed before, up to taking the mirror image we may assume that $K = K_{p/q}$, where $1 < p < q$, p is even, and q is odd. Note that if $p/q = (q-1)/q$ then K is a $(2, q)$ torus knot. Hence if K is a non-torus knot admitting some small Seifert fibered surgery, then by Lemma 3 and Lemma 4 we must have $p/q = [b_1, b_2]$ for some integers b_1, b_2. Note that $b_i \neq \pm 1$, otherwise K is a torus knot. Therefore by Lemma 5 one of the b_i must be ± 2. In other words, K is a twist knot.

Consider the case $p/q = [b, 2]$. The proof for $p/q = [b, -2]$ is similar. Since $[b, 2] = [b-1, -2]$, we may further assume that $b = 2n$ is even. Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p'/q' = [2, 2, -2]$. Notice that after $-1/n$ surgery on k_1, the knot k_2 becomes the knot $K = K_{[b,2]}$ in $S^3 = L(-1/n, \infty)$. Therefore by Lemma 6, $K(\gamma) = L(-1/n, \gamma)$ are small Seifert fibered spaces for $\gamma = -1$, -2 and -3.

Part (3) follows from [Br]. □

Lemma 8. *Let K be a non-torus 2-bridge knot. Suppose $K(\gamma)$ is toroidal. Then*

(1) $K = K_{[b_1, b_2]}$ for some b_1, b_2.

(2) If $|b_i| > 2$ *for* $i = 1, 2$*, there is exactly one such* γ*. When both* b_i *are even,* $\gamma = 0$*. When* b_1 *is odd and* b_2 *is even,* $\gamma = 2b_2$*.*

(3) If $K = K_{[2n,2]}$ *and* $|n| > 1$*,* $K(\gamma)$ *is toroidal if and only if* $\gamma = 0$ *or* -4*. For* $K = K_{[2n,-2]}$*,* $\gamma = 0$ *or* 4*.*

(4) If $K = K_{[2,-2]}$*,* $\gamma = 0,\ 4,$ *or* -4*.*

Proof. We refer the reader to [HT] for notations. If $K(\gamma)$ is toroidal, there is an essential punctured torus T in the knot exterior. By Theorem 1 of [HT], T is carried by some $\Sigma[b_1, \dots, b_k]$, where $[b_1, \dots, b_k]$ is an expansion of p/q. By the proof of Theorem 2 of [HT], we have $0 = 2 - 2g = n(2-k)$. Therefore $k = 2$. This proves (1). The rest follows by determining all the possible expansions of type $[b_1, b_2]$ for p/q. The boundary slopes of the surfaces can be calculated using Proposition 2 of [HT]. By the proof of [Pr, Corollary 2.1], an incompressible punctured torus T in the exterior of a 2-bridge knot will become an essential torus after surgery along the slope of ∂T. $\square$

Lemma 9. *If* K *is a nontorus 2-bridge knot, then* $K(\gamma)$ *has a geometric decomposition, i.e. it is either toroidal, or Seifert fibered, or hyperbolic.*

Proof. A p/q 2-bridge knot can be obtained by taking two arcs of slope p/q on the "pillowcase", then joining the ends with two trivial arcs. From this picture it is easy to see that K is a strongly invertible knot, i.e, there is an involution φ of S^3 such that $\varphi(K) = K$, and the fixed point set of φ is a circle S intersecting K at two points. φ restricts to an involution of $E(K) = S^3 - \text{Int}N(K)$, which can be extended to an involution $\widehat{\varphi}$ of the surgered manifold. $\widehat{\varphi}$ has nonempty fixed point. Since $K(\gamma)$ is irreducible [HT], the result follows from Thurston's orbifold geometrization theorem [Th3], which says that if an irreducible, closed 3-manifold admits an orientable preserving periodic map with nonempty fixed point set, then M has a geometric decomposition. See the thesis of Q. Zhou [Zh] for a proof. $\square$

Proof of Theorem 1. Part (5) is well known: If K is a (p, q) torus knot, then $E(K)$ is a Seifert fibered space, so all but one Dehn filling are Seifert fibered. The exceptional one is the one with slope pq, producing a connected sum of two lens spaces. See [Mo].

Part (4) is also well known: If K is the Figure 8 knot, by [Th1] $K(\gamma)$ is hyperbolic unless γ is an integer between -4 and 4. Since $K = K_{[2,-2]} = K_{[-2,2]}$, by Corollary 7 and Lemma 8 $K(\gamma)$ is toroidal for $\gamma = 0, -4, 4$, and is Seifert fibered for $\gamma = -3, -2, -1, 1, 2, 3$.

If K is not a torus knot or twist knot, then by Corollary 7 and Lemma 8, $K(\gamma)$ is not Seifert fibered, and it is toroidal if and only if K and γ are as described in (2). So (1) and (2) follows from Lemma 9.

It remains to prove (3). Consider the knot $K = K_{[2n,2]}$. Using Corollary 7, Lemmas 8 and 9, we need only show that $K(\gamma)$ is hyperbolic if γ is an integer not between 0 and -4. Since $K = K_{[2n,2]}$ is a hyperbolic knot [HT], and since $K(0)$ in non hyperbolic, the 2π-theorem of Gromov-Thurston (see [BH]) and Lemma 9 imply that $K(\gamma)$ is hyperbolic unless $\Delta(0,\gamma) = |\gamma| < 23$.

Let $L = k_1 \cup k_2$ be the 2-bridge link associated to the rational number $p/q = [2, 2, -2]$. We use $L(-, \gamma_2)$ to denote the manifold obtained by removing k_1 and performing γ_2 surgery on k_2. As in the proof of Lemma 6, we have $K_{[2n,2]}(\gamma) = L(-1/n, \gamma)$.

Neumann and Reid [NR] proved that $L(-, \gamma)$ is hyperbolic unless γ is an integer between 0 and -4. Since $L(\infty, \gamma)$ is a lens space, hence non hyperbolic, the 2π-theorem and Lemma 9 imply that if γ is not between 0 and -4, then $L(-1/n, \gamma)$ is hyperbolic unless $\Delta(\infty, -1/n) = |n| < 23$. Combining this with the above result, we see that $K_{[2n,2]}(\gamma)$ is hyperbolic unless $|n| < 23$ and $|\gamma| < 23$, so there are only finitely many surgeries left to check. Finally, we use Jeff Weeks' *SnapPea* program [We] to check this finite set of surgeries. It has been verified that if $|n| > 1$ then $K_{[2n,2]}(\gamma)$ is hyperbolic unless $\gamma = 0, -1, -2, -3, -4$.

The knot $K = K_{[2n,-2]}$ is the mirror image of $K_{[-2n,2]}$, so $K_{[2n,-2]}(\gamma)$ is homeomorphic to $K_{[-2n,2]}(\gamma)$. Therefore, if $|n| > 1$ then $K(\gamma)$ is hyperbolic unless $\gamma = 0, 1, 2, 3, 4$. □

Acknowledgment. We are grateful to Alan Reid for some very useful comments to an earlier version of this paper.

References

[BH] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn filling*, Knot '90, Walter de Gruyter & Co., Berlin, New York, 1992, pp. 425 – 433.

[Br] M. Brittenham, *Small Seifert-fibered spaces and Dehn surgery on 2-bridge knots,*, preprint.

[De1] C. Delman, *Essential laminations and Dehn surgery on 2-bridge knots*, Topology and its Appl..

[De2] ———, *Constructing essential laminations which survive all Dehn surgeries*, preprint.

[GO] D. Gabai and U. Oertel, *Essential laminations in 3-manifolds*, Annals of Math. **130** (1989), 41–73.

[HT] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Inv. Math. **79** (1985), 225–246.

[Ho] C. Hodgson, *Geometric structures on 3-dimensional orbifolds: Notes on Thurston's proof,*, preprint.

[KOS] S. Kojima, K. Ohshika and T. Soma, *Towards a proof of Thurston's geometrization theorem for orbifolds*, Hyperbolic Geometry and Three Manifolds, Res. Inst. Math. Sci., Kokyuroku.

[Mo] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.

[NR] W. Neumann and A. Reid, *Arithmetic of Hyperbolic manifolds*, Topology '90, Walter de Gruyter, Berlin, New York 1992, pp. 273–310.

[Pr] J. Przytycki, *Incompressibility of surfaces after Dehn surgery*, Michigan Math. J. **30** (1983), 289 – 308.

[Th1] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University, 1978.

[Th2] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

[Th3] ———, *Three-manifolds with symmetry*, preprint.

[We] J. Weeks, *Snappea*.

[Wu] Y-Q. Wu, *Dehn surgery on arborescent links*, preprint.

[Zh] Q. Zhou, *Three dimensional geometric cone structures*, UCLA thesis (1990).

Department of Mathematics, University of Texas at Austin, Austin, TX 78712
E-mail address: britten@math.utexas.edu

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: wu@math.uiowa.edu