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An SIRS model with a nonlinear incidence rate

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Abstract

The global dynamics of an SIRS model with a nonlinear incidence rate is investigated. We establish a threshold for a disease to be extinct or endemic, analyze the existence and asymptotic stability of equilibria, and verify the existence of bistable states, i.e., a stable disease free equilibrium and a stable endemic equilibrium or a stable limit cycle. In particular, we find that the model admits stability switches as a parameter changes. We also investigate the backward bifurcation, the Hopf bifurcation and Bogdanov–Takens bifurcation and obtain the Hopf bifurcation criteria and Bogdanov–Takens bifurcation curves, which are important for making strategies for controlling a disease.

1. Introduction

In classical epidemiological models, the incidence rates are bilinear, i.e., proportional to the product of the number of infective individuals and the number of susceptible individuals [4,12,17,25–27]. Such models always admit a globally asymptotically stable disease free equilibrium or endemic equilibrium, corresponding to the disease free steady state or endemic steady state. However, actual data and evidences observed for many diseases show that dynamics of disease transmission are not always as simple as shown in these models. Thus, in recent years, researchers [1,5–8,10,11,15,18,20,21,23,24,28] have taken into account oscillations in incidence rates and proposed many nonlinear incidence rates. With these nonlinear incidence rates, many interesting and complicated transmission dynamics of epidemics have been shown, such as multiple equilibria, periodic orbits, Hopf and Bogdanov–Takens bifurcations, which state clearer and more reasonable qualitative description of the disease dynamics and give better suggestions for the prediction or control of diseases.

Let $S(t)$ be the number of susceptible individuals at time t, $I(t)$ be the number of infective individuals at time t, and $R(t)$ be the number of recovered individuals at time t. Capasso and Serio [5] introduced a saturated incidence rate $g(I)S$ in an epidemic model when they studied the cholera epidemic in 1973, where $g(I)$ is decreasing when I is large enough.

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Liu et al. studied the codimension-1 bifurcation for SEIRS and SIRS models with the incidence rate $\beta I^p S^q$ in [13,14]. Lizana and Rivero [16], Glendinning and Perry [8] and Derrick and van den Driessche [7] studied saddle-node bifurcation, Hopf bifurcation and Bogdanov–Takens bifurcation of SIRS or SIR models with the incidence rate of $\beta I^p S^q$. Ruan and Wang [21] studied saddle-node bifurcation, Hopf bifurcation, Bogdanov–Takens bifurcation and the existence of none, one and two limit cycles of an SIRS model with an incidence rate of $kI^2S/(1+\alpha I^2)$, which was also proposed by Liu et al. [13]. Derrick and van den Driessche [6] considered a very general nonlinear incidence rate. van den Driessche and Watmough [23,24] studied an incidence rate of the form $\beta I(1+\nu I^{k-1})S$, where $\beta > 0$, $\nu \geqslant 0$ and $k > 0$. When $\nu = 0$ this incidence rate is the bilinear incidence rate βIS [4]. In [23], they studied an SIS model with this incidence rate for $\nu > 0$ and $k = 2$, showing the backward bifurcation, local stability and global stability of equilibria. In [24], they further introduced the same incidence rate in an SIRS model. However, although they obtained saddle-node bifurcation for the model, they only focused on numerical examples to show possibilities of Hopf bifurcation and homoclinic orbit and to analyze the attractive area of an endemic equilibrium, paying no attention to theoretical analysis. Alexander and Moghadas [1] analyzed an SIV model with a generalized nonlinear incidence rate $f(I;v)$ and showed the existence of bistability and various Hopf bifurcation, especially for the incidence rate $\beta\ I(1+\nu I^q)S$ with $\beta > 0$, $\nu \geqslant 0$ and $0 < q \leqslant 1$. However, they neglected such dynamics as Bogdanov–Takens bifurcation. In view that former works on the incidence rate $\beta I(1+\nu I^{k-1})S$ are not complete and in detail for the effect it may have on the global transmission dynamics of a disease, in this paper we would like to continue their work and give detailed theoretical analysis of the SIRS model with the same incidence rate, so as to clearly show the effect of this nonlinear incidence rate on the transmission dynamics of epidemics. In fact, by our analysis, we have not only theoretically obtained backward bifurcation, Hopf bifurcation and Bogdanov–Takens bifurcation for this SIRS model, but also shown bistable steady states and explicit conditions for asymptotic stability of equilibria, even for globally asymptotic stability, and especially found stability switches for one of the endemic equilibria.

We still consider the incidence rate $\beta I(1+\nu I^{k-1})S$ with $\beta > 0$, $\nu > 0$ and $k = 2$ and investigate the following SIRS model:

$$\begin{aligned}
\frac{dS}{dt} &= B - dS - \beta I(1+\nu I)S + \gamma R,\\
\frac{dI}{dt} &= \beta I(1+\nu I)S - (d+\alpha)I,\\
\frac{dR}{dt} &= \alpha I - (d+\gamma)R,
\end{aligned} \tag{1.1}$$

where B is the recruitment rate of the population, d is the death rate of the population, α is the recovery rate of infective individuals, $B > 0$, $d > 0$, and $\alpha > 0$. Assume susceptible individuals will be infective after contacting with infective ones and infective ones recover from the disease with temporary immunity and then return to the susceptible class when losing immunity. γ is the rate that recovered individuals lose immunity and return to the susceptible class.

We present global qualitative and bifurcation analysis for the model and obtain rich dynamics. A disease free equilibrium exists for all parameters and is asymptotically stable when the basic reproduction number $R_0 < 1$ and unstable when $R_0 > 1$. When $R_0 > 1$ there is a unique endemic equilibrium, but when $R_0 < 1$ there may be none, one or two endemic equilibria and a backward bifurcation may emerge. As for the two endemic equilibria, one is always a saddle; the other may be stable or unstable. When R_0 increases, the stability of this latter equilibrium may change from stable to unstable then back to stable, which indicates the existence of stability switches. The system may admit bistable steady states: a stable disease free equilibrium and a stable endemic equilibrium or a stable disease free equilibrium and a stable limit cycle. In this case, the initial condition is very important for the eventual steady state the system settles into. When all equilibria are unstable, at least one stable limit cycle emerges and the disease is destined to break out periodically. The disease free equilibrium or the endemic equilibrium may be globally asymptotically stable for suitable parameters. The system may also undergo Hopf bifurcation or Bogdanov–Takens bifurcation, which are important for strategies for control of a disease. The criteria for Hopf bifurcation and appropriate curves of Bogdanov–Takens bifurcation have been obtained.

To simplify the model, adding all equations of (1.1) and denoting the number of the total population by $N(t)$ ($N = S + I + R$), we obtain

$$\frac{dN}{dt} = B - dN.$$

Then for any initial condition, $N(t)$ will tend to a constant $N_0 \equiv B/d$ when t tends to infinity. In this paper, we assume the population is in equilibrium and investigate the behavior of the system on the plane $S + I + R = N_0 > 0$. So (1.1) becomes

$$\frac{dI}{dt} = \beta I(1 + \nu I)(N_0 - I - R) - (d + \alpha)I,$$
$$\frac{dR}{dt} = \alpha I - (d + \gamma)R. \tag{1.2}$$

Rescaling (1.2) by $I_1 = \beta I/(d+\gamma)$, $R_1 = \beta R/(d+\gamma)$, $t_1 = (d+\gamma)t$ and still writing (I_1, R_1, t_1) as (I, R, t), we obtain

$$\frac{dI}{dt} = I(1 + pI)(A - I - R) - \frac{A}{R_0}I,$$
$$\frac{dR}{dt} = qI - R, \tag{1.3}$$

where $p = \nu(d+\gamma)/\beta$, $A = \beta N_0/(d+\gamma)$, $R_0 = \beta N_0/(d+\alpha)$, $q = \alpha/(d+\gamma)$, $p > 0$, $A > 0$, $R_0 > 0$, $q > 0$.

The organization of this paper is as follows. In the next section, we present a qualitative analysis of (1.3). We analyze the existence and stability of equilibria, find stability switches for an endemic equilibrium and the existence of a stable limit cycle and bistable states, and analyze the nonexistence of limit cycles and globally asymptotic stability of equilibria. In Section 3, we show that the system admits Hopf bifurcation. In Section 4, we show that the system undergoes a Bogdanov–Takens bifurcation at the degenerate equilibrium.

2. Qualitative analysis

2.1. Disease free equilibrium

Eq. (1.3) admits a unique disease free equilibrium $(0,0)$. The Jacobian matrix of (1.3) at $(0,0)$ is

$$M_0 = \begin{pmatrix} A - \frac{A}{R_0} & 0 \\ q & -1 \end{pmatrix}.$$

It is easy to obtain the following result:

Theorem 2.1. $(0,0)$ *is asymptotically stable if* $R_0 < 1$, *and unstable if* $R_0 > 1$.

2.2. Endemic equilibria

To obtain endemic equilibria of (1.3), it suffices to consider the following equations:

$$I(1 + pI)(A - I - R) - \frac{A}{R_0}I = 0,$$
$$qI - R = 0, \tag{2.1}$$

which yield

$$p(q+1)I^2 + (q + 1 - Ap)I + \frac{A}{R_0} - A = 0. \tag{2.2}$$

This equation may admit positive solutions

$$I_1 = \frac{Ap - q - 1 - \sqrt{(q + 1 - Ap)^2 - 4Ap(q+1)(1 - R_0)/R_0}}{2p(q+1)},$$
$$I_2 = \frac{Ap - q - 1 + \sqrt{(q + 1 - Ap)^2 - 4Ap(q+1)(1 - R_0)/R_0}}{2p(q+1)}.$$

Set $R_1 = q\,I_1$, $R_2 = q\,I_2$, $I_* = (I_1 + I_2)/2$, $R_* = qI_*$ and $R_0^* \equiv 4p(q+1)A/(q+1+Ap)^2$. Note that $R_0^* \leqslant 1$. We can state the following results

(1) Let $p \leqslant (q+1)/A$. Then (1.3) admits a unique endemic equilibrium (I_2, R_2) when $R_0 > 1$, and has no endemic equilibrium when $R_0 \leqslant 1$.
(2) Let $p > (q+1)/A$. Then (1.3) admits a unique endemic equilibrium (I_2, R_2) when $R_0 \geqslant 1$, no endemic equilibrium when $R_0 < R_0^*$, and two endemic equilibria $E_1 = (I_1, R_1)$ and $E_2 = (I_2, R_2)$ when $R_0^* \leqslant R_0 < 1$. When $R_0 = R_0^*$, $E_1 = E_2 = E_* = (I_*, R_*)$.

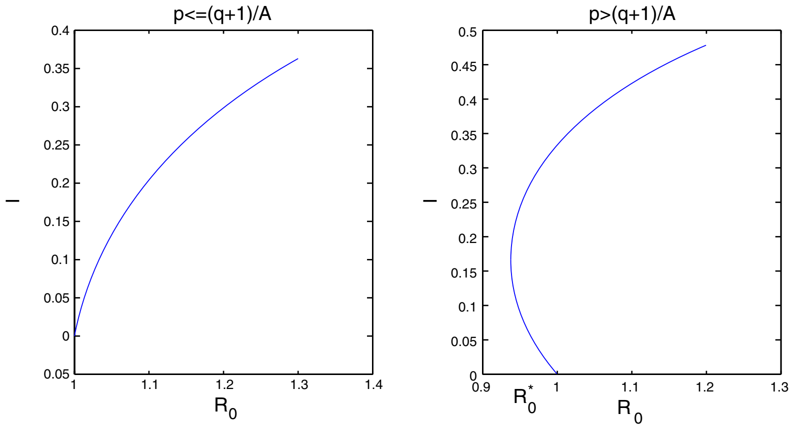


Fig. 1. The relationship between R_0 and endemic equilibria (I).

From the conclusion (2), we see that (1.3) has a backward bifurcation (see the right graph of Fig. 1).

Let us now consider the stability of the endemic equilibrium E_1. The Jacobian matrix of (1.3) at E_1 is

$$M_1 = \begin{pmatrix} A - qI_1 - \frac{A}{R_0} - 2I_1 - 3pI_1^2 + 2pAI_1 - 2pqI_1^2 & -I_1 - pI_1^2 \\ q & -1 \end{pmatrix}.$$

It is easy to obtain

$$\det(M_1) = -\sqrt{(q+1-Ap)^2 - 4Ap(q+1)(1-R_0)/R_0}I_1 < 0$$

when $R_0 > R_0^*$. Hence E_1 is a saddle.

The Jacobian matrix of (1.3) at E_2 is

$$M_2 = \begin{pmatrix} A - qI_2 - \frac{A}{R_0} - 2I_2 - 3pI_2^2 + 2pAI_2 - 2pqI_2^2 & -I_2 - pI_2^2 \\ q & -1 \end{pmatrix}.$$

Since

$$\det(M_2) = \sqrt{(q+1-Ap)^2 - 4Ap(q+1)(1-R_0)/R_0}I_2 > 0$$

when $R_0 > R_0^*$, E_2 is a node or a focus.

To obtain the stability of E_2, we consider

$$\begin{aligned} \mathrm{tr}(M_2) &= A - \frac{A}{R_0} - qI_2 - 2I_2 - 3pI_2^2 + 2pAI_2 - 2pqI_2^2 - 1 \\ &= \frac{1}{q+1}\left\{[(q+1)^2 - Ap]I_2 + \frac{A(1-R_0)}{R_0}(q+2) - (q+1)\right\}. \end{aligned}$$

Let us locate the set where $\mathrm{tr}(M_2) = 0$. When $(q+1)^2 - Ap \neq 0$, by the expression of I_2 we see that $\mathrm{tr}(M_2) = 0$ if and only if

$$aR_0^2 - bR_0 + c = 0, \tag{2.3}$$

$$R_0 \geqslant R_0^*, \tag{2.4}$$

$$G(R_0) \geqslant 0, \tag{2.5}$$

where

$$\begin{aligned} a &= p(q^2 + Aq + 2q + A^2p + A + 1)(p + Ap + 1 + q), \\ b &= Ap(q^2Ap + 2pq^2 + q^2 + 2q + 6pAq + 6pq + p^2A^2 + 6Ap + 1 + 4p), \\ c &= p^2A^2(q+2)^2, \\ G(R_0) &\equiv \frac{((q+1)^3 + 2pq^2 + q^2Ap + 3pAq + 4pq + 2Ap + 2p + p^2A^2)R_0 - 2Ap(q+2)(q+1)}{R_0(2q - Ap + q^2 + 1)}. \end{aligned}$$

We consider (2.3) as a quadratic equation of R_0. Set $\Delta_1 = b^2 - 4ac$. Then the solutions of (2.3) are

$$R_{01} = \frac{A(2pq^2 + q^2Ap + q^2 + 6pq + 6Apq + 2q + 4p + 1 + 6Ap + A^2p^2 - \sqrt{\Delta_1})}{2(p + Ap + 1 + q)(q^2 + Aq + 2q + A^2p + A + 1)},$$

$$R_{02} = \frac{A(2pq^2 + q^2Ap + q^2 + 6pq + 6Apq + 2q + 4p + 1 + 6Ap + A^2p^2 + \sqrt{\Delta_1})}{2(p + Ap + 1 + q)(q^2 + Aq + 2q + A^2p + A + 1)}.$$

Let

$$p_1 \equiv \frac{A + 2q + 4 - 2\sqrt{(A + q + 2)(q + 2)}}{A^2},$$

$$p_2 \equiv \frac{A + 2q + 4 + 2\sqrt{(A + q + 2)(q + 2)}}{A^2},$$

$$p_0 \equiv \frac{(4q + 4 + 2\sqrt{4q^2 + 8q + 4 + A^2q^2})(q + 1)}{2A^2q},$$

$$A_0 \equiv \frac{4(1 + q)^2}{q^2(q + 2)}.$$

Then if $(q + 1)^2 - Ap \neq 0$, (2.3) admits two positive solutions R_{01} and R_{02} when $p \geqslant p_2$ or $p \leqslant p_1$, and admits no positive solution when $p_1 < p < p_2$. Moreover, $R_0^* \leqslant R_{01} \leqslant R_{02}$ and $R_0^* = R_{01}$ when $p = p_0$. By further elementary calculation of $G(R_0)$, including the case when $(q + 1)^2 - Ap = 0$, we can state the set where $\mathrm{tr}(M_2) = 0$ as follows:

Since $\det(M_2) > 0$, the stability of E_2 is decided by the sign of $\mathrm{tr}(M_2)$. It is locally asymptotically stable when $\mathrm{tr}(M_2) < 0$ and unstable when $\mathrm{tr}(M_2) > 0$. Thus it suffices to investigate the sign of $\mathrm{tr}(M_2)$ to obtain the locally asymptotic stability of E_2. First, we point out relationships among some quantities: $p_2 = p_0 = \frac{(q+1)^2}{A}$ if $A = A_0$; $\frac{(q+1)^2}{A} < p_2 < p_0$ if $A < A_0$; $p_2 < p_0 < \frac{(q+1)^2}{A}$ if $A > A_0$. Notice that $\mathrm{tr}(M_2)$ is a continuous function of p and R_0 when $R_0 \geqslant R_0^*$; $\mathrm{tr}(M_2)|_{R_0=R_0^*}$ is increasing in p and $\mathrm{tr}(M_2)|_{R_0=R_0^*} = 0$ when $p = p_0$; $\mathrm{tr}(M_2) < 0$ when $R_0 \to +\infty$. Then by virtue of the set where $\mathrm{tr}(M_2) = 0$, we can show the stability of E_2 in Tables 2 and 3.

Remark 2.1. From above analysis and Table 3, we see that there exist stability switches for E_2. If $p_2 < p_0 < \frac{(q+1)^2}{A}$ and $p_2 < p < p_0$ (case (b)), then as R_0 increases from R_0^*, E_2 changes its stability from being stable to being unstable at $R_0 = R_{01}$ and then from being unstable to being stable at $R_0 = R_{02}$, which may result in two Hopf bifurcations and is important for the strategies for control of the disease.

Now we give some results about the nonexistence of limit cycles and globally asymptotic stability of $(0,0)$ and E_2.

Lemma 2.1. *The solutions of (1.3) with $(I(0), R(0)) \in R_+^2$ are bounded.*

Proof. By the first equation of (1.3), $\mathrm{d}I/\mathrm{d}t < 0$ when I is large enough. Then for any solution $(I(t), R(t))$ of (1.3), there exists an $I^0 > 0$ and $t_0 > 0$ such that $I(t) \leqslant I^0$ when $t \geqslant t_0$. Similarly, there exists an $R^0 > 0$ and $t_1 > 0$ such that $R(t) \leqslant R^0$ when $t \geqslant t_1$. Note that $\mathrm{d}I/\mathrm{d}t = 0$ when $I = 0$ and $\mathrm{d}R/\mathrm{d}t \geqslant 0$ when $R = 0, I \geqslant 0$. Thus all positive semi-flows of (1.3) with $I(0) \geqslant 0$, $R(0) \geqslant 0$ finally lie in $[0, I^0] \times [0, R^0]$. □

Theorem 2.2. *For (1.3), if $p \leqslant \frac{q+1}{A}$ and $R_0 < 1$ or $p \geqslant \frac{q+1}{A}$ and $R_0 < R_0^*$, then $(0,0)$ is globally asymptotically stable.*

Proof. In both of these two cases, $(0,0)$ is the unique equilibrium of (1.3). By the right side of (1.3), R axis is invariable and orbits do not go across it. Thus (1.3) admits no limit cycles. Then by Lemma 2.1 and Poincaré–Bendixson Theorem, the disease free equilibrium $(0,0)$ is globally asymptotically stable. □

Theorem 2.3. *If $p < \frac{1}{A}$ or $\frac{(q+1)^2}{A} \leqslant p_2 \leqslant p_0$, then for (1.3) E_2 is globally asymptotically stable when $R_0 > 1$.*

Proof. Suppose $p < \frac{1}{A}$. Taking Dulac function $D = 1/(IR)$, we obtain

$$\frac{\partial(Df_1)}{\partial I} + \frac{\partial(Df_2)}{\partial R} = -\frac{1}{R} + \frac{Ap}{R} - 2pI - p - \frac{q}{R^2},$$

where (f_1, f_2) is the vector field of (1.3). Obviously, $\frac{\partial(Df_1)}{\partial I} + \frac{\partial(Df_2)}{\partial R} < 0$ when $p < \frac{1}{A}$. Then by the Dulac's criteria, (1.3) admits no limit cycles or separatrix cycles. Since (1.3) admits only two equilibria $(0,0)$ and E_2 when $R_0 > 1$ and $(0,0)$ is unstable, by Lemma 2.1 and the Poincaré–Bendixson Theorem, E_2 is globally asymptotically stable.

Let us now consider $\frac{(q+1)^2}{A} \leqslant p_2 \leqslant p_0$. It suffices to consider the case where $p \geqslant \frac{1}{A}$. If $p \geqslant \frac{q+1}{A}$ and $R_0 > 1$, the unique endemic equilibrium E_2 of 1.3) is locally asymptotically stable and $(0,0)$ is a saddle. Then if the system admits a limit cycle, it must surround E_2 and lies in the first quadrant of I–R plane. Suppose Γ is a closed orbit surrounding E_2 in the positive I–R plane. Then

$$\begin{aligned}\Delta &= \int_\Gamma \operatorname{div}\left(\frac{dI}{dt}, \frac{dR}{dt}\right) dt = \int_\Gamma \left[(1+2pI)(A-I-R) - \frac{A}{R_0} - I - pI^2 - 1\right] dt \\ &= \int_\Gamma \left[(2+2pI)(A-I-R) - 2\frac{A}{R_0} + \frac{A}{R_0} - (A-I-R) - I - pI^2 - 1\right] dt \\ &= \int_\Gamma \frac{2dI}{I} + \int_\Gamma \left(\frac{A}{R_0} - A + R - pI^2 - 1\right) dt = \int_\Gamma \left(\frac{A}{R_0} - A\right) dt + \int_\Gamma (qI - pI^2 - 1)\, dt.\end{aligned}$$

By $R_0 > 1$, we have $\int_\Gamma \left(\frac{A}{R_0} - A\right) dt < 0$. Since $p \geqslant \frac{q+1}{A} \geqslant \frac{q^2(q+2)(q+1)}{4(q+1)^2} > \frac{q^2}{4}$, we have $-pI^2 + qI - 1 < 0$. Hence $\Delta < 0$. By the Dulac's criteria Γ is asymptotically stable. Thus, any limit cycle that surrounds E_2 is stable. Note that E_2 is stable. We have a contradiction. Thus (1.3) admits no limit cycles. Then by the Poincaré–Bendixson Theorem, E_2 is globally asymptotically stable.

When $\frac{1}{A} \leqslant p \leqslant \frac{q+1}{A}$, if $0 \leqslant q \leqslant \frac{-1+\sqrt{5}}{2}$, then $p \geqslant \frac{1}{A} \geqslant \frac{q^2(q+2)}{4(q+1)^2} \geqslant \frac{q^2}{4}$ since $A \leqslant A_0$. Thus $-pI^2 + qI - 1 < 0$. Then previous discussions show that E_2 is globally asymptotically stable. If $q > \frac{-1+\sqrt{5}}{2}$, the Dulac function $1/(IR)$ leads to the global stability of E_2. □

2.3. Bistable states

To obtain conditions where bistable states exist, we need to relate the stability condition of $(0,0)$ to that of E_2. Since the stability of $(0,0)$ is decided by whether $R_0 > 1$ or $R_0 < 1$, and the stability of E_2 is decided by the sign of $\operatorname{tr}(M_2)$,

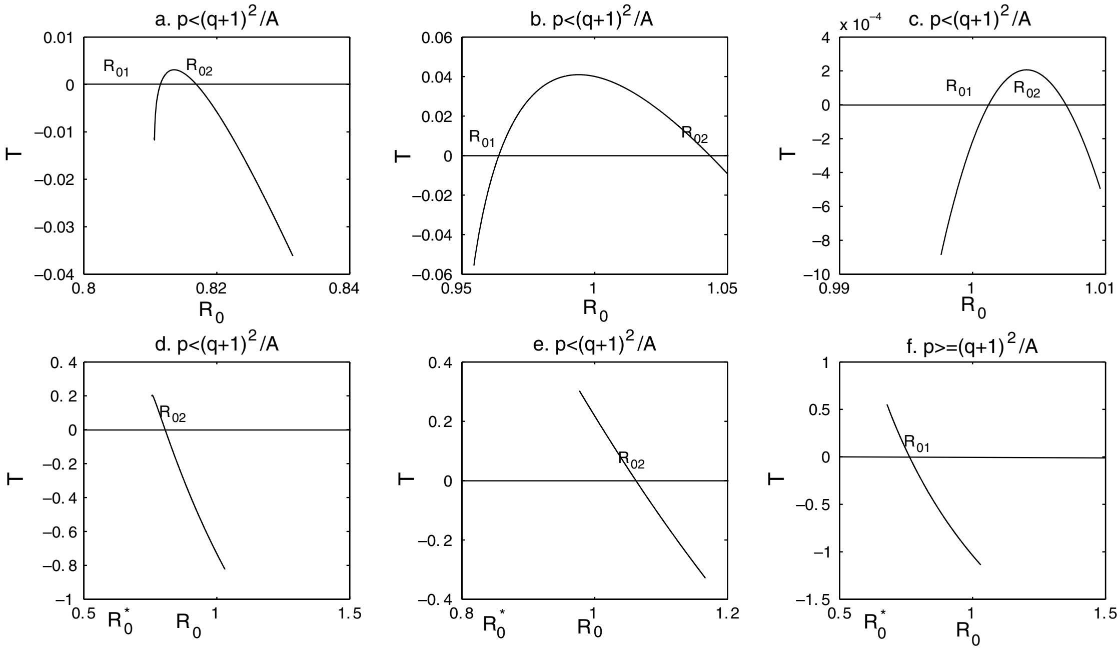


Fig. 2. Possible relationships between R_{0i} $(i = 1,2)$ and *1*, shown in the graph of $T \equiv \operatorname{tr}(M_2)$. For (a), $A = 2.577539755$, $q = 2.5$, $p = 3.45$, and $R_{01} < R_{02} < 1$; for (b), $A = 4.2$, $q = 3.8$, $p = 1.813965302$, and $R_{01} < 1 < R_{02}$; for (c), $A = 4.2$, $q = 3.8$, $p = 1.759434070$, and $1 < R_{01} < R_{02}$; for (d), $A = 2.577539755$, $q = 2.5$, $p = 4$, and $R_{02} < 1$; for (e), $A = 4.2$, $q = 3.8$, $p = 2.791922839$, and $1 < R_{02}$; for (f), $A = 2.577539755$, $q = 2.5$, $p = 4.85$, and $R_{01} < 1$.

which depends on whether $R_0 > R_{0i}$ or $R_0 < R_{0i}$, $i = 1$ or 2, it suffices to decide if $R_{0i} < 1$ or $R_{0i} > 1$ to know whether bistability exists. Since the relationship between $\mathrm{tr}(M_2)$ and R_0 has been completely known, we can do this by identifying the sign of $\mathrm{tr}(M_2)$ at $R_0 = 1$.

By the expression of $\mathrm{tr}(M_2)$, we see that $\mathrm{tr}(M_2) < 0$ at $R_0 = 1$ if $p \geqslant \frac{(q+1)^2}{A}$. Then by Tables 2 and 3, we see $R_{01} < 1$ when $p \geqslant \frac{(q+1)^2}{A}$. Thus if $p \geqslant \frac{(q+1)^2}{A}$, then when $R_{01} < R_0 < 1$, $R_0 < 1$ and $\mathrm{tr}(M_2) < 0$ hold simultaneously, which implies that both $(0,0)$ and E_2 are locally asymptotically stable (an example is shown in Fig. 3). Then the initial state of the disease decides whether it will be eradicated or endemic.

When $p < \frac{(q+1)^2}{A}$, calculations show that $\mathrm{tr}(M_2)$ may be positive or negative at $R_0 = 1$. Since we know the relationship between $\mathrm{tr}(M_2)$ and R_0, we can mark the possible position of $R_0 = 1$ on the graph of $\mathrm{tr}(M_2)$ with respect to R_0 (see Fig. 2). By means of this, it is easy to see whether $R_{0i} < 1$ or $R_{0i} > 1$ ($i = 1$ or 2) so as to find the condition where bistable states exist. In fact, From Fig. 2 we see that if $p < \frac{(q+1)^2}{A}$, then $R_0 < 1$ and $\mathrm{tr}(M_2) < 0$ also hold simultaneously for many parameter relationships, which also implies the bistability of $(0,0)$ and E_2.

Furthermore, we show that in some cases, (1.3) even admits a stable disease free equilibrium $(0,0)$ and a stable limit cycle simultaneously. Thus the initial state of the disease decides whether it will be eradicated or break out periodically. For example, if $p_2 < p_0 < \frac{(q+1)^2}{A}$ (i.e., $A > A_0$) and $p = \frac{(q+1)^2}{A}$, then when $R_0^* < R_0 < R_{01}$, (1.3) admits a stable equilibrium $(0,0)$ and a stable limit cycle simultaneously (this will be proved in Section 3). Another example is also shown in Fig. 4, where $p_2 < p_0 < \frac{(q+1)^2}{A}$, $p_0 < p < \frac{(q+1)^2}{A}$ and $R_0^* < R_0 < 1$ (Fig. 2e).

Since E_1 must be unstable, by Poincaré–Bendixson Theorem, if both $(0,0)$ and E_2 are unstable (see Fig. 2c), the system must admit at least one stable closed orbit surrounding E_2. Then the ω – limit set for all positive solutions except E_1, E_2 and stable manifolds of $(0,0)$ and E_1 must be limit cycles, which indicates that the disease is destined to break out periodically. An example is shown in Fig. 5.

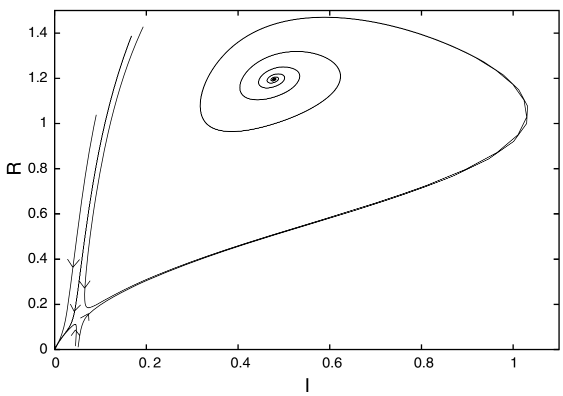


Fig. 3. Bistable equilibria. $A = 2.577539755$, $q = 2.5$, $p = 4.85$, $R_0 = 0.85918$. Equilibria of (1.3) are: $(0,0)$ (stable node), $(0.052044, 0.13011)$ (saddle) and $(0.47821, 1.1955)$ (stable focus).

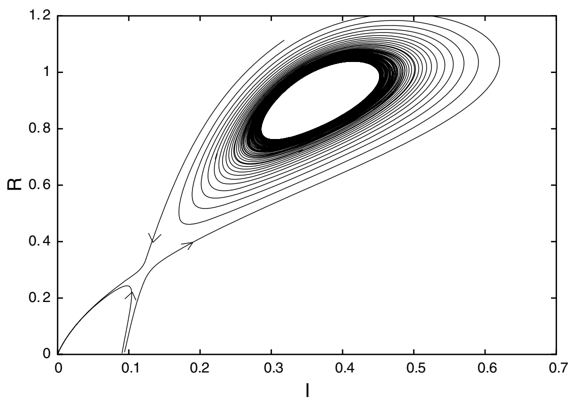


Fig. 4. A stable equilibrium and a stable limit cycle. $A = 2.577539755$, $q = 2.5$, $p = 4$, $R_0 = 0.8054811734$. Equilibria of (1.3) are $(0,0)$ (stable node), $(0.122, 0.305)$ (saddle) and $(0.36444, 0.9111)$ (unstable focus).

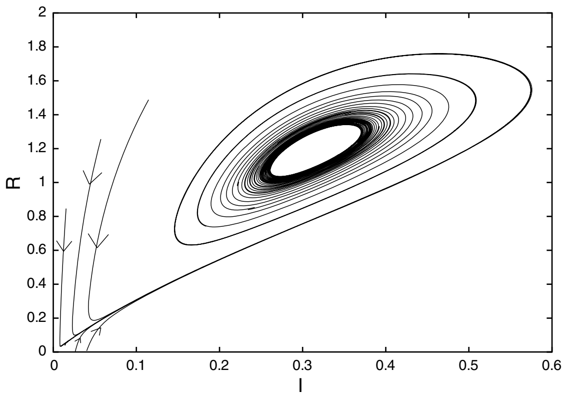


Fig. 5. Two unstable equilibria and a stable limit cycle. $A = 4.2$, $q = 3.8$, $p = 1.759434070$, $R_0 = 1.004117831$. Equilibria of (1.3) are $(0,0)$ (saddle) and $(I_2, R_2) = (0.3131482413, 1.189963317)$ (unstable focus).

3. Hopf bifurcation

In this section, we study the Hopf bifurcation of (1.3). Since R axis is invariant and E_1 must be a saddle, there is no closed orbit surrounding $(0,0)$ or E_1. Thus Hopf bifurcations only emerge at E_2. For (1.3), let $x = I - I_2$, $y = R - R_2$ to translate E_2 to $(0,0)$. Then (1.3) becomes

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= a_{11}x + a_{12}y + f_1(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= a_{21}x + a_{22}y,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}a_{11} &= -3pI_2^2 - R_2 - 2I_2 + 2pI_2A + A - 2pI_2R_2 - \frac{A}{R_0},\\ a_{12} &= -I_2 - pI_2^2,\\ a_{21} &= q,\\ a_{22} &= -1,\\ f_1(x,y) &= (-1 - 2pI_2)xy - px^3 + (pA - 1 - py - 3pI_2 - pR_2)x^2.\end{aligned}$$

To obtain the Hopf bifurcation, we fix parameters such that $\mathrm{tr}(M_2) = 0$, which is equivalent to $a_{11} = 1$.

Let $X = x,\ Y = x + a_{12}y$. Then (3.1) is reduced to

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= Y + f_1\left(X, \frac{Y - X}{a_{12}}\right),\\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= -kX + f_1\left(X, \frac{Y - X}{a_{12}}\right),\end{aligned} \tag{3.2}$$

where $k = -(1 + qa_{12})$. In fact, $k = \det(M_2)$. Then $k = 0$ when $R_0 = R_0^*$ and $k > 0$ when $R_0 > R_0^*$. We will find in the next section that (1.3) may admit Bogdanov–Takens bifurcation when $k = 0$.

Here we assume $k > 0$.

Let $u = -X, v = Y/\sqrt{k}$. We obtain the normal form for the Hopf bifurcation

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= -\sqrt{k}v + F_1(u,v),\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \sqrt{k}u + F_2(u,v),\end{aligned} \tag{3.3}$$

where $F_1(u,v) = -f_1(-u, (v\sqrt{k} + u)/a_{12})$, $F_2(u,v) = -F_1(u,v)/\sqrt{k}$.

Set the Liapunov number by

$$\sigma = \frac{1}{16}\left[\frac{\partial^3 F_1}{\partial u^3} + \frac{\partial^3 F_1}{\partial u \partial v^2} + \frac{\partial^3 F_2}{\partial u^2 \partial v} + \frac{\partial^3 F_2}{\partial v^3}\right] + \frac{1}{16}\left[\frac{\partial^2 F_1}{\partial u \partial v}\left(\frac{\partial^2 F_1}{\partial u^2} + \frac{\partial^2 F_1}{\partial v^2}\right) - \frac{\partial^2 F_2}{\partial u \partial v}\left(\frac{\partial^2 F_2}{\partial u^2} + \frac{\partial^2 F_2}{\partial v^2}\right) - \frac{\partial^2 F_1}{\partial u^2}\frac{\partial^2 F_2}{\partial u^2} + \frac{\partial^2 F_1}{\partial v^2}\frac{\partial^2 F_2}{\partial v^2}\right]$$

which can be reduced to

$$\sigma = -\frac{3}{8}p + \frac{p}{4a_{12}} + \frac{1}{16k}\left(6pI_2 + 2pR_2 - 2pA + 2 - 2\frac{1+2pI_2}{a_{12}}\right)[6pI_2 + 2pR_2 - 2pA + 2 + q(1+2pI_2)].$$

Note that for values of R_0 that satisfy $\mathrm{tr}(M_2) = 0$, $\frac{\mathrm{d}(\mathrm{tr}(M_2))}{R_0}|_{R_0=R_{01}} > 0$ when $p_2 < p_0 < \frac{(q+1)^2}{A}$ and $p_2 < p < p_0$; otherwise, $\frac{\mathrm{d}(\mathrm{tr}(M_2))}{R_0}|_{R_0=R_{0i}} < 0$, $i = 1$ or 2. Then by results in [9] or [19] and the conditions for $\mathrm{tr}(M_2) = 0$ in the last section, we have the following Hopf bifurcation results:

Theorem 3.1. *For* (1.3), *assume* $\mathrm{tr}(M_2) = 0$, *i.e., one of the following holds* (*see* Tables 1–3):

(1) $p_2 < p_0 < \frac{(q+1)^2}{A}$, $p_2 < p \leqslant \frac{(q+1)^2}{A}$, $R_0 = R_{02}$;
(2) $p_2 < p_0 < \frac{(q+1)^2}{A}$, $p_2 < p < p_0$, $R_0 = R_{01}$;

Table 1
The set where $\mathrm{tr}(M_2) = 0$

Conditions	Values of R_0 satisfying $\mathrm{tr}(M_2) = 0$
$p_2 \leqslant p < \frac{(q+1)^2}{A}$	R_{02}
$p < \min\left\{\frac{(q+1)^2}{A}, p_0\right\}$ or $p > \max\left\{\frac{(q+1)^2}{A}, p_0\right\}$	R_{01}
$p = \frac{(q+1)^2}{A}$ and $A > A_0$	$R_0 = \frac{A(q+2)}{A(q+2)+q+1} (= R_{01})$

Table 2
The locally asymptotic stability of E_2 when $\frac{(q+1)^2}{A} \leqslant p_2 \leqslant p_0$ (i.e., $A \leqslant A_0$)

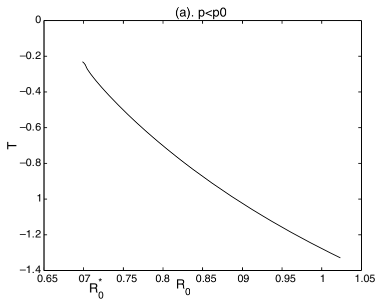
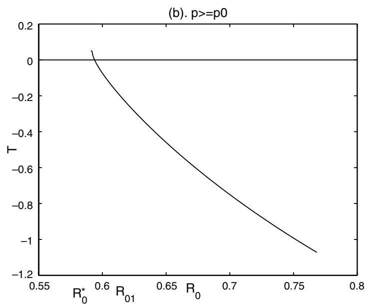
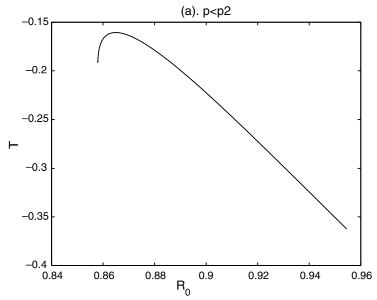
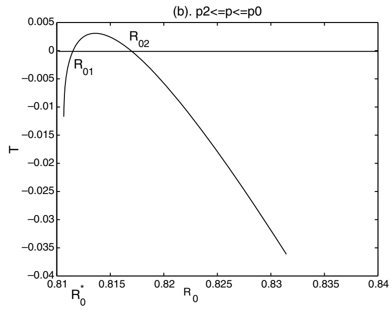
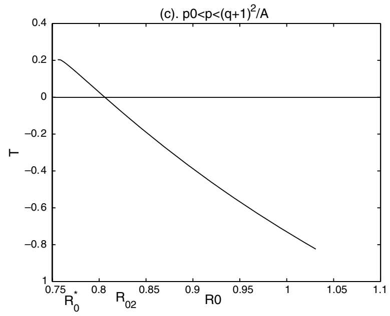
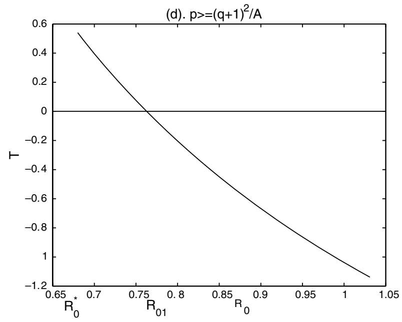
Conditions	Stability of E_2	Graph of $T = \mathrm{tr}(M_2)$
(a) $p < p_0$	E_2 is locally stable	
(b) $p \geqslant p_0$	When $R_0^* < R_0 < R_{01}$, E_2 is unstable; when $R_0 > R_{01}$, E_2 is locally stable	

Table 3
The locally asymptotic stability of E_2 when $p_2 < p_0 < \frac{(q+1)^2}{A}$ (i.e., $A > A_0$)

Conditions	Stability of E_2	Graph of $T = \mathrm{tr}(M_2)$
(a) $p < p_2$	E_2 is locally stable	
(b) $p_2 < p \leqslant p_0$	When $R_0^* < R_0 < R_{01}$, or $R_0 > R_{02}$, E_2 is locally stable; when $R_{01} < R_0 < R_{02}$, E_2 is unstable	
(c) $p_0 < p < \frac{(q+1)^2}{A}$	When $R_0^* < R_0 < R_{02}$, E_2 is unstable; when $R_0 > R_{02}$, E_2 is locally stable	
(d) $p \geqslant \frac{(q+1)^2}{A}$	When $R_0^* < R_0 < R_{01}$, E_2 is unstable; when $R_0 > R_{01}$, E_2 is locally stable	

(3) $p_2 < p_0 < \frac{(q+1)^2}{A}$, $p > \frac{(q+1)^2}{A}$, $R_0 = R_{01}$;
(4) $\frac{(q+1)^2}{A} \leqslant P_2 \leqslant p_0$, $p > p_0$, $R_0 = R_{01}$.

Suppose R_{0h} is the exact value of R_0 in above conditions which implies $\mathrm{tr}(M_2) = 0$. *Then if $\sigma < 0$, for* (1), (3) *and* (4) ((2)), *a stable limit cycle bifurcates from E_2 as R_0 decreases* (*increases*) *from R_{0h}*; *if $\sigma > 0$, for* (1), (3) *and* (4) ((2)), *an unstable limit cycle bifurcates from E_2 as R_{0h} increases* (*decreases*) *from R_{0h}.*

Example 3.1. Suppose $p_2 < p_0 < \frac{(q+1)^2}{A}$ and $p = \frac{(q+1)^2}{A}$,

$$\sigma = -\frac{(Aq^3 - 4q^2 + 2q^2A - 8q - 4)(q^2A + 2qA + \sqrt{A(q+2)(Aq^3 - 4q^2 + 2q^2A - 8q - 4)})}{2(q+2)^2(q+1)A}.$$

We can obtain $\sigma < 0$ by $A > A_0$ and the existence condition of E_2. Thus when R_0 decreases from $R_{0h} = \frac{A(q+2)}{(A(q+2)+q+1)}$, a stable limit cycle bifurcates from E_2. Then as has mentioned in Section 2, a stable equilibrium $(0,0)$ and a stable limit cycle exist simultaneously in (1.3).

4. Bogdanov–Takens bifurcation

The Bogdanov–Takens bifurcation (for short, BT bifurcation) is a type of codimension-2 bifurcation that emerges when (1.3) admits a unique degenerate equilibrium. Assume the following two assumptions hold.

(S1) $Ap - q - 1 > 0$.
(S2) $R_0 = R_0^*$.

Then (1.3) admits a unique endemic equilibrium: $E_* = (I_*, R_*) = \left(\frac{Ap-q-1}{2p(q+1)}, \frac{(Ap-q-1)q}{2p(q+1)}\right)$.

Let $x = I - I_*$, $y = R - R_*$ to translate E_* to $(0,0)$. (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= b_{11}x + b_{12}y + g_1(x,y), \\ \frac{dy}{dt} &= b_{21}x + b_{22}y, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} b_{11} &= -3pI_*^2 - R_* - 2I_* + 2pI_*A + A - 2pI_*R_* - A/R_0, \\ b_{12} &= -I_* - pI_*^2, \\ b_{21} &= q, \\ b_{22} &= -1, \\ g_1(x,y) &= (-1 - 2pI_*)xy - px^3 + (pA - 1 - py - 3pI_* - pR_*)x^2. \end{aligned}$$

It is not difficult to see

$$\begin{aligned} b_{11} &= \frac{(Ap-q-1)(Ap+q+1)q}{4p(q+1)^2} > 0, \\ b_{12} &= -\frac{(Ap-q-1)(Ap+q+1)}{4p(q+1)^2} < 0. \end{aligned}$$

Then by (S1) and (S2), the linearized matrix of (1.3) at E_* is

$$M_* = \begin{pmatrix} b_{11} & b_{12} \\ q & -1 \end{pmatrix}$$

and $\det(M_*) = -b_{11} - b_{12}q = 0$.

To obtain a codimension-2 bifurcation, we further assume $\mathrm{tr}(M_*) = 0$, i.e., $b_{11} = 1$. By (S2) and analysis in Section 2, this is equivalent to

(S3) $p = p_0$.

Then $b_{12} = -1/q$. By (S1)–(S3), (4.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= x - (1/q)y + c_{21}x^2 + c_{22}xy + P(x,y), \\ \frac{dy}{dt} &= qx - y, \end{aligned} \tag{4.2}$$

where $c_{21} = pA - 1 - 3pI_* - pR_*$, $c_{22} = -1 - 2pI_*$, $P(x,y) = -px^3 - px^2y$.

Let $X = x,\ Y = x - y/q$. (4.2) becomes

$$\begin{aligned} \frac{dX}{dt} &= Y + (c_{21} + qc_{22})X^2 - c_{22}qXY + P_1(X, Y), \\ \frac{dY}{dt} &= (c_{21} + qc_{22})X^2 - c_{22}qXY + P_1(X, Y), \end{aligned} \tag{4.3}$$

where $P_1(X, Y) = -p(q+1)X^3 + pqX^2Y$.

Let $u = X + c_{22}q/2,\ v = Y + (c_{21} + qc_{22})X^2$. We obtain

$$\begin{aligned} \frac{du}{dt} &= v + Q_1(u, v), \\ \frac{dv}{dt} &= (c_{21} + qc_{22})u^2 + (2c_{21} + qc_{22})uv + Q_2(u, v), \end{aligned} \tag{4.4}$$

where $Q_1(u, v),\ Q_2(u, v)$ are polynomials in (u, v) of at least the third order. Then we have

$$c_{21} + qc_{22} = pA - 1 - 3pI_* - pR_* - q - 2pqI_* = -\frac{Ap - q - 1}{2} < 0,$$

$$2c_{21} + qc_{22} = 2pA - 2 - 6pI_* - 2pR_* - q - 2pqI_* = \frac{(q+1)^2 - Ap}{q+1}.$$

By Theorem 3 in Section 2.11 in [19], we see that $(0, 0)$ of (4.4) (i.e., E_* of (1.3)) is a cusp. If $(q+1)^2 - Ap \neq 0$, then $2c_{21} + qc_{22} \neq 0$ and E_* is of codimension 2. See Figs. 6 and 7.

Theorem 4.1. *Assume* $(S1)$, $(S2)$ *and* $(S3)$. *Then the interior equilibrium* E_* *of* (1.3) *is a cusp, and it is a Bogdanov–Takens singularity with codimension* 2 *when* $p \neq \frac{(q+1)^2}{A}$.

In the following we will investigate the approximating BT bifurcation curves. Further assume

(S4) $p \neq \frac{(q+1)^2}{A}$.

We choose p and R_0 as bifurcation parameters. Suppose $A,\ p_0,\ q,\ R_0^*$ satisfy (S1)–(S4). Let $p = p_0 + \lambda_1$, $R_0 = R_0^* + \lambda_2, I_* = \frac{Ap_0 - q - 1}{2p_0(q+1)}$, $R_* = qI_*$. Clearly if $\lambda_1 = \lambda_2 = 0$, E_* is a degenerate equilibrium. Let $x = I - I_*$, $y = R - R_*$. E_* is translated to $(0, 0)$ and (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= e_0 + e_1x + e_2y + e_3x^2 + e_4xy + e_5x^2y + e_6x^3, \\ \frac{dy}{dt} &= qx - y, \end{aligned} \tag{4.5}$$

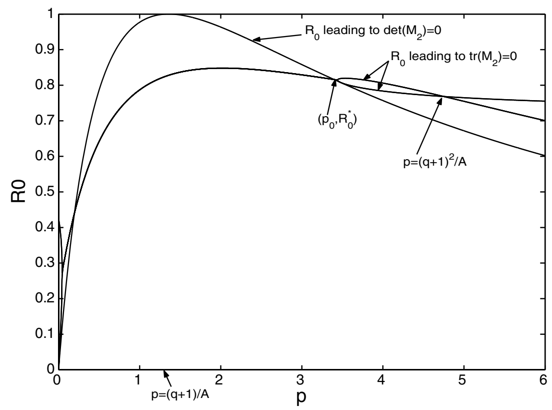


Fig. 6. Values of p and R_0 for $\det(M_2) = 0$ and $\operatorname{tr}(M_2) = 0$, where $A = 2.577539755$, $q = 2.5$. When $p = p_0 = 3.480000001$, $R_0 = R_0^* = 0.8075901343$, the equilibrium E_* is a cusp.

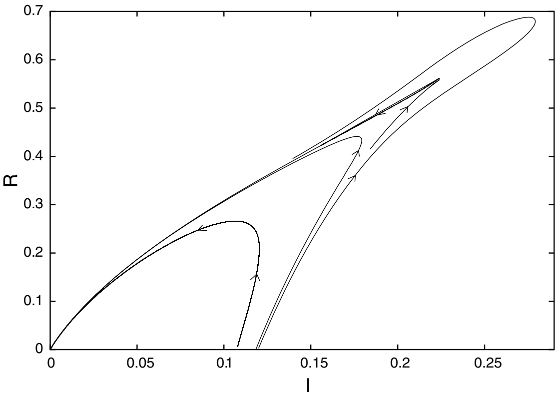


Fig. 7. A cusp of (1.3). $A = 2.577539755$, $q = 2.5$, $p = p_0 = 3.480000001$, $R_0 = R_0^* = 0.8075901343$, $E_* = (0.22455, 0.56136)$ is a cusp.

where

$$\begin{aligned} e_0 &= I_*(1 + (p_0 + \lambda_1)I_*)(A - I_* - R_*) - A/(R_0^* + \lambda_2)I_*, \\ e_1 &= -I_*(1 + (p_0 + \lambda_1)I_*) + (2(p_0 + \lambda_1)I_* + 1)(A - I_* - R_*) - A/(R_0^* + \lambda_2), \\ e_2 &= -I_*(1 + (p0 + r1) * I_*), \\ e_3 &= -2(p_0 + \lambda_1)I_* - 1 + (p_0 + \lambda_1)(A - I_* - R_*), \\ e_4 &= -2(p_0 + \lambda_1)I_* - 1, \\ e_5 &= -p_0 - \lambda_1, \\ e_6 &= -p_0 - \lambda_1. \end{aligned}$$

Let $X = x, Y = e_0 + e_1x + e_2y + e_3x^2 + e_4xy + e_5x^2y + e_6x^3$. Then (4.5) becomes

$$\begin{aligned} \frac{dX}{dt} &= Y, \\ \frac{dY}{dt} &= e_0 + (e_2q + e_1)X + e_{11}Y + (e_3 + e_4q)X^2 + e_{12}XY + \frac{e_4}{e_2}Y^2 + W_1(X, Y, \lambda), \end{aligned} \tag{4.6}$$

where $e_{11} = e_1 - 1 - e_0e_4/e_2$, $e_{12} = 2e_3 + e_4^2e_0/e_2^2 - e_1e_4/e_2$ and $W_1(X, Y, \lambda)$ is a smooth function of X, Y, and λ at least of order three.

Let $x = X + e_{11}/e_{12}$, $y = Y$. (4.6) becomes

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= h_0 + h_1x + h_2x^2 + h_3xy + h_4y^2 + W_2(x, y, \lambda), \end{aligned} \tag{4.7}$$

where $h_0 = e_0 - (e_2q + e_1)e_{11}/e_{12} + (e_3 + e_4q)e_{11}^2/e_{12}^2$, $h_1 = e_2q + e_1 - 2(e_3 + e_4q)\,e_{11}/e_{12}$, $h_2 = e_3 + e_4q$, $h_3 = e_{12}$, $h_4 = e_4/e_2$ and $W_2(x, y, \lambda)$ is a smooth function of x, y, and λ at least of order three. It is not difficult to obtain that when $\lambda_1 \to 0$, $\lambda_2 \to 0$,

$$\begin{aligned} h_0 &\to 0, \\ h_1 &\to 0, \\ h_2 &\to -\frac{1}{2}Ap_0 + \frac{1}{2}q + \frac{1}{2} < 0, \\ h_3 &\to \frac{(q+1)^2 - Ap_0}{q+1} \neq 0, \\ h_4 &\to -4\frac{p_0^2A(q+1)}{(q+1)^2 - A^2p_0^2} > 0. \end{aligned}$$

We introduce a new time variable τ, which satisfies $\mathrm{d}t = (1 - h_4 x)\,\mathrm{d}\tau$, and still write τ as t. Then (4.7) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y(1 - h_4 x), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (1 - h_4 x)(h_0 + h_1 x + h_2 x^2 + h_3 xy + h_4 y^2) + W_3(x, y, \lambda), \end{aligned} \tag{4.8}$$

where $W_3(x, y, \lambda)$ is a smooth function of x, y, and λ at least of order three. Let $X = x$, $Y = y(1 - h_4 x)$. Still write (X, Y) as (x, y). Then we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= c_1 + c_2 x + c_3 x^2 + c_4 xy + W_4(x, y, \lambda), \end{aligned} \tag{4.9}$$

where $c_1 = h_0$, $c_2 = h_1 - 2h_0 h_4$, $c_3 = h_0 h_4^2 - 2h_1 h_4 + h_2$, $c_4 = h_3$ and $W_4(x, y, \lambda)$ is a smooth function of x, y, and λ at least of order three. Then when $\lambda_1 \to 0$, $\lambda_2 \to 0$,

$$\begin{aligned} c_1 &\to 0, \\ c_2 &\to 0, \\ c_3 &\to -\frac{1}{2} A p_0 + \frac{1}{2} q + \frac{1}{2} < 0, \\ c_4 &\to \frac{(q+1)^2 - A p_0}{q + 1} \neq 0. \end{aligned}$$

Let $X = (c_4^2/c_3)x + c_2 c_4^2/2c_3^2$, $Y = (c_4^3/c_3^2)y$, $\tau = (c_3/c_4)t$. Write (X, Y, τ) as (x, y, t). Then (4.9) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \tau_1 + \tau_2 y + x^2 + xy + W_5(x, y, \lambda), \end{aligned} \tag{4.10}$$

where $\tau_1 = c_1 c_4^4/c_3^3 - c_2^2 c_4^4/4c_3^4$, $\tau_2 = -c_2 c_4^2/2c_3^2$ and $W_5(x, y, \lambda)$ is a smooth function of x, y, and λ at least of order three.

By the theorems in Bogdanov [2,3] and Takens [22], we obtain the following local representations of bifurcation curves in a small neighborhood of the origin of (4.10) (i.e., E_* of (1.3)).

Theorem 4.2. *Let* (*S*1)–(*S*4) *hold. Then* (1.3) *admits the following bifurcation behavior*:

(1) *there is a saddle-node bifurcation curve* $SN = \{(\lambda_1, \lambda_2) : 4c_1 c_3 = c_2^2 + 0(\|\lambda\|)^2\}$;
(2) *there is a Hopf bifurcation curve* $H = \{(\lambda_1, \lambda_2) : c_1 = 0 + 0(\|\lambda\|)^2, c_2 < 0\}$;
(3) *there is a homoclinic curve* $HL = \{(\lambda_1, \lambda_2) : 25c_1 c_3 + 6c_2^2 = 0 + 0(\|\lambda\|)^2\}$.

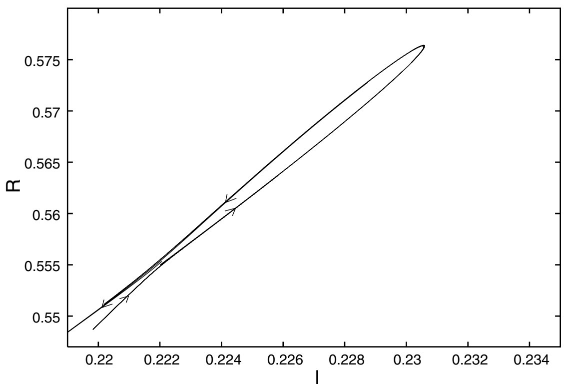


Fig. 8. Homoclinic bifurcation, where $A = 2.577539755$, $q = 2.5$, $p_0 = 3.480000001$, $R_0^* = 0.8075901343$, $\lambda_1 = 0.005$, $\lambda_2 = -0.0004842477903$, $R_0 = 0.8071058862$.

Example 4.1. Let $A = 2.577539755$, $q = 2.5$. Then $p_0 = 3.480000001$, $R_0^* = 0.8075901342$. By Theorem 4.2, the homoclinic curve is given by

$$0.2836444579\lambda_2^2 - 0.2458974305\lambda_1\lambda_2 - 0.01391895108\lambda_1^2 - 0.001353584153\lambda_1 - 0.01329719021\lambda_2 + o(\|\lambda\|^2) = 0.$$

For $\lambda_1 = 0.005$ and $\lambda_2 = -0.0004842477903$, using XPPAUT, we obtain a homoclinic orbit of (1.3) (see Fig. 8).

5. Discussion

In this paper we study an SIRS epidemic model with a nonlinear incidence rate $\beta I(1 + vI^{k-1})S$ with $\beta > 0$, $k = 2$, and $v > 0$, which was introduced in epidemic models in [23,24]. Previous studies of analogous model with (1.1) in [24] mainly focused on simulations and obtained such dynamics as saddle-node, Hopf and Bogdanov–Takens bifurcations, and the endemic basin. In contrast, we analyze the model theoretically and found richer dynamics of the model, from which the effect of the nonlinear incidence rate on the spread of the disease is more clearly shown.

We study (1.3), the limit state of (1.1), assuming the population is in equilibrium. We see that the disease free equilibrium $(0,0)$ always exists and is locally stable when $R_0 < 1$ and unstable when $R_0 > 1$, where R_0 is introduced as the basic reproduction number. When $R_0 > 1$ there is a unique endemic equilibrium E_2 , whereas when $R_0 < 1$ there may exist none, one (E_2) or two endemic equilibria (E_1 and E_2), which implies the existence of the backward bifurcation. Of the two endemic equilibria, E_1 is always a saddle, and E_2 may be stable or unstable. Detailed analysis of the stability of E_2 shows that stability switches for E_2 exist , which indicates that when R_0 increases the stability of E_2 will change from stable to unstable then back to stable. Combining stability conditions for $(0,0)$ and E_2, we find the existence of bistable states which may be two stable equilibria ($(0,0)$ and E_2) or a stable equilibrium $(0,0)$ and a stable limit cycle. In this case, the eventual behavior of the system is sensitive to the initial positions, which makes the model realistic. Such combination also shows that, on some occasions all equilibria are unstable and the $\omega-$ limit set for all positive solutions except equilibria and their stable manifolds must be limit cycles, which indicates that the disease is destined to break out periodically. By Dulac's criteria, energy integration method and Poincaré–Bendixson Theorem, we obtain globally asymptotic stability of $(0,0)$ or E_2 under some suitable conditions. We also study the Hopf bifurcation and obtain the criteria to judge its stability. By showing the existence and approximating curves of Bogdanov–Takens bifurcation, we have very clear description of the dynamics of the system near the unique degenerate equilibrium. Finally, we notice that all above dynamics of the model are sensitive to the parameter p. However, the expression of p shows that it plays the same role as v in the model. Therefore, it is in deed the nonlinear incidence rate that produces the complicated dynamics of epidemic models and makes the models more reasonable and practical.

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