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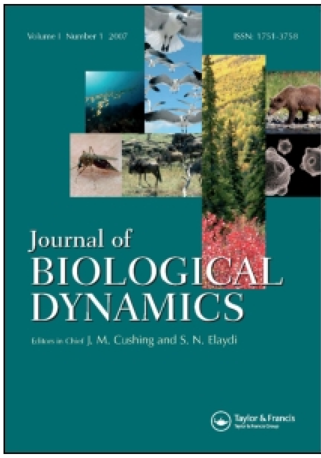

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Bistable waves for a class of cooperative reaction-diffusion systems

Yu Jin [a]; Xiao-Qiang Zhao [a]
[a] Department of Mathematics and Statistics, Memorial University of Newfoundland, NL, Canada

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Bistable waves for a class of cooperative reaction–diffusion systems

Yu Jin and Xiao-Qiang Zhao*

Department of Mathematics and Statistics, Memorial University of Newfoundland, St John's, NL, Canada

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Dedicated to Hal Smith on the occasion of his 60th birthday

In this paper, we consider a class of coupled cooperative reaction–diffusion systems, in which one population (or subpopulation) diffuses while the other is sedentary. We use the shooting method to prove the existence of the bistable travelling wave, and then obtain its global attractivity with phase shift and uniqueness (up to translation) via the dynamical system approach. The results are applied to some specific examples of reaction–diffusion population models.

Keywords: bistable waves; global attractivity; shooting method; monotone semiflow

AMS Subject Classifications: 35K57; 37N25; 92B05

1. Introduction

Classic reaction–diffusion models usually take the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \dfrac{\partial^2 u}{\partial x^2} + F(u, v), \\ \dfrac{\partial v}{\partial t} = D_2 \dfrac{\partial^2 v}{\partial x^2} + G(u, v), \end{cases} \tag{1}$$

where u and v are densities of two populations (or subpopulations of a population, or two particles) at location x and time t, D_1 and D_2 are positive diffusion constants, and $F(u, v)$ and $G(u, v)$ are reaction functions. This model has been generally applied to describe the dispersal dynamics of populations, disease transmission dynamics, chemical reactions, and so on (see, *e.g.*, [1,9,10,12] and references therein), and it seems to work well in most cases.

However, it has been recently noticed that in some situations during the dispersal process, one of the reacting populations diffuses so slowly in the habitat, compared with the other, such that its diffusion can almost be neglected. This phenomenon is very interesting in the study of

*Corresponding author. Email: xzhao@math.mun.ca

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population dispersal, and it naturally suggests that only one of the diffusion constants be positive while the other be zero, when we describe the dispersal dynamics in reaction–diffusion models. To model fecally orally transmitted diseases such as cholera, typhoid fever, infections hepatitis, polyometitis, etc., Capasso and Maddalena [1] assumed that the bacteria diffuse randomly in the habitat, while the diffusion of the human population can be neglected with respect to that of bacteria. As a result, they studied the following model:

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d\dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + a_{12}u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{2}$$

where u_1 and u_2 denote the spatial densities of the infectious agent and infective human population, respectively; $d > 0$ is the diffusion constant of bacteria; $1/a_{11}$ is the mean lifetime of the agent in the environment; $1/a_{22}$ is the mean infectious period of the human infectives; a_{12} is the multiplicative factor of the infectious agent due to the human population; and $g(u_1)$ is the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of the epidemic.

Other examples come from single species reaction–diffusion population models. Recently, some authors assumed that only part of the population is migrating and the other part is sedentary. Cook [11] studied a Verhulst-type population model with sedentary and migrating subpopulations, assuming that there is a joint carrying capacity for both subpopulations and that the offspring of both groups forms one pool that is then distributed to both subpopulations at constant proportions:

$$\begin{cases} v_t = r_v(u+v)\left(1 - \dfrac{u+v}{K}\right) + Dv_{xx}, \\ w_t = r_w(u+v)\left(1 - \dfrac{u+v}{K}\right). \end{cases} \tag{3}$$

Lewis and Schmitz [8] studied another Verhulst-type model in which it was assumed that individuals switch between mobile and stationary states during their lifetime and that the migrants have a positive mortality while the sedentary subpopulation reproduces and is subject to a finite carrying capacity:

$$\begin{cases} v_t = D\Delta v - \mu v - \gamma_2 v + \gamma_1 w, \\ w_t = f(w) - \gamma_1 w + \gamma_2 v, \end{cases} \tag{4}$$

where $f(w) = rw(1 - w/K)$. Hadeler and Lewis [5] proposed a Fisher-type equation with a quiescent state:

$$\begin{cases} v_t = D\Delta v + f(v) - \gamma_2 v + \gamma_1 w, \\ w_t = \gamma_2 v - \gamma_1 w, \end{cases} \tag{5}$$

where individuals in state v move and interact as in the standard Fisher's equation, while those in state w are quiescent. Such behaviour is typical for invertebrates living in small ponds in arid climates, which dry up and reappear subject to rainfall [5].

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Motivated by Equations (2)–(5) and other recent works (see, *e.g.*, [3,4,14,16,17]), we consider the following general reaction–diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + F(u, v), \\ \dfrac{\partial v}{\partial t} = G(u, v), \end{cases} \tag{6}$$

where $u(t, x)$ and $v(t, x)$ are densities of the migrating and sedentary populations at the location x and time t, respectively; $D > 0$ is the diffusion constant of the migrating population; and $F(u, v)$ and $G(u, v)$ are reaction functions.

Systems (2)–(5) have been investigated extensively, and many results have been established in the monostable case, where the corresponding reaction system admits zero equilibrium and a stable nontrivial equilibrium. The stability of the trivial and nontrivial equilibria for Equation (2) was studied in [1]; the existence of monotone travelling waves and the minimal wave speed for Equation (2) were established in [19], and it was shown in [13] that this minimal wave speed is also the spreading speed for solutions with initial functions having compact supports. The minimal speed for monotone travelling waves for Equation (4) was determined in [8] from the linearization at the zero equilibrium, under the assumption that the emigration rate is less than the intrinsic growth rate for the sedentary class; the authors of [5] studied the spreading speed, minimal wave speed, and the persistence of the population in different domains for Equation (4); and the spreading speed and travelling waves of Equation (4) were also established in [15]. System (5) was briefly discussed in [5], and it was shown in [17] that the spreading speed coincides with the minimal wave speed for monotone travelling waves.

In epidemiology, large outbreaks usually tend to a nontrivial endemic state, while small outbreaks tend to extinction. This may explain why, even though we are exposed to many infections, only some diseases have evolved into an endemic state [3]. Mathematically, this leads to the study of epidemic models with bistable nonlinearities. There are some results for Equation (2) in the bistable case: a saddle point structure was obtained under Neumann boundary conditions in [2] and under Dirichlet boundary conditions in [7]; a complete analysis of the steady states was obtained under Dirichlet boundary conditions in [3]; and the existence, uniqueness, and global exponential stability with phase shift of bistable travelling waves were established in [16].

In this paper, we are interested in the existence, uniqueness, and global attractivity of travelling waves of the general system (6) with bistable nonlinearity. The organization of this paper is as follows. In Section 2, we establish the existence of bistable travelling waves for Equation (6) by the shooting method (see, *e.g.*, [14]). In Section 3, we obtain the global attractivity with phase shift and uniqueness (up to translation) of travelling waves via the dynamical system approach (see [16,18]). In Section 4, we present some specific examples of Equation (6) to illustrate the applicability of our main results.

2. Existence of bistable waves

Throughout this paper, we make the following assumptions to obtain the cooperative and bistable nonlinearity for system (6).

(H1) There exist three points $E_- = (0, 0)$, $E_0 = (a_1, b_1)$, and $E_+ = (a_2, b_2)$ with $0 < a_1 < a_2$ and $0 < b_1 < b_2$ such that
   (i) $F, G \in C^1(\mathbb{R}_+^2, \mathbb{R})$, $F_v(u, v) \geq 0$, $G_u(u, v) \geq 0$, and $G_v(u, v) < 0$ on $\mathbb{R}_+^2$, and $G_u(0, 0) > 0$;

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(ii) E_-, E_0, and E_+ are only zeros of $f(u, v) := (F(u, v), G(u, v))$ in the order interval $[E_-, E_+]$;

(iii) all eigenvalues of the Jacobian matrices $\mathrm{Df}(E_-)$ and $\mathrm{Df}(E_+)$ have negative real parts, and $\mathrm{Df}(E_0)$ has an eigenvalue with positive real part and another with negative real part;

(iv) $F_v(u, v) > 0$ for $(u, v) \in [0, a_2] \times [0, b_2]$.

Consider the spatially homogeneous system associated with Equation (6)

$$\begin{cases} u'(t) = F(u, v), \\ v'(t) = G(u, v). \end{cases} \tag{7}$$

By the assumption (H1), Equation (7) has only three equilibria: E_-, E_0, and E_+ in $[E_-, E_+]$, where E_- and E_+ are stable and E_0 is a saddle. In this paper, we will study the existence of bistable waves of Equation (6), *i.e.*, travelling wave solutions connecting E_- and E_+.

Let $\tau = x + ct$ and $(u(t, x), v(t, x)) = (U(x + ct), V(x + ct))$ be a travelling wave solution of Equation (6). Then, the wave front profile $(U(\tau), V(\tau))$ satisfies

$$\begin{cases} cU' = DU'' + F(U, V), \\ cV' = G(U, V), \end{cases} \tag{8}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}\tau$. Since we are interested in travelling waves connecting E_- and E_+, we impose the following asymptotic boundary conditions on Equation (8)

$$\begin{cases} U(-\infty) = 0, \quad V(-\infty) = 0, \quad U'(-\infty) = 0, \\ U(+\infty) = a_2, \quad V(+\infty) = b_2, \quad U'(+\infty) = 0. \end{cases} \tag{9}$$

In the case where $c = 0$, Equation (8) becomes

$$\begin{cases} DU'' + F(U, V) = 0, \\ G(U, V) = 0, \end{cases} \tag{10}$$

which is equivalent to

$$\begin{cases} U' = W, \\ W' = -\dfrac{F(U, V^*(U))}{D}, \end{cases} \tag{11}$$

where $V^*(U)$ satisfies $G(U, V^*(U)) = 0$. By (H1) and the implicit function theorem, it is not difficult to see that $V^*(U)$ is continuously differentiable on $[0, \infty)$. Since E_- and E_+ are stable for Equation (7), we can easily see that $(0, 0)$ and $(a_2, 0)$ are saddles of Equation (11). Then a travelling wave of Equation (6) connecting E_- and E_+ with wave speed $c = 0$ corresponds to a heteroclinic orbit of Equation (11) connecting $(0, 0)$ and $(a_2, 0)$. The solutions of Equation (11) through $(0, 0)$ can be expressed as

$$\frac{W^2}{2} = -\frac{1}{D}\int_0^U F(t, V^*(t))\mathrm{d}t.$$

Thus, Equation (11) admits a heteroclinic orbit connecting $(0, 0)$ and $(a_2, 0)$ if and only if $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U = 0$.

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In what follows, we mainly consider the case where $c > 0$. It is easy to see that Equations (8) and (9) are equivalent to

$$\begin{cases} U' = W, \\ V' = \dfrac{G(U,V)}{c}, \\ W' = \dfrac{cW - F(U,V)}{D}, \end{cases} \tag{12}$$

with boundary conditions

$$U(-\infty) = 0, \quad V(-\infty) = 0, \quad W(-\infty) = 0, \tag{13}$$

$$U(+\infty) = a_2, \quad V(+\infty) = b_2, \quad W(+\infty) = 0. \tag{14}$$

Clearly, Equation (12) has three equilibria $(E_-, 0)$, $(E_0, 0)$, and $(E_+, 0)$. Thus, a travelling wave solution of Equation (6) connecting E_- and E_+ with positive wave speed corresponds to a solution of Equation (12) connecting $(E_-, 0)$ and $(E_+, 0)$, *i.e.*, a solution of Equations (12)–(14).

LEMMA 2.1 *For any $c > 0$, the Jacobian matrix of Equation* (12) *at* $(E_-, 0)$ *has one positive eigenvalue $\lambda(c)$ and two eigenvalues with negative real parts.*

Proof The Jacobian matrix of Equation (12) at $(E_-, 0)$ is

$$J_0 = \begin{bmatrix} 0 & 0 & 1 \\ \dfrac{G_u(0,0)}{c} & \dfrac{G_v(0,0)}{c} & 0 \\ -\dfrac{F_u(0,0)}{D} & -\dfrac{F_v(0,0)}{D} & \dfrac{c}{D} \end{bmatrix}.$$

The characteristic equation of J_0 is

$$\left(\lambda - \frac{G_v(0,0)}{c}\right)\left(\lambda^2 - \frac{c\lambda}{D} + \frac{F_u(0,0)}{D}\right) + \frac{G_u(0,0)F_v(0,0)}{Dc} = 0.$$

Consider $f(\lambda, m) = (\lambda - G_v(0,0)/c)(\lambda^2 - c\lambda/D + F_u(0,0)/D) + m$. Then we have

$$f\left(\lambda, \frac{F_u(0,0)G_v(0,0)}{Dc}\right) = \lambda\left[\lambda^2 - \left(\frac{c}{D} + \frac{G_v(0,0)}{c}\right)\lambda + \frac{F_u(0,0) + G_v(0,0)}{D}\right].$$

Since $\mathrm{Df}(E_-)$ is stable, it follows that

$$F_u(0,0) + G_v(0,0) < 0, \quad F_u(0,0)G_v(0,0) - F_v(0,0)G_u(0,0) > 0.$$

Then it is easy to see that $f(\lambda, F_u(0,0)G_v(0,0)/Dc)$ has three solutions $0, \lambda_1$, and λ_2 with $\lambda_1 < 0 < \lambda_2$, and hence, $f(\lambda, G_u(0,0)F_v(0,0)/Dc)$ has one positive solution and two solutions with negative real parts. Thus, J_0 has one positive eigenvalue $\lambda(c)$ and two eigenvalues with negative real parts. ∎

By Lemma 2.1, Equation (12) has a one-dimensional unstable manifold corresponding to $\lambda(c)$ at $(E_-, 0)$. Let $\bar{X} = (X_1, X_2, X_3)$ be an eigenvector of J_0 corresponding to $\lambda(c)$. Then there is a nonconstant solution of Equations (12) and (13), which tends to $(E_-, 0)$ as $\tau \to -\infty$ and

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whose tangent vector at $\tau = -\infty$ is the eigenvector $\bar{X}$ or $-\bar{X}$. It follows from the equation $J_0\bar{X} = \lambda(c)\bar{X}$ that

$$\begin{cases} X_3 = \lambda(c)X_1, \\ \dfrac{G_u(0,0)X_1}{c} + \dfrac{G_v(0,0)X_2}{c} = \lambda(c)X_2, \\ -\dfrac{F_u(0,0)X_1}{D} - \dfrac{F_v(0,0)X_2}{D} + \dfrac{cX_3}{D} = \lambda(c)X_3. \end{cases}$$

Without loss of generality, we can assume that $\bar{X} = (1, G_u(0,0)/(c\lambda(c) - G_v(0,0)), \lambda(c))$. Since $G_u(0,0) > 0$, $G_v(0,0) < 0$, and $\lambda(c) > 0$, $c > 0$, we have $X_i > 0$, $i = 1, 2, 3$.

In the rest of this section, we assume that (U, V, W) is a solution of Equations (12) and (13) with the tangent vector $\bar{X}$ at $\tau = -\infty$. By the above analysis, we can easily obtain the following result.

Lemma 2.2 *Let* (U, V, W) *be a solution of Equations* (12) *and* (13). *Then near* $\tau = -\infty$, (U, V, W) *satisfies* $U > 0$, $V > 0$, $W = U' > 0$, *and* $V' > 0$.

Definition 2.1 *Let* (U, V, W) *be a solution of Equations* (12) *and* (13). *Let* $\tau_0 = \tau_0(c)$ *be the first zero of* U' *if it exists and* $\bar{u} = U(\tau_0)$ ($\bar{u}$ *may be* $+\infty$).

Since $U > 0$ and $U' > 0$ on $(-\infty, \tau_0)$, we can express V and W as functions of U for $U \in (0, \bar{u})$. Let $\mathcal{V}(U) = V(\tau(U))$ and $\mathcal{W}(U) = W(\tau(U))$ for $U \in (0, \bar{u})$. Then $\mathcal{V}$ and $\mathcal{W}$ satisfy the following equations

$$\mathcal{V}' = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}U} = \frac{G(U, \mathcal{V})}{c\mathcal{W}}, \tag{15}$$

$$\mathcal{W}' = \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}U} = \frac{c\mathcal{W} - F(U, \mathcal{V})}{D\mathcal{W}}, \tag{16}$$

for $U \in (0, \bar{u})$ with the initial conditions

$$\mathcal{V}(0) = 0, \quad \mathcal{W}(0) = 0. \tag{17}$$

Lemma 2.3 *Let* (U, V, W) *be a solution of Equations* (12) *and* (13). *Then* $V'(\tau) > 0$ *for all* $\tau \in (-\infty, \tau_0)$.

Proof Since $V'(\tau) > 0$ near $\tau = -\infty$ and $U'(\tau) > 0$ for $\tau \in (-\infty, \tau_0)$, we have $\mathcal{V}'(U) > 0$ and hence $G(U, \mathcal{V}(U)) > 0$ for all $U \in (0, u_0)$ for some $u_0 \in (0, \bar{u}]$. If $u_0 < \bar{u}$ and $G(u_0, \mathcal{V}(u_0)) = 0$, then $\mathcal{V}'(u_0) = 0$. However, $\mathrm{d}G/\mathrm{d}U|_{U=u_0} < 0$, that is, $[\partial G/\partial U + (\partial G/\partial \mathcal{V})(\partial \mathcal{V}/\partial U)]_{U=u_0} < 0$, which implies that $\mathcal{V}'(u_0) > 0$, a contradiction. Therefore, $\mathcal{V}'(U) > 0$ for all $U \in (0, \bar{u})$, and hence, $V'(\tau) > 0$ for all $\tau \in (-\infty, \tau_0)$. ■

Lemma 2.4 *Let* (U, V, W) *be a solution of Equations* (12) *and* (13). *If* $U(\tau) \in (0, a_2)$ *for some* $\tau \in (-\infty, \tau_0)$, *then* $V(\tau) \in (0, b_2)$.

Proof Clearly, we have $U(\tau) \in (0, a_2)$ and $V(\tau) \in (0, b_2)$ when τ is near $-\infty$. Suppose that there exists $\tilde{\tau} \in (-\infty, \tau_0)$ such that $U(\tilde{\tau}) \in (0, a_2)$ and $V(\tilde{\tau}) = b_2$. Then $G(U(\tilde{\tau}), V(\tilde{\tau})) = G(U(\tilde{\tau}), b_2) \leq G(a_2, b_2) = 0$ since $G_u \geq 0$. Thus, we have $V'(\tilde{\tau}) \leq 0$, a contradiction to $V'(\tau) > 0$ on $(-\infty, \tau_0)$. ■

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LEMMA 2.5 *There exists no nontrivial solution* (U, V, W) *of Equations* (12) *and* (13) *satisfying* $U(\tau_0) = a_2$, $U'(\tau_0) = 0$, *and* $U''(\tau_0) \le 0$ *for some finite* τ_0.

Proof Suppose that there is such a solution. Then either $U''(\tau_0) < 0$ or $U''(\tau_0) = 0$. If $U''(\tau_0) = 0$, then by Equation (8) we have $F(U(\tau_0), V(\tau_0)) = 0$. Since $U(\tau_0) = a_2$, we have $V(\tau_0) = b_2$, which contradicts the uniqueness of solutions. If $U''(\tau_0) < 0$, then $F(U(\tau_0), V(\tau_0)) > 0$, *i.e.*, $F(a_2, V(\tau_0)) > 0 = F(a_2, b_2)$. Since $F_v \ge 0$, we have $V(\tau_0) > b_2$. By $G_v < 0$, we have $G(U(\tau_0), V(\tau_0)) = G(a_2, V(\tau_0)) < G(a_2, b_2) = 0$. Thus, we have $V'(\tau_0) < 0$, which contradicts $V'(\tau_0) \ge 0$. ■

THEOREM 2.1 *System* (6) *has a monotone increasing travelling wave solution* $(U(x + ct), V(x + ct))$ *connecting* E_- *to* E_+ *for some real number c such that the wave speed c has the same sign as the integral* $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U$, *where* $V^*(U)$ *satisfies* $G(U, V^*(U)) = 0$.

Proof Since $V^*(U)$ is continuously differentiable on $[0, a_2]$, $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U$ is well defined.

In the case where $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U = 0$, we have shown that Equation (11) has a heteroclinic orbit connecting $(0, 0)$ and $(a_2, 0)$, and hence Equation (6) has a monotone travelling wave solution with $c = 0$.

Next we consider the case where $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U > 0$. We proceed with the following four steps.

Step 1 We claim that $\bar{u} > a_1$. Indeed, it follows from Lemma 2.3 that $\mathcal{V}'(U) > 0$, that is, $G(U, \mathcal{V}(U)) > 0$ for all $U \in (0, \bar{u})$. Since $G_v < 0$ and $G(U, V^*(U)) = 0$, we have $\mathcal{V}(U) < V^*(U)$ for all $U \in (0, \bar{u})$. By (H1) and the implicit function theorem, we can also find a continuously differentiable function $V_F^*(U)$ on $[0, a_2]$ such that $F(U, V_F^*(U)) = 0$ for all $U \in [0, a_2]$. Again by (H1) and the qualitative analysis of Equation (7), it is not difficult to obtain $V^*(U) < V_F^*(U)$ for all $U \in (0, a_1)$. Assume, by contradiction, that $\bar{u} \le a_1$. Then $F(U, \mathcal{V}(U)) < F(U, V^*(U)) < F(U, V_F^*(U)) = 0$ on $(0, \bar{u})$, and hence by Equation (16), $\mathcal{W}'(U) > c/D > 0$ on $(0, \bar{u})$, which implies that $\mathcal{W}(\bar{u}) > 0$. This contradicts $\mathcal{W}(\bar{u}) = 0$.

Step 2 We claim that if c is sufficiently large, then there exists a finite $\hat{\tau}$ such that $U'(\tau) > 0$ on $(-\infty, \hat{\tau}]$ and $U(\hat{\tau}) = a_2$.

Choose $c_0 > \sqrt{(2|m_1|D/a_1)}$, where $m_1 = \max_{(u,v)\in[0,a_2]\times[0,b_2]} F(u, v)$. We claim that $\bar{u} > a_2$ for $c > c_0$. Suppose that this is not true. Then there is some $\tilde{c} > c_0$ such that $\bar{u} \le a_2$ and $\widetilde{\mathcal{W}}(\bar{u}) = 0$, where $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}})$ is the solution of Equations (15) and (16) corresponding to $\tilde{c}$. By analysis in step 1, $\widetilde{\mathcal{W}}' \ge \tilde{c}/D$ for $U \in (0, a_1]$. Then $\widetilde{\mathcal{W}}(a_1) \ge (\tilde{c}/D)a_1$. Since $\widetilde{\mathcal{W}}(\bar{u}) = 0$ and $\bar{u} \le a_2$, we have $\widetilde{\mathcal{W}}(U) \ge (\tilde{c}/2D)a_1$ for all $U \in [a_1, \hat{u}]$ for some $\hat{u} \in (a_1, a_2)$ and $\widetilde{\mathcal{W}}(\hat{u}) = (\tilde{c}/2D)a_1 < \widetilde{\mathcal{W}}(a_1)$. On the other hand, for all $U \in [a_1, \hat{u})$, $\widetilde{\mathcal{W}}'(U) = \tilde{c}/D - F(U, \widetilde{\mathcal{V}})/D\widetilde{\mathcal{W}} \ge \tilde{c}/D - m_1/D\widetilde{\mathcal{W}} \ge \tilde{c}/D - m_1/(D(\tilde{c}/2D)a_1) = \tilde{c}/D - 2m_1/\tilde{c}a_1 = (\tilde{c}^2 - 2m_1D/a_1)/D\tilde{c} > 0$. Thus, $\widetilde{\mathcal{W}}$ increases in $U \in [a_1, \hat{u})$, and hence $\widetilde{\mathcal{W}}(\hat{u}) > \widetilde{\mathcal{W}}(a_1)$, a contradiction. Thus, if $c > c_0$, we have $\bar{u} > a_2$, which indicates that there exists some finite $\hat{\tau}$ such that $U'(\tau) > 0$ on $(-\infty, \hat{\tau}]$ and $U(\hat{\tau}) = a_2$.

Let $(U(\tau), V(\tau), W(\tau))$ be a solution of Equations (12) and (13). Set

$$P_1 = \{c > 0 : U'(\tau) > 0 \text{ on } (-\infty, \hat{\tau}] \text{ and } U(\hat{\tau}) = a_2 \text{ for some finite } \hat{\tau}\}.$$

Then P_1 is not empty and is open by continuous dependence on c. Moreover, $P_1 \supseteq (c_0, +\infty)$.

Step 3 We claim that for sufficiently small $c > 0$, there is a finite $\bar{\tau}$ with $U'(\bar{\tau}) = 0$ and $U(\bar{\tau}) \in (a_1, a_2)$.

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Suppose that this is not true. Then there exists a sequence $\{c_i\}_{i\in N} \subseteq \mathbb{R}^+$ with $\lim_{i\to+\infty} c_i = 0$ such that the corresponding solutions (U_i, V_i, W_i) of Equations (12) and (13) satisfy $(U_i)' > 0$ on $(-\infty, \bar{\tau}_i)$ and $U_i(\bar{\tau}_i) = a_2$ for some $\bar{\tau}_i$ ($\bar{\tau}_i$ may be infinity). By Equation (16), we have

$$\frac{\mathcal{W}_i^2(U)}{2} = \int_0^U \left[\frac{c_i}{D}\mathcal{W}_i(s) - \frac{F(s, \mathcal{V}_i(s))}{D}\right] \mathrm{d}s, \quad \forall U \in [0, a_2]. \tag{18}$$

Let

$$A_i = \sup_{U\in[0,a_2]} \mathcal{W}_i(U), \quad m_2 = \min_{(u,v)\in[0,a_2]\times[0,b_2]} F(u, v).$$

Then $A_i > 0, m_2 < 0$, and $A_i^2/2 \le c_i A_i a_2/D - m_2 a_2/D$. Thus, $A_i/2 + m_2 a_2/DA_i \le c_i(a_2/D)$. Since $x/2 + m_2 a_2/Dx$ increases in $x \in (0, +\infty)$ and $c_i \to 0$ as $i \to +\infty$, we obtain that for large i, A_i is uniformly bounded. This implies that $\mathcal{W}_i(U)$ is uniformly bounded on $[0, a_2]$ for large $i \in N$.

Let $M_i(U) = G(U, \mathcal{V}_i(U))$, for $U \in [0, a_2]$. Then

$$M_i'(U) = G_u + G_v \mathcal{V}_i'(U) = G_u + G_v \frac{M_i(U)}{c\mathcal{W}_i}$$

and

$$\begin{aligned} M_i(U) &= M_i(0) \exp\left(\int_0^U \frac{G_v(s, \mathcal{V}_i(s))}{c_i\mathcal{W}_i(s)}\mathrm{d}s\right) + \int_0^U G_u(s, \mathcal{V}_i(s)) \exp\left(\int_s^U \frac{G_v(t, \mathcal{V}_i(t))}{c_i\mathcal{W}_i(t)}\mathrm{d}t\right)\mathrm{d}s \\ &= \int_0^U G_u(s, \mathcal{V}_i(s)) \exp\left(\int_s^U \frac{G_v(t, \mathcal{V}_i(t))}{c_i\mathcal{W}_i(t)}\mathrm{d}t\right)\mathrm{d}s. \end{aligned} \tag{19}$$

Since $\mathcal{W}_i(U)$ is uniformly bounded on $[0, a_2]$ for large $i \in N$, $c_i\mathcal{W}_i(U) \to 0$ as $i \to \infty$ uniformly for $U \in [0, a_2]$. $G_u(U, \mathcal{V}_i(U))$ is bounded for $U \in [0, a_2]$ since $G(u, v) \in C^1(\mathbb{R} \times \mathbb{R})$. Moreover, there exists $\delta > 0$ such that $G_v \le -\delta$ for $(U, V) \in [0, a_2] \times [0, b_2]$. Then we finally obtain $M_i(U) \to 0$ as $i \to \infty$ uniformly for $U \in [0, a_2]$. This, by the definitions of $M_i(U)$ and $V^*(U)$, implies that $G(U, \mathcal{V}_i(U)) \to G(U, V^*(U))$ as $i \to \infty$, uniformly for $U \in [0, a_2]$, that is,

$$\lim_{i\to\infty} |G(U, \mathcal{V}_i(U)) - G(U, V^*(U))| = 0, \quad \text{uniformly for } U \in [0, a_2]. \tag{20}$$

For any $U \in [0, a_2]$ and $i \in \mathbb{N}$, we have

$$\begin{aligned} \delta_1|\mathcal{V}_i(U) - V^*(U)| &\le \left|\int_0^1 G_v(U, \mathcal{V}_i(U) + s(V^*(U) - \mathcal{V}_i(U)))\mathrm{d}s(V^*(U) - \mathcal{V}_i(U))\right| \\ &= \left|G(U, \mathcal{V}_i(U)) - G(U, V^*(U))\right|, \end{aligned}$$

where $\delta_1 > 0$ is such that $G_v \le -\delta_1$ on $[0, a_2] \times [0, b_2]$. Then by Equation (20), we have $\mathcal{V}_i(U) \to V^*(U)$ as $i \to \infty$ uniformly for $U \in [0, a_2]$, and hence, $F(U, \mathcal{V}_i(U)) \to F(U, V^*(U))$ as $i \to \infty$, uniformly for $U \in [0, a_2]$. Then, letting $U = a_2$ and $i \to \infty$ in Equation (20), we obtain

$$\lim_{i\to\infty} -\frac{D\mathcal{W}_i^2(a_2)}{2} = \lim_{i\to\infty} \int_0^{a_2} F(s, \mathcal{V}_i(s))\mathrm{d}s = \int_0^{a_2} F(s, V^*(s))\mathrm{d}s. \tag{21}$$

Thus, $\int_0^{a_2} F(s, V^*(s))\mathrm{d}s \le 0$, which contradicts our assumption that $\int_0^{a_2} F(s, V^*(s))\mathrm{d}s > 0$.

Let $(U(\tau), V(\tau), W(\tau))$ be a solution of Equations (12) and (13). Set

$$P_2 = \{c > 0 : U'(\bar{\tau}) = 0 \text{ for some finite } \bar{\tau} \in \mathbb{R} \text{ and } U(\bar{\tau}) \in (0, a_2]\}.$$

Then P_2 is nonempty and contains $(0, \bar{c})$ for some finite $\bar{c} \in \mathbb{R}$.

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Moreover, for each $c \in P_2$, since $\tau_0 = \tau_0(c)$ is the first zero of U', we have $U''(\tau_0) \le 0$. Then by Lemma 2.5, $U(\tau_0) \in (a_1, a_2)$. Suppose that $U''(\tau_0) = 0$. By Equation (12), we have $F(U(\tau_0), V(\tau_0)) = 0$ and $G(U(\tau_0), V(\tau_0)) \ge 0$. However, by (H1)(ii), we have $G(U(\tau_0), V(\tau_0)) > 0$, that is, $V'(\tau_0) > 0$. It then follows that

$$\begin{aligned} U'''(\tau_0) &= W''(\tau_0) \\ &= \frac{c}{D}W'(\tau_0) - \frac{1}{D}F_u U'(\tau_0) - \frac{1}{D}F_v V'(\tau_0) \\ &= -\frac{1}{D}F_v(U(\tau_0), V(\tau_0))V'(\tau_0) < 0. \end{aligned}$$

This contradicts the definition of τ_0. Thus, we have $U''(\tau_0) < 0$, and hence, P_2 is open.

Step 4 Let $c^* = \sup P_2$. By the above two steps, c^* exists and $c^* \in \mathbb{R}^+ \setminus (P_1 \cup P_2)$. Let $(U^*(\tau), V^*(\tau), W^*(\tau))$ be a solution of Equations (12) and (13) corresponding to c^*. Then $U^{*\prime} > 0$ and $V^{*\prime} > 0$ on $\mathbb{R}$. Moreover, $U^*(\tau) \in (0, a_2)$ and $V^*(\tau) \in (0, b_2)$ for all $\tau \in \mathbb{R}$. Then as τ tends to $+\infty$, $U^*(\tau)$ and $V^*(\tau)$ have limits. Since $\bar{u} > a_1$, we have $\lim_{\tau\to+\infty} U^*(\tau) = a_2$ and $\lim_{\tau\to+\infty} V^*(\tau) = b_2$. Thus, $(U^*(\tau), V^*(\tau)$ is the bistable travelling waves of Equation (6) connecting E_- to E_+ with positive speeds c^* when $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U > 0$.

Finally, we consider the case where $\int_0^{a_2} F(U, V^*(U))\mathrm{d}U < 0$. By a change of variables $\bar{u} = a_2 - u$ and $\bar{v} = b_2 - v$, Equation (6) reduces to

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = D\dfrac{\partial^2 \bar{u}}{\partial x^2} + \bar{F}(\bar{u}, \bar{v}), \\ \dfrac{\partial \bar{v}}{\partial t} = \bar{G}(\bar{u}, \bar{v}), \end{cases} \tag{22}$$

where $\bar{F}(\bar{u}, \bar{v}) = -F(a_2 - \bar{u}, b_2 - \bar{v})$ and $\bar{G}(\bar{u}, \bar{v}) = -G(a_2 - \bar{u}, b_2 - \bar{v})$. Letting $\bar{G}(\bar{u}, \bar{v}) = 0$, we have $\bar{v} = \bar{v}^*(\bar{u}) := b_2 - V^*(a_2 - \bar{u})$. Then $\bar{f}(\bar{u}, \bar{v}) := (\bar{F}(\bar{u}, \bar{v}), \bar{G}(\bar{u}, \bar{v}))$ has only three zeros $E_- = (0, 0)$, $E_+ = (a_2, b_2)$, and $\bar{E}_0 = (a_2 - a_1, b_2 - b_1)$ on $[E_-, E_+]$. Moreover, it is easy to see that Equation (22) satisfies (H1) and $\int_0^{a_2} \bar{F}(\bar{u}, \bar{v}^*(\bar{u}))\mathrm{d}\bar{u} = -\int_0^{a_2} F(U, V^*(U))\mathrm{d}U > 0$. It follows from what we have proved that Equation (22) has a monotone increasing travelling wave solution $(\bar{U}(x + ct), \bar{V}(x + ct))$ connecting E_- and E_+ for some $c > 0$. Define $U(\xi) = a_2 - \bar{U}(-\xi)$ and $V(\xi) = b_2 - \bar{V}(-\xi)$, $\forall \xi \in \mathbb{R}$. Clearly, $(U(-\infty), V(-\infty)) = (0, 0)$ and $(U(\infty), V(\infty)) = (a_2, b_2)$. It then follows that $(U(x - ct), V(x - ct))$ is a monotone increasing travelling wave solution of Equation (6) connecting E_- and E_+. ■

3. Attractivity and uniqueness of bistable waves

In this section, we discuss the global attractivity with phase shift and uniqueness (up to translation) of the bistable travelling wave of Equation (6). In addition to (H1), we further impose the following conditions on F and G.

(H2) F and G can be extended to the domain $(-l, \infty)^2$ for some $l > 0$ such that
- (i) $F, G \in C^2((-l, \infty)^2, \mathbb{R})$, $F_u(u, v) < 0$, $F_v(u, v) > 0$, $G_u(u, v) \ge 0$, and $G_v(u, v) < 0$ for $(u, v) \in (-l, \infty)^2$.
- (ii) There exists $L > 0$ such that for any $l_2 > L$, there exists $l_1 > 0$ such that $F(l_1, l_2) < 0$.

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Let $\mathbb{X} = \mathrm{BUC}(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the usual supreme norm. Let $\mathbb{X}_+ = \{(\psi_1, \psi_2) \in \mathbb{X} : \psi_i(x) \geq 0, \forall x \in \mathbb{R}, i = 1, 2\}$. Then $\mathbb{X}_+$ is a closed cone of $\mathbb{X}$ and its induced partial ordering makes $\mathbb{X}$ into a Banach lattice. For any $\psi^1 = (\psi_1^1, \psi_2^1)$ and $\psi^2 = (\psi_1^2, \psi_2^2) \in \mathbb{X}$, we write $\psi^1 \leq_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in \mathbb{X}_+$, $\psi^1 <_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in \mathbb{X}_+ \setminus \{0\}$, and $\psi^1 \ll_{\mathbb{X}} \psi^2$ if $\psi^2 - \psi^1 \in \mathrm{Int}(\mathbb{X}_+)$.

By the arguments similar to those in [16, Lemma 3.1], we can prove the following result for Equation (6).

LEMMA 3.1 *For any $\psi \in \mathbb{X}_+$, Equation* (6) *has a unique bounded and nonnegative solution $(\Psi(t)\psi)(x) := (u(t, x, \psi), v(t, x, \psi))$ with $\Psi(0)\psi = \psi$, and the solution semiflow $\Psi(t)$ of Equation* (6) *is monotone on $\mathbb{X}_+$. Moreover, $(\Psi(t)\psi^1)(x) \ll (\Psi(t)\psi^2)(x)$ for all $t > 0$ and $x \in \mathbb{R}$ whenever $\psi^1, \psi^2 \in \mathbb{X}_+$ with $\psi^1 <_{\mathbb{X}} \psi^2$.*

In view of Section 2, we assume that $\phi(x - ct) = (\phi_1(x - ct), \phi_2(x - ct))$ is a strictly increasing travelling wave solution of Equation (6) connecting E_- and E_+. Letting $z = x - ct$, we transform Equation (6) into the following system:

$$\begin{cases} u_t(t, z) = cu_z(t, z) + Du_{zz}(t, z) + F(u(t, z), v(t, z)), \\ v_t(t, z) = cv_z(t, z) + G(u(t, z), v(t, z)). \end{cases} \tag{23}$$

It is easy to see that $\phi(z)$ is an equilibrium of system (23). Denote $(\Phi(t)\psi)(z) := (u(t, z, \psi), v(t, z, \psi))$ as the solution of Equation (23) with $\Phi(0)\psi = \psi \in \mathbb{X}_+$. Then the solution $(\Psi(t)\psi)(x)$ of Equation (6) with the initial value ψ is given by $(\Psi(t)\psi)(x) = (\Phi(t)\psi)(x - ct)$. Moreover, the comparison principle holds for Equation (6) and hence for Equation (23). By constructing the upper and lower solutions for Equation (23) in the same way as in [16], we can obtain the following result.

LEMMA 3.2 *The wave profile $\phi(z)$ is a Liapunov stable equilibrium of Equation* (23).

Since $\Phi(t) : \mathbb{X}_+ \to \mathbb{X}_+$ is the solution semiflow of Equation (23), it follows that $\Phi(t) : [E_-, E_+] \to [E_-, E_+]$ is monotone, and for any $s \in \mathbb{R}$, $\phi(\cdot + s)$ is a stable equilibrium of $\Phi(t)$. Consequently, by using the convergence theorem [18, Theorem 2.2.4] and the similar arguments as in the proof of [16, Theorem 3.1], we can establish the following result on the global attractivity with phase shift and uniqueness (up to translation) of the bistable wave of Equation (6).

THEOREM 3.1 *Let $\phi(x - ct)$ be a monotone travelling wave solution of system* (6) *and $\Psi(t, x, \psi) := (u(t, x, \psi), v(t, x, \psi))$ be the solution of Equation* (6) *with $\Psi(0, \cdot, \psi) = \psi \in \mathbb{X}_+$. Then for any $\psi \in \mathbb{X}_+$ with*

$$\limsup_{\xi \to -\infty} \psi(\xi) \ll E_0 \ll \liminf_{\xi \to \infty} \psi(\xi), \tag{24}$$

there exists $s_\psi \in \mathbb{R}$ such that $\lim_{t \to +\infty} \|\Psi(t, x, \psi) - \phi(x - ct + s_\psi)\| = 0$ uniformly for $x \in \mathbb{R}$. Moreover, any travelling wave solution of system (6) *connecting E_- and E_+ is a translate of ϕ.*

Remark 3.1 By the spectrum analysis of the linearization operator of Equation (23) at the equilibrium solution $\phi(z)$, as in [16, Section 4], we can obtain the local exponential stability with phase shift of the bistable wave $\phi(x - ct)$ with $c \neq 0$. This, together with Theorem 3.1, implies the global exponential stability with phase shift of the bistable wave $\phi(x - ct)$ with $c \neq 0$ of Equation (6).

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4. Examples

In this section, we apply the results in Sections 2 and 3 to some reaction–diffusion population models and show the existence and global exponential stability of bistable waves.

Example 4.1 Consider a reaction–diffusion epidemic model (see, *e.g.*, [1,16])

$$\begin{cases} \dfrac{\partial U_1(x,t)}{\partial t} = d\dfrac{\partial^2 U_1(x,t)}{\partial x^2} - U_1(x,t) + \alpha U_2(x,t), \\ \dfrac{\partial U_2(x,t)}{\partial t} = -\beta U_2(x,t) + g(U_1(x,t)), \end{cases} \tag{25}$$

where d, α, and β are positive constants; U_1 and U_2 denote the spatial densities of an infectious agent and the infective human population, respectively; and $F(U_1, U_2) = -U_1(x,t) + \alpha U_2(x,t)$ and $G(U_1, U_2) = -\beta U_2(x,t) + g(U_1(x,t))$. The existence, uniqueness (up to translation), and global exponential stability with phase shift of the bistable travelling wave with nonzero wave speed were established for Equation (25) in [16]. However, it seems that the claim in step 2 of the proof of Theorem 2.1 (for the existence) in [16] needs to be readdressed since the inequality

$$\dot{V}_c(\eta) = \frac{c}{d} - \frac{\eta - \alpha u_2}{md(\eta - b)} \le \frac{c}{d} + \frac{b - \eta}{md(\eta - b)} = \frac{1}{d}\left(c - \frac{1}{m}\right)$$

cannot be obtained as the authors stated there. By Theorem 2.1, we can establish the existence of the bistable wave under the assumptions (A1)–(A3) in [16].

Example 4.2 Consider a reaction–diffusion model with quiescent phases (see, *e.g.*, [4,8,15])

$$\begin{cases} v_t = D\Delta v - \mu v - \gamma_2 v + \gamma_1 w, \\ w_t = g(w) - \gamma_1 w + \gamma_2 v, \end{cases} \tag{26}$$

where v and w are densities of two particles, g is smooth, and $D, \mu, \gamma_1, \gamma_2 > 0$. For system (26), $F(v,w) = -\mu v - \gamma_2 v + \gamma_1 w$ and $G(v,w) = g(w) - \gamma_1 w + \gamma_2 v$. Then $F_v = -\mu - \gamma_2 < 0$, $F_w = \gamma_1 > 0$, and $G_v = \gamma_2 > 0$. Assume that $g \in C^2(-l, \infty)$ for some $l > 0$ such that $g(0) = 0$, $g'(w) > 0$, $\forall w > 0$, and $g'(w) < \gamma_1$ on $(-l, +\infty)$; $g(w) = \gamma_1 \mu w/(\mu + \gamma_2)$ has only three zeros 0, b_1, and b_2 on $[0, b_2]$; and $g''(w) > 0$ for $w \in (0, b_1)$ and $g''(w) < 0$ for $w > b_1$. It is easy to check that system (26) satisfies (H1) and (H2). By Theorems 2.1 and 3.1, it then follows that system (26) admits a bistable travelling wave, which is globally attractive with phase shift (or even globally exponentially stable with phase shift when the wave speed $c \neq 0$) and uniqueness (up to translation).

Example 4.3 Consider a reaction–diffusion model with a quiescent stage (see, *e.g.*, [5,17])

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D\Delta u_1(t,x) + f(u_1(t,x)) - \gamma_2 u_1(t,x) + \gamma_1 u_2(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = \gamma_2 u_1(t,x) - \gamma_1 u_2(t,x)), \end{cases} \tag{27}$$

where u_1 and u_2 are densities of the dispersal and nondispersal subpopulations, respectively; $D > 0$; $f(u)$ is a nonlinear continuous function; and γ_1 and γ_2 are the emigration and immigration rates, respectively. For system (27), $F(u_1, u_2) = f(u_1) - \gamma_2 u_1 + \gamma_1 u_2$ and $G(u_1, u_2) = \gamma_2 u_1 - \gamma_1 u_2$. Then $F_{u_2} = \gamma_1 > 0$, $G_{u_1} = \gamma_2 > 0$, and $G_{u_2} = -\gamma_1 < 0$. If we further assume that

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$f \in C^2(-l, \infty)$ for some $l > 0$ such that $f'(u_1) - \gamma_2 < 0$ for $u_1 \in (-l, \infty)$; $f(u_1)$ has only three zeros 0, a_1, a_2 on $[0, a_2]$; and $f'(0) < 0$, $f'(a_1) > 0$, and $f'(a_2) < 0$; then Equation (27) satisfies (H1) and (H2). Thus, Theorems 2.1 and 3.1 imply that system (27) admits a bistable travelling wave, which is globally attractive with phase shift (or even globally exponentially stable with phase shift when the wave speed $c \neq 0$) and uniqueness (up to translation). As a particular example, f can be chosen as $f(u_1) = u_1(u_1 - a)(1 - u_1)$ for some $0 < a < 1$.

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