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SPATIAL DYNAMICS OF A NONLOCAL PERIODIC REACTION-DIFFUSION MODEL WITH STAGE STRUCTURE*

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Abstract. In this paper, we investigate a nonlocal periodic reaction-diffusion population model with stage structure. In the case of unbounded spatial domain, we establish the existence of the asymptotic speed of spread and show that it coincides with the minimal wave speed for monotone periodic traveling waves. In the case of bounded spatial domain, we obtain a threshold result on the global attractivity of either zero or a positive periodic solution.

Key words. nonlocal periodic model, spreading speed, traveling waves, positive periodic solution, global attractivity

AMS subject classifications. 34K05, 35K57, 37N25, 92B05

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1. Introduction.

Age structure has been an interesting topic in population dynamics (see, e.g., [1, 2, 3, 5, 6, 7, 9, 14, 17, 18, 20] and the references therein), since we can investigate the separate quantities of immature and mature populations in an age-structured population model. To derive a model for a single species of population with age structure and diffusion, we usually assume that individuals move around not only after maturity, but also while immature. For a standard argument, Metz and Diekmann [14] give

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - \mu(a)u,$$

where $u(t, a, x)$ is the density of the population of the species at time $t \geq 0$, age $a \geq 0$, and location x in a spatial domain Ω; $D(a) \geq 0$ and $\mu(a) \geq 0$ are the diffusion rate and the death rate of the population at age a, respectively.

To study the behaviors of immature individuals and mature individuals, we can also divide the population of a species into two groups: immature population and mature population. For simplicity, we assume that the maturation time (or the length of the juvenile period) is the same for all juvenile individuals, denoted by $\tau \geq 0$. For distributed maturation delay, see, e.g., [2, 3] and the references therein. Assume that the diffusion rate and death rate are age-dependent for immature individuals, but age-independent for mature individuals. As a result, we have the following system for a single species of population with age structure and diffusion (see also [5, 17, 18, 20]):

(1.1)
$$\begin{cases} \partial_t u(t,a,x) + \partial_a u(t,a,x) = d_j(a)\Delta u - \mu_j(a)u(t,a,x), \quad t>0, 0<a<\tau, x\in\Omega, \\ u(t,0,x) = f(u_m(t,x)), \quad t \geq -\tau, x\in\Omega, \\ \partial_t u_m(t,x) = d_m \Delta u_m - g(u_m(t,x)) + u(t,\tau,x), t>0, x\in\Omega, \end{cases}$$

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where $u(t,a,x)$ is the density of the population at time $t \geq -\tau$, age $a \geq 0$, location $x \in \Omega$, $u_m(t,x)$ is the density of the mature population, $f(u_m)$ and $g(u_m)$ are the birth rate and the mortality rate of mature individuals, respectively, $d_j(a) \geq 0$ is the diffusion rate of the immature individuals at age $a \in (0,\tau)$, $d_m \geq 0$ is the diffusion rate of the mature individuals, $\mu_j(a) > 0$ denotes the per capita mortality rate of juveniles at age a, $u(t,\tau,x)$ is the adults recruitment term for those of maturation age τ, Δ is the Laplacian operator.

In fact, the dynamics of many populations is influenced greatly by the time varying environments (e.g., due to seasonal variation). For example, in a one year period, the birth rate may be high in spring and summer and low in winter, while in winter more individuals might be at risk of death because of low temperature, lack of food, or some other reasons. Moreover, populations usually like to move in warm weather during the spring and summer time. Therefore, it is more realistic to consider a nonautonomous version of (1.1) for population dynamics. In particular, a periodic model, in which the birth rate, mortality rates, and diffusion rates are assumed to be periodic in time, is probably the simplest but nonetheless interesting and realistic case. In this paper, we consider the following model:

(1.2)

$$\begin{cases} \partial_t u(t,a,x) + \partial_a u(t,a,x) = d_j(t,a)\Delta u - \mu_j(t,a)u(t,a,x), \quad t>0, a\in(0,\tau),\ x\in\Omega, \\ u(t,0,x) = f(t,u_m(t,x)), \quad t \geq -\tau,\ x\in\Omega, \\ \partial_t u_m(t,x) = d_m(t)\Delta u_m - g(t,u_m(t,x)) + u(t,\tau,x), \quad t>0,\ x\in\Omega, \end{cases}$$

where $d_j(t,a) \geq 0$ and $\mu_j(t,a) \geq 0$ denote the diffusion rate and the per capita mortality rate of juveniles at age a at time t, respectively; $d_m(t) \geq 0$ denotes the diffusion rate of mature individuals at time t; $f(t,u_m)$ and $g(t,u_m)$ are the birth and mortality rates of mature individuals at time t, respectively.

Similarly as in [18] (see also [16]), we integrate along characteristics to reduce the system (1.2) to one equation with nonlocal terms. Let $v(r,a,x) = u(a+r,a,x)$, where r is regarded as a parameter. It follows that

$$\begin{cases} \partial_a v(r,a,x) = \left[\partial_t u(t,a,x) + \partial_a u(t,a,x)\right]_{t=r+a} \\ \qquad\qquad\quad = d_j(a+r,a)\Delta v(r,a,x) - \mu_j(a+r,a)v(r,a,x), \\ v(r,0,x) = f(r,u_m(r,x)). \end{cases}$$

Integrating the last equation, we obtain

$$v(r,a,x) = \int_\Omega \Gamma(\zeta(r,a),x-y)F(r,a)f(r,u_m(r,y))dy,$$

where Γ is the fundamental solution associated with the partial differential operator $\partial_t - \Delta$ and

$$\zeta(r,a) = \int_0^a d_j(s+r,s)ds, \qquad F(r,a) = \exp\left(-\int_0^a \mu_j(s+r,s)ds\right). \tag{1.3}$$

Since $u(t,a,x) = v(t-a,a,x)$, it follows that

$$u(t,a,x) = \int_\Omega \Gamma(\zeta(t-a,a),x-y)F(t-a,a)f(t-a,u_m(t-a,y))dy. \tag{1.4}$$

Set

$$a(t) := \zeta(t-\tau,\tau), \quad b(t) := F(t-\tau,\tau), \quad f_{-\tau}(t,u) := f(t-\tau,u).$$

Substituting (1.4) into the equation for u_m in (1.2), we finally reduce the age-structured population model (1.2) to the following time-delayed reaction-diffusion equation for mature individuals:

(1.5)

$$\begin{cases} \partial_t u_m(t,x) \\ = d_m(t)\Delta u_m - g(t,u_m(t,x)) + b(t)\int_\Omega \Gamma(a(t),x-y) f_{-\tau}(t,u_m(t-\tau,y))dy, \\ u_m(s,x) = \phi(s,x), \quad s\in[-\tau,0],\ x\in\Omega, \end{cases}$$

where $\phi(t,x)$ is an initial function to be specified later. For simplicity, dropping all m's and writing $u_m(t,x)$ as $u(t,x)$, we investigate the following system:

(1.6)

$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u - g(t,u(t,x)) + b(t)\displaystyle\int_\Omega \Gamma(a(t),x-y) f_{-\tau}(t,u(t-\tau,y))dy, \\ u(s,x) = \phi(s,x), \quad s\in[-\tau,0],\ x\in\Omega. \end{cases}$$

Basically we assume that $d_j(t,a)$ and $\mu_j(t,a)$ are periodic in $t \geq 0$ with the period $\omega > 0$ for $a \in (0,\tau)$, and that $d(t)$, $g(t,u)$, and $f(t,u)$ are periodic in t with the period $\omega > 0$ for $u \in \mathbb{R}_+$. This implies that $a(t) = a(t+\omega)$, $d(t) = d(t+\omega)$, $b(t) = b(t+\omega)$, $g(t,u) = g(t+\omega,u)$, and $f(t,u) = f(t+\omega,u)$ for all $t \geq 0$, $u \in \mathbb{R}_+$. Moreover, we assume $d(t) \geq d > 0$ for all $t \geq 0$, and

(H1) $f \in C^1([-\tau,+\infty)\times\mathbb{R}_+,\mathbb{R}_+)$, $g \in C^1(\mathbb{R}_+^2,\mathbb{R}_+)$, $f(t,0) = 0$ for $t \geq -\tau$, $f_u(t,u) > 0$ for all $t \geq -\tau$ and $u \geq 0$, $g(t,0) = 0$ for $t \geq 0$, and there exists $l_1 > 0$ such that $|g(t,u) - g(t,v)| \leq l_1|u-v|$ for all $t \geq 0$ and $u, v \in \mathbb{R}_+$;

(H2) $G(t,u,v) := -g(t,u) + b(t)f_{-\tau}(t,v)$ is strictly subhomogeneous in (u,v) in the sense that for any $\alpha \in (0,1)$, $G(t,\alpha u,\alpha v) > \alpha G(t,u,v)$ for all $u, v \geq 0$;

(H3) there exists positive number $L > 0$ such that $G(t,\bar{L},\bar{L}) \leq 0$ for all $t \geq 0$, $\bar{L} \geq L$.

The purpose of this paper is to study the asymptotic speed of spread and periodic traveling waves of (1.6) in the infinite spatial domain, and the global attractivity of zero or a positive periodic solution of (1.6) in a bounded spatial domain. The asymptotic speed of spread (in short, spreading speed) was first introduced by Aronson and Weinberger [4] for reaction-diffusion equations and has been an important ecological metric in a wide range of ecological applications; see, e.g., [10, 11, 18] and the references therein. Intuitively, the spreading speed c^* in a spatial epidemic model can be interpreted as: if one runs at a speed $c > c^*$, then one will leave the epidemic behind; whereas if one runs at a speed $c < c^*$, then one will eventually be surrounded by the epidemic. Traveling wave solutions have also been investigated extensively for a variety of evolution systems; see, e.g., [5, 10, 11, 17, 18] and the references therein. For the autonomous case of (1.6), the dynamics, including spreading speed and traveling waves, have been studied extensively. So, Wu and Zou [17] investigated traveling wave fronts in the case where $\Omega = \mathbb{R}$, $g(u) = \beta u$. Gourley and Kuang [5] established the linear stabilities of two spatially homogeneous equilibrium solutions, studied traveling wave fronts in the case where $\Omega = \mathbb{R}$, $f(u) = \alpha u$, and $g(u) = \beta u^2$, and obtained a global convergence theorem in the case of bounded intervals. Thieme and Zhao [18] studied the traveling wave solutions, minimal wave speed, and asymptotic speed of spread in the case of $\Omega = \mathbb{R}^n$. Xu and Zhao [20] established a threshold dynamic and global attractivity of the positive steady state when Ω is a bounded domain in $\mathbb{R}^n$.

This paper is organized as follows. In section 2, we first establish the well-posedness and the comparison principle for (1.6) with $\Omega = \mathbb{R}$, then prove the existence of the spreading speed c^* for solutions of (1.6) with initial data having compact supports, and show that it coincides with the minimal wave speed for monotone periodic traveling waves, by appealing to the theory of the spreading speed and traveling waves for monotone periodic semiflows developed in [10, 11]. In section 3, we use the theory of monotone and subhomogeneous dynamical systems to investigate the global dynamics of (1.6) in a bounded domain $\Omega \subseteq \mathbb{R}^n$, and obtain a threshold result for global attractivity of either zero or a positive periodic solution.

2. Spreading speed and traveling waves. In this section, we consider that the population diffuses in an unbounded spatial domain and study (1.6) with $\Omega = \mathbb{R}$:

(2.1)
$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u - g(t,u(t,x)) + b(t)\displaystyle\int_{\mathbb{R}} \Gamma(a(t), x-y) f_{-\tau}(t, u(t-\tau, y))dy, \\ u(t,x) = \phi(t,x), \quad t \in [-\tau, 0],\ x \in \mathbb{R}. \end{cases}$$

In the following, we first apply the threshold dynamics in a scalar periodic and time-delayed equation, developed by Xu and Zhao [21], to the spatially homogeneous system associated with (2.1) to find a periodic solution of (2.1). Then we use the theory of abstract functional differential equations and reaction-diffusion systems to establish the existence of solutions to (2.1) and a comparison principle. Finally, we prove that the solution periodic semiflow of (2.1) satisfies all the assumptions on monotone periodic semiflows in [10], and hence, we obtain the existence of the spreading speed and traveling wave solutions for (2.1).

Let $\mathbb{Y}$ be the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}$ with the usual supreme norm $\|\cdot\|_{\mathbb{Y}}$ (i.e., $\mathbb{Y} = C([-\tau,0],\mathbb{R})$), and let $\mathbb{Y}_+ = C([-\tau,0],\mathbb{R}_+)$. Then $(\mathbb{Y},\mathbb{Y}_+)$ is an ordered Banach space. For $\varphi, \psi \in \mathbb{Y}$, we write $\varphi \le \psi$ if $\psi - \varphi \in \mathbb{Y}_+$, $\varphi < \psi$ if $\psi - \varphi \in \mathbb{Y}_+ \setminus \{0\}$, $\varphi \ll \psi$ if $\psi - \varphi \in \operatorname{int}(\mathbb{Y}_+)$. Moreover, we define $\mathbb{Y}_r = \{\varphi \in \mathbb{Y} : 0 \le \varphi \le r\}$ for any $r \in \mathbb{Y}$ with $r \gg 0$.

Let $\mathbb{X}$ be the set of all bounded and continuous functions from $\mathbb{R}$ into $\mathbb{R}$ and $\mathbb{X}_+ = \{\varphi \in \mathbb{X}; \varphi(x) \ge 0 \text{ for all } x \in \mathbb{R}\}$. For $\varphi, \psi \in \mathbb{X}$, we write $\varphi \le \psi (\varphi \ll \psi)$ if $\varphi(x) \le \psi(x)$ $(\varphi(x) < \psi(x))$ for all $x \in \mathbb{R}$, $\varphi < \psi$ if $\varphi \le \psi$ but $\varphi \ne \psi$. It is easy to see that $\mathbb{X}_+$ is a positive cone of $\mathbb{X}$. Define $\mathbb{X}_r = \{\varphi \in \mathbb{X} : 0 \le \varphi \le r\}$ for any $r \in \mathbb{X}$ with $r \gg 0$. We equip $\mathbb{X}$ with the compact open topology, i.e., $u^m \to u$ in $\mathbb{X}$ means that the sequence of $u^m(x)$ converges to $u(x)$ as $m \to \infty$ uniformly for x in any compact set on $\mathbb{R}$. Define

$$\| u \|_{\mathbb{X}} = \sum_{k=1}^{\infty} \frac{\max_{|x|\le k} |u(x)|}{2^k} \qquad \forall u \in \mathbb{X},$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}$. Then $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a normed space. Let $d_{\mathbb{X}}(\cdot,\cdot)$ be the distance induced by the norm $\|\cdot\|_{\mathbb{X}}$. It follows that the topology in the metric space $(\mathbb{X}, d_{\mathbb{X}})$ is the same as the compact open topology in $\mathbb{X}$. Moreover, $(\mathbb{X}_r, d_{\mathbb{X}})$ is a complete metric space.

Let C be the set of continuous functions from $[-\tau, 0]$ into $\mathbb{X}$, $C_+ = \{\varphi \in C, \varphi(s) \in \mathbb{X}_+, s \in [-\tau, 0]\}$ and $C_r = \{\varphi \in C : 0 \le \varphi \le r\}$ for any $r \in \mathbb{Y}$ with $r \gg 0$. Then C_+ is a positive cone of C. For convenience, we also identify an element $\varphi \in C$ as a function from $[-\tau, 0] \times \mathbb{R}$ into $\mathbb{R}$ defined by $\varphi(s, x) = \varphi(s)(x)$ for any $s \in [-\tau, 0]$ and $x \in \mathbb{R}$. For $\varphi, \psi \in C$, we write $\varphi \le \psi (\varphi \ll \psi)$ if $\varphi(s, x) \le \psi(s, x)$ $(\varphi(s, x) < \psi(s, x))$ for all $s \in [-\tau, 0], x \in \mathbb{R}$, $\varphi < \psi$ if $\varphi \le \psi$ but $\varphi \ne \psi$. For any continuous function $w(\cdot) : [-\tau, b) \to \mathbb{X}$, $b > 0$, we define $w_t \in C$ by $w_t(s) = w(t + s)$ for all $t \in [0, b)$, $s \in [-\tau, 0]$. It is then easy to see that $t \to w_t$ is a continuous function from $[0, b)$ to C. Moreover, we also equip C with the compact open topology and define the norm on C:

$$\| u \|_C = \sum_{k=1}^{\infty} \frac{\max_{|x| \le k, s \in [-\tau, 0]} |u(s, x)|}{2^k} \qquad \forall u \in C,$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}$.

For any constant $N > 0$, $\widehat{N}$ denotes the constant function with value N in $\mathbb{Y}$, $\mathbb{X}$, or C.

Now we consider the spatially homogeneous system associated with (2.1). Letting $u(t, x) = w(t)$, we have

$$\text{(2.2)} \qquad \begin{cases} \dfrac{dw(t)}{dt} = -g(t, w(t)) + b(t) f_{-\tau}(t, w(t - \tau)), \\ w(t) = \varphi(t), \quad t \in [-\tau, 0], \quad \varphi \in \mathbb{Y}_+. \end{cases}$$

The linearized equation associated with (2.2) at $w = 0$ is

$$\text{(2.3)} \qquad \begin{cases} \dfrac{dw(t)}{dt} = -g_u(t, 0) w(t) + b(t) \partial_u f_{-\tau}(t, 0) w(t - \tau), \\ w(t) = \varphi(t), \quad t \in [-\tau, 0], \quad \varphi \in \mathbb{Y}_+. \end{cases}$$

Since g, b, and $f_{-\tau}$ are periodic functions in $t \ge 0$, we can easily see that for any $\varphi \in \mathbb{Y}_+$, (2.3) admits a unique solution $w(t, \varphi)$ existing for all $t \ge -\tau$ with $w(s, \varphi) = \varphi(s)$ for $s \in [-\tau, 0]$, and $w_t(\varphi) \in \mathbb{Y}_+$ for all $t \ge 0$, where $\{w_t\}_{t \ge 0}$ is the solution semiflow for (2.3) defined by $w_t(\varphi)(s) = w(t + s, \varphi)$ for all $s \in [-\tau, 0]$, $t > 0$.

Define the Poincaré map of (2.3) $P : \mathbb{Y}_+ \to \mathbb{Y}_+$ by $P(\varphi) = w_\omega(\varphi)$ for all $\varphi \in \mathbb{Y}_+$, and let $r = r(P)$ be the spectral radius of P. The following two results come from [21].

PROPOSITION 2.1 (see [21, Proposition 2.1]). *$r = r(P)$ is positive and is an eigenvalue of P with a positive eigenfunction φ^*. Moreover, if $\tau = k\omega$ for some integer $k \ge 0$, then $r - 1$ has the same sign as $\int_0^\omega [-g_u(t, 0) + b(t) \partial_u f_{-\tau}(t, 0)] dt$.*

THEOREM 2.2 (see [21, Theorem 2.1]). *Let* (H1)–(H3) *hold. The following statements are valid.*

(i) *If $r \le 1$, then zero solution is globally asymptotically stable for* (2.2) *with respect to $\mathbb{Y}_+$.*
(ii) *If $r > 1$, then* (2.2) *has a unique positive ω-periodic solution $\beta^*(t)$, and $\beta^*(t)$ is globally asymptotically stable with respect to $\mathbb{Y}_+ \setminus \{0\}$.*

In the remainder of this section, we further assume that

(H4) $r = r(P) > 1$.

By the proof of [21, Theorem 2.1] and (H3), it is easy to see that $\beta^*(t) \in [0, L]$ for all $t \ge -\tau$ and $[\hat{0}, \hat{L}]$ is positively invariant for (2.1). Define $\beta_0^* \in \mathbb{Y}_{\hat{L}}$ as $\beta_0^*(s) = \beta^*(s)$ for all $s \in [-\tau, 0]$.

Consider

$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u, \quad t>0, \\ u(0,x) = \phi(x), \quad x\in\mathbb{R}, \phi\in\mathbb{X}. \end{cases} \tag{2.4}$$

The solution of (2.4) can be expressed as

$$u(t,x,\phi) = \int_{\mathbb{R}} \Gamma(\eta(t), x-y)\phi(y)dy, \quad t\geq 0, \tag{2.5}$$

where $\eta(t) = \int_0^t d(s)ds$. According to [8, Chapter II], (2.4) admits an evolution operator $U(t,s):\mathbb{X}\to\mathbb{X}$, $0\leq s\leq t$, which satisfies $U(t,t)=I$, $U(t,s)U(s,\rho)=U(t,\rho)$ for all $0\leq\rho\leq s\leq t$, and $U(t,0)(\phi)(x) = u(t,x,\phi)$ for $t\geq 0$, $x\in\mathbb{R}$, and $\phi\in\mathbb{X}$, where $u(t,x,\phi)$ is the solution of (2.4). Moreover, for any $0\leq s<t$, $U(t,s)$ is a compact and positive operator on $\mathbb{X}$, and $U(t,s)(\phi)(x)>0$ for all $0\leq s<t$, $x\in\mathbb{R}$, and $\phi\in\mathbb{X}$ provided that $\phi(x)\geq 0$ and $\phi\not\equiv 0$.

Define $B:[0,\infty)\times C\to\mathbb{X}$ by $B(t,\phi) := -g(t,\phi(0,\cdot)) + b(t)\int_{\mathbb{R}}\Gamma(a(t),\cdot - y)f_{-\tau}(t,\phi(-\tau,y))dy$ for any $t\in[0,\infty)$, $\phi\in C$. Then (2.1) becomes

$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u + B(t,u_t), \quad t>0, \\ u(t,x) = \phi(t,x), \quad t\in[-\tau,0],\ x\in\mathbb{R}, \end{cases} \tag{2.6}$$

which can be written as an integral equation

$$u(t,\cdot,\phi) = U(t,0)\phi(0,\cdot) + \int_0^t U(t,s)B(s,u_s)ds, \quad t\geq 0, \quad \phi\in C, \tag{2.7}$$

whose solutions are called mild solutions to (2.6).

THEOREM 2.3. *Let* (H1)–(H4) *hold. For any* $\phi\in C_{\hat{L}}$, *system* (2.1) *has a unique mild solution* $u(t,x,\phi)$ *with* $u_0(\cdot,\cdot,\phi)=\phi$ *and* $u_t(\cdot,\cdot,\phi)\in C_{\hat{L}}$ *for all* $t\geq 0$, *and* $u(t,x,\phi)$ *is a classic solution when* $t>\tau$. *Moreover, if* $\hat{u}(t,x)$ *and* $\bar{u}(t,x)$ *are a pair of lower and upper solutions of* (2.1), *respectively, with* $\hat{u}_0(\cdot,\cdot)\leq\bar{u}_0(\cdot,\cdot)$, *then* $\hat{u}_t(\cdot,\cdot)\leq\bar{u}_t(\cdot,\cdot)$ *for all* $t\geq 0$.

Proof. We first show that B is quasi-monotone on $[0,\infty)\times C_{\hat{L}}$ in the sense that

$$\lim_{h\to 0^+} d(\psi(0,\cdot)-\phi(0,\cdot)+h[B(t,\psi)-B(t,\phi)],\mathbb{X}_+) = 0 \tag{2.8}$$

for all $\phi,\psi\in C_{\hat{L}}$ with $\phi(s,x)\leq\psi(s,x)$ for all $s\in[-\tau,0]$, $x\in\mathbb{R}$. In fact, for any $\phi,\psi\in C_{\hat{L}}$ with $\phi(s,x)\leq\psi(s,x)$ for all $(s,x)\in[-\tau,0]\times\mathbb{R}$, we have

$$\begin{aligned}
&\psi(0,\cdot)-\phi(0,\cdot)+h[B(t,\psi)-B(t,\phi)] \\
&= \psi(0,\cdot)-\phi(0,\cdot)+h[-(g(t,\psi(0,\cdot))-g(t,\phi(0,\cdot)))] \\
&+ h\left[\int_{\mathbb{R}}\Gamma(a(t),\cdot - y)b(t)(f_{-\tau}(t,\psi(-\tau,y))-f_{-\tau}(t,\phi(-\tau,y)))dy\right] \\
&\geq \psi(0,\cdot)-\phi(0,\cdot)-h(g(t,\psi(0,\cdot))-g(t,\phi(0,\cdot))) \\
&\geq (1-hl_1)(\psi(0,\cdot)-\phi(0,\cdot)).
\end{aligned}$$

Thus, for $1-hl_1>0$, $\psi(0,\cdot)-\phi(0,\cdot)+h[B(t,\psi)-B(t,\phi)]\in\mathbb{X}_+$, and hence, (2.8) holds. Then by [13, Corollary 5] (for $v^-=\hat{0}$, $v^+=\hat{L}$, $S^+=S^-=S=T\equiv U$,

$B^+ = B^- = B$), (2.1) admits a unique mild solution $u(t, \cdot, \phi)$ on $[-\tau, \infty)$ for any $\phi \in C_{\hat{L}}$ and $u_t(\cdot, \cdot, \phi) \in C_{\hat{L}}$ for all $t \geq 0$. Moreover, the comparison principle holds for lower and upper solutions. ☐

In order to study the spreading speed and traveling waves, we introduce the assumptions in [10, 11]. Let $u \in C$. Define the reflection operator $\mathcal{R}$ by $\mathcal{R}[u](\theta, x) := u(\theta, -x)$ for all $\theta \in [-\tau, 0], x \in \mathbb{R}$. Given $y \in \mathbb{R}$, define the translation operator T_y by $T_y[u](\theta, x) := u(\theta, x - y)$ for all $\theta \in [-\tau, 0], x \in \mathbb{R}$. Let $Q : C_{b^*} \to C_{b^*}$ be a map, where $b^* \in \mathbb{Y}$ with $b^* \gg 0$. Assume the following:

(A1) $Q[\mathcal{R}[u]] = \mathcal{R}[Q[u]]$, $T_y[Q[u]] = Q[T_y[u]]$ $\forall y \in \mathbb{R}$.

(A2) $Q : C_{b^*} \to C_{b^*}$ is continuous with respect to the compact open topology.

(A4) $Q : C_{b^*} \to C_{b^*}$ is monotone in the sense that $Q[u] \geq Q[v]$ whenever $u \geq v$ in C_{b^*}.

(A5) $Q : \mathbb{Y}_{b^*} \to \mathbb{Y}_{b^*}$ admits exactly two fixed points 0 and b^*, and for any positive number ε, there is an $\alpha \in \mathbb{Y}_{b^*}$ with $\| \alpha \|_{\mathbb{Y}} < \varepsilon$ such that $Q[\alpha] \gg \alpha$.

(A6) One of the following two conditions holds:

(a) $Q[C_{b^*}]$ is precompact in C_{b^*}.

(b′) The set $Q[C_{b^*}](0, \cdot)$ is precompact in $\mathbb{X}$, and there is a positive number $\varsigma \leq \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for $-\tau \leq \theta \leq -\varsigma$, the operator

$$S[u](\theta, x) := \begin{cases} u(0, x), & -\tau \leq \theta < -\varsigma, \\ Q[u](\theta, x), & -\varsigma \leq \theta \leq 0 \end{cases}$$

is continuous on C_{b^*}, and $S[D](\cdot, 0)$ is precompact in $\mathbb{Y}$ for any T-invariant set $D \subseteq C_{b^*}$ with $D(0, \cdot)$ being precompact in $\mathbb{X}$. A set $W \subseteq C_{b^*}$ is said to be T-invariant if $T_y W = W$ for all $y \in \mathbb{R}$.

Recall that a family of operators $\{\Phi_t\}_{t \geq 0}$ is an ω-periodic semiflow on a metric space (X, ρ) with the metric ρ, provided that $\{\Phi_t\}$ satisfies

(i) $\Phi_0(v) = v$ $\forall v \in X$;

(ii) $\Phi_t(\Phi_\omega(v)) = \Phi_{t+\omega}(v)$ $\forall t \geq 0, v \in X$;

(iii) $\Phi(t, v) = \Phi_t(v)$ is continuous in (t, v) on $[0, +\infty) \times X$.

Define a family of operators $\{Q_t\}_{t \geq 0}$ on $C_{\hat{L}}$ by

$$Q_t(\phi)(s, x) = u(t + s, x, \phi) \quad \forall t \geq 0, \quad s \in [-\tau, 0], \quad x \in \mathbb{R}, \quad \phi \in C_{\hat{L}},$$

where $u(t, x, \phi)$ is the mild solution of (2.1) with $u(s, x) = \phi(s, x)$ for $s \in [-\tau, 0]$, $x \in \mathbb{R}$. Note that for any $(t_0, \phi_0) \in \mathbb{R}_+ \times C_{\hat{L}}$, we have

$$\| Q_t(\phi) - Q_{t_0}(\phi_0) \|_C \leq \| Q_t(\phi) - Q_t(\phi_0) \|_C + \| Q_t(\phi_0) - Q_{t_0}(\phi_0) \|_C.$$

Note that $U(t, 0)\varphi$ is continuous in $(t, \varphi) \in [0, \infty) \times \mathbb{X}$ with respect to the compact open topology. By a similar argument as in [12, Theorem 8.5.2], it follows that $Q_t(\phi)$ is continuous at (t_0, ϕ_0) with respect to the compact open topology. Thus, $\{Q_t\}_{t \geq 0}$ is an ω-periodic semiflow on $C_{\hat{L}}$.

LEMMA 2.4. *For each $t > 0$, Q_t is strictly subhomogeneous.*

Proof. For any $\phi \in C_{\hat{L}}$ with $\phi \not\equiv 0$, let $u(t, x, \phi)$ be the solution of (2.1) with $u(s, x) = \phi(s, x)$ for $s \in [-\tau, 0]$, $x \in \mathbb{R}$. Fix $k \in (0, 1)$. Since $G(t, u, v)$ is strictly subhomogeneous in (u, v), we have

$$\begin{aligned}
&\partial_t(ku(t, x)) \\
&= d(t)\Delta(ku) - kg(t, u(t, x)) + kb(t) \int_{\mathbb{R}} \Gamma(a(t), x - y) f_{-\tau}(t, u(t - \tau, y)) dy \\
&\leq d(t)\Delta(ku) - g(t, ku(t, x)) + b(t) \int_{\mathbb{R}} \Gamma(a(t), x - y) f_{-\tau}(t, ku(t - \tau, y)) dy.
\end{aligned}$$

Thus, $ku(t,x,\phi)$ is a lower solution of (2.1) with $ku(s,x,\phi) = k\phi(s,x)$ for $s \in [-\tau,0], x \in \mathbb{R}$. Then, $ku(t,x,\phi) \le u(t,x,k\phi)$ for $t \ge 0$, where $u(t,x,k\phi)$ is the solution of (2.1) with $u(s,x,k\phi) = k\phi(s,x)$ for $(s,x) \in [-\tau,0] \times \mathbb{R}$.

Let $w(t,x) = u(t,x,k\phi) - ku(t,x,\phi)$. Then $w(s,x) = 0$ for $(s,x) \in [-\tau,0] \times \mathbb{R}$ and $w(s,x) \ge 0$ for $(s,x) \in [-\tau,\infty) \times \mathbb{R}$. We further show that $w(t,x) > 0$ for all $t > 0, x \in \mathbb{R}$. For simplicity, we write $\widetilde{F}(t,u(t,x),v(t,x)) = -g(t,u(t,x)) + b(t)\int_{\mathbb{R}} \Gamma(a(t),x-y) f_{-\tau}(t,v(t,y))dy$. It follows that

(2.9)
$$\begin{aligned}
&\frac{\partial w(t,x)}{\partial t}\\
&= \frac{\partial u(t,x,k\phi)}{\partial t} - k\frac{\partial u(t,x,\phi)}{\partial t}\\
&= d(t)\Delta u(t,x,k\phi) + \widetilde{F}(t,u(t,x,k\phi),u(t-\tau,x,k\phi))\\
&-k[d(t)\Delta u(t,x,\phi) + \widetilde{F}(t,u(t,x,\phi),u(t-\tau,x,\phi))]\\
&= d(t)\Delta w(t,x) + [\widetilde{F}(t,u(t,x,k\phi),u(t-\tau,x,k\phi)) - \widetilde{F}(t,ku(t,x,\phi),ku(t-\tau,x,\phi))]\\
&+[\widetilde{F}(t,ku(t,x,\phi),ku(t-\tau,x,\phi)) - k\widetilde{F}(t,u(t,x,\phi),u(t-\tau,x,\phi))]\\
&= d(t)\Delta w(t,x) - g(t,u(t,x,k\phi)) + g(t,ku(t,x,\phi)) + h(t,x)\\
&+b(t)\int_{\mathbb{R}} \Gamma(a(t),x-y)[f_{-\tau}(t,u(t-\tau,y,k\phi)) - f_{-\tau}(t,ku(t-\tau,y,\phi))]dy\\
&\ge d(t)\Delta w(t,x) - g(t,u(t,x,k\phi)) + g(t,ku(t,x,\phi)) + h(t,x)\\
&\ge d(t)\Delta w(t,x) - l_1 w(t,x) + h(t,x),
\end{aligned}$$

where $h(t,x) = \widetilde{F}(t,ku(t,x,\phi),ku(t-\tau,x,\phi)) - k\widetilde{F}(t,u(t,x,\phi),u(t-\tau,x,\phi))$. Let $\widetilde{U}(t,s) : \mathbb{X} \to \mathbb{X}$, $0 \le s \le t$, be the evolution operator of

$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u - l_1 u(t,x), & t > 0,\\ u(0,x) = \psi(x), & x \in \mathbb{R}, \psi \in \mathbb{X}. \end{cases}$$

Then $\widetilde{U}(t,s)(\psi)(x) = e^{-l_1(t-s)}U(t,s)(\psi)(x)$ for all $t \ge s \ge 0$, $x \in \Omega$, $\psi \in C$, where $U(t,s)$ is the evolution operator of (2.4). Thus, the equation

$$(2.10)\qquad \begin{cases} \partial_t u(t,x) = d(t)\Delta u - l_1 u(t,x) + h(t,x),\\ u(0,x) = \psi(x), \quad x \in \mathbb{R}, \psi \in \mathbb{X} \end{cases}$$

can be written as

$$(2.11)\qquad u(t,x,\psi) = \widetilde{U}(t,0)(\psi)(x) + \int_0^t \widetilde{U}(t,s)h(s,x)ds, \quad t \ge 0, x \in \mathbb{R}, \psi \in C.$$

By (H2), we have $h(t,x) > 0$ for all $t > 0, x \in \mathbb{R}$. It then follows from (2.11) and the property of $\widetilde{U}(t,s)$ that for any $\psi \ge 0$ with $\psi \not\equiv 0$, the solution of (2.10) satisfies $u(t,x,\psi) > 0$ for all $t > 0, x \in \mathbb{R}$. Then by (2.9) and the comparison principle, we have $w(t,x) > 0$ for all $t > 0, x \in \mathbb{R}$. Therefore, $u(t,x,k\phi) > ku(t,x,\phi)$ for all $t > 0, x \in \mathbb{R}$, and hence, $Q_t(k\phi) > kQ_t(\phi)$ for all $t > 0$, which indicates that for each $t > 0$, Q_t is strictly subhomogeneous. ∎

LEMMA 2.5. *For any $\varphi \in C_{\widehat{L}}$ with $\varphi \not\equiv 0$, $u(t,x,\varphi) > 0$ for all $t \geq \tau$, $x \in \mathbb{R}$.*

Proof. Let $\varphi \in C_{\widehat{L}}$ with $\varphi \not\equiv 0$. By Theorem 2.3, $u(t,x,\varphi) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. It follows from (H1) that for any $t > 0$, $u(t,x,\varphi)$ satisfies

$$\begin{aligned}\partial_t u(t,x) &= d(t)\Delta u - g(t,u(t,x,\varphi)) + b(t)\int_{\mathbb{R}} \Gamma(a(t),x-y) f_{-\tau}(t,u(t-\tau,y,\varphi)) \\ &\geq d(t)\Delta u - g(t,u(t,x,\varphi)) \\ &\geq d(t)\Delta u - l_1 u(t,x,\varphi).\end{aligned}$$

By [19, Theorem 5.5.4], $u(t,x,\varphi) > 0$ for all $t > 0$, $x \in \mathbb{R}$, provided that $\varphi(0,\cdot) > 0$.

Now we show that for any $\varphi \in C_{\widehat{L}}$ with $\varphi \not\equiv 0$ and $\varphi(0,\cdot) = 0$, there exists $t_0 = t_0(\varphi) \in [0,\tau]$ such that $u(t_0,\cdot,\varphi) > 0$. Assume, by contradiction, that for some $\varphi \in C_{\widehat{L}}$ with $\varphi \not\equiv 0$ and $\varphi(0,\cdot) = 0$ we have $u(t,\cdot,\varphi) \equiv 0$ for all $t \in [0,\tau]$. It follows from (2.7) that

$$0 = \int_0^t U(t,s)b(s)\int_{\mathbb{R}} \Gamma(a(s),x-y) f_{-\tau}(s,u_s(-\tau,y))dyds, \quad t \in [0,\tau],$$

which implies that $\int_{\mathbb{R}} \Gamma(a(s),x-y) f_{-\tau}(s,u_s(-\tau,y))dy = 0$ for any $s \in [0,\tau]$, and hence, $f_{-\tau}(s,u_s(-\tau,y)) = 0$ for any $s \in [0,\tau]$, $y \in \mathbb{R}$. Then by (H1), $u_s(-\tau,y) = 0$ for any $s \in [0,\tau]$, $y \in \mathbb{R}$. That is, $\varphi \equiv 0$, a contradiction. Thus, we have $u(t_0,\cdot,\varphi) > 0$ for some $t_0 = t_0(\varphi) \in [0,\tau]$. Then for any $t > t_0$, $\widetilde{U}(t,t_0)[u(t_0,\cdot,\varphi)](x) = e^{-l_1(t-t_0)}U(t,t_0)[u(t_0,\cdot,\varphi)](x) > 0$, and hence, by the comparison principle, we have $u(t,x,\varphi) > 0$ for all $t > t_0$, $x \in \mathbb{R}$.

Therefore, for any $\varphi \in C_{\widehat{L}}$ with $\varphi \not\equiv 0$, $u(t,x,\varphi) > 0$ for all $t > \tau, x \in \mathbb{R}$. ∎

LEMMA 2.6. *For any $t > 0$, Q_t satisfies* (A1), (A2), (A4), *and* (A6) *with $b^* = \widehat{L}$, and Q_ω satisfies* (A5) *with $b^* = \beta_0^*$, where $\beta_0^* \in \mathbb{Y}_{\widehat{L}}$ with $\beta_0^*(s) = \beta^*(s)$ for all $s \in [-\tau,0]$.*

Proof. It is easy to see that Q_t satisfies (A1), (A2), and (A4) with $b^* = \widehat{L}$ for any $t > 0$.

Let $\hat{Q}_t = Q_t|_{\mathbb{Y}_{\widehat{L}}}$. Then $\hat{Q}_t : \mathbb{Y}_{\widehat{L}} \to \mathbb{Y}_{\widehat{L}}$ is the ω-periodic semiflow generated by (2.2). Moreover, it is not difficult to see that $\hat{Q}_t$ is strictly monotone for any $t \geq \tau$ and strongly monotone for any $t \geq 2\tau$ on $\mathbb{Y}_{\widehat{L}}$. Note that (2.2) has a positive ω-periodic solution $\beta^*(t)$ which is globally asymptotically stable in $\mathbb{Y}_{\widehat{L}} \setminus \{0\}$. We see that $\hat{Q}_\omega$ has only two fixed points 0 and β_0^* in $\mathbb{Y}_{\widehat{L}}$, where $\beta_0^*(s) = \beta^*(s)$ for all $s \in [-\tau,0]$. Thus, by the Dancer–Hess connecting orbit lemma (see, e.g., [23]), the map $\hat{Q}_\omega$ admits a strictly monotone full orbit $\{\varphi_n\}_{-\infty}^{\infty} \subseteq \mathbb{Y}_{\beta_0^*}$ connecting 0 to β_0^* and $\varphi_n < \varphi_{n+1}$ for any $n = 0, \pm 1, \pm 2, \ldots$. For any $\bar{n} \in \mathbb{N}$ such that $\bar{n}\omega \geq 2\tau$, since $\hat{Q}_{\bar{n}\omega}$ is strongly monotone, we have $\hat{Q}_{\bar{n}\omega}(\varphi_n) = \hat{Q}_\omega^{\bar{n}}(\varphi_n) \ll \hat{Q}_\omega^{\bar{n}}(\varphi_{n+1}) = \hat{Q}_{\bar{n}\omega}(\varphi_{n+1})$ for any $n = 0, \pm 1, \pm 2, \ldots$. That is, $\varphi_{n+\bar{n}\omega} \ll \varphi_{n+1+\bar{n}\omega}$ for any $n = 0, \pm 1, \pm 2, \ldots$. Therefore, $\varphi_n \ll \varphi_{n+1}$ for any $n = 0, \pm 1, \pm 2, \ldots$, and hence, Q_ω satisfies (A5) with $b^* = \beta_0^*$.

Now we show that Q_t satisfies (A6)(a) with $b^* = \widehat{L}$ for $t > \tau$. Fix $t_0 > \tau$ and set $a = t_0 - \tau$, $b = t_0$. Let $u(t,\varphi)$ be the solution of (2.1) with $u_0(\varphi) = \varphi \in C_{\widehat{L}}$ and define the Kuratowski measure of noncompactness of a subset A of $\mathbb{X}$ as

$$\alpha(A) = \inf\{r > 0 : A \text{ has a finite cover of diameter } \leq r\}.$$

First we prove that $\overline{\{u(t,\varphi) : a \le t \le b, \varphi \in C_{\widehat{L}}\}}$ is compact in $\mathbb{X}$. By (2.7), for any $\epsilon \in (0,a), t \in [a,b]$, and $\varphi \in C_{\widehat{L}}$, we have

$$\begin{aligned}
&u(t,\varphi)\\
&= U(t,0)\varphi(0,\cdot) + \int_0^{t-\epsilon} U(t,s)B(s,u_s)ds + \int_{t-\epsilon}^t U(t,s)B(s,u_s)ds\\
&= U(t,t-\epsilon)\left[U(t-\epsilon,0)\varphi(0,\cdot) + \int_0^{t-\epsilon} U(t-\epsilon,s)B(s,u_s)ds\right] + \int_{t-\epsilon}^t U(t,s)B(s,u_s)ds\\
&= U(t,t-\epsilon)u(t-\epsilon,\varphi) + \int_{t-\epsilon}^t U(t,s)B(s,u_s)ds.
\end{aligned}$$

Since $\{u(t-\epsilon,\varphi), t \in [a,b], \varphi \in C_{\widehat{L}}\}$ is bounded in $\mathbb{X}_+$ and $U(t,t-\epsilon)$ is compact, we have

$$\alpha(\{U(t,t-\epsilon)u(t-\epsilon,\varphi), t \in [a,b], \varphi \in C_{\widehat{L}}\}) = 0.$$

It is easy to see $\{U(t,s)B(s,u_s) : t \in [a,b], s \in [0,t], \varphi \in C_{\widehat{L}}\}$ is bounded in $\mathbb{X}_+$. Let $N > 0$ such that $\| U(t,s)B(s,u_s) \|_{\mathbb{X}} \le N$ for all $t \in [a,b], s \in [0,t], \varphi \in C_{\widehat{L}}$. By the fact of $\alpha(A) \le \delta(A)$, where $\delta(A)$ is the diameter of $A \subseteq \mathbb{X}$, we have

$$\alpha\left(\left\{\int_{t-\epsilon}^t U(t,s)B(s,u_s)ds : t \in [a,b], s \in [t-\epsilon,t], \varphi \in C_{\widehat{L}}\right\}\right) \le 2\epsilon N.$$

Thus,

$$\begin{aligned}
&\alpha(\{u(t,\varphi) : t \in [a,b], \varphi \in C_{\widehat{L}}\})\\
&\le \alpha(\{U(t,t-\epsilon)u(t-\epsilon,\varphi), t \in [a,b], \varphi \in C_{\widehat{L}}\})\\
&\quad + \alpha\left(\left\{\int_{t-\epsilon}^t U(t,s)B(s,u_s)ds : t \in [a,b], s \in [t-\epsilon,t], \varphi \in C_{\widehat{L}}\right\}\right)\\
&\le 2\epsilon N.
\end{aligned}$$

Letting $\epsilon \to 0$, we have $\alpha(\{u(t,\varphi) : t \in [a,b], \varphi \in C_{\widehat{L}}\}) = 0$, and hence, $\{u(t,\varphi) : t \in [a,b], \varphi \in C_{\widehat{L}}\}$ is precompact in $\mathbb{X}$.

Given a compact interval $I \subseteq \mathbb{R}$, let $K = \min\{K_1 > 0 : I \subseteq [-K_1, K_1]\}$. Since $\{u(t,\varphi) : t \in [a,b], \varphi \in C_{\widehat{L}}\}$ is precompact in $\mathbb{X}$, $\{u(t,\varphi)|_I : t \in [a,b],\ \varphi \in C_{\widehat{L}}\}$ is equicontinuous in $\mathbb{X}$, and hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|u(t,x_1,\varphi) - u(t,x_2,\varphi)| < \epsilon \tag{2.12}$$

for all $t \in [a,b]$ and $\varphi \in C_{\widehat{L}}$, provided that $x_1, x_2 \in I$ and $|x_1 - x_2| < \delta$.

Let $[a_1, b_1]$ be any bounded interval on $\mathbb{R}$ with $a_1 > 0$ and let $U_0(t)$ be the semigroup generated by $u_t = \Delta u$. Then $U_0(t)\varphi(x) = \int_{-\infty}^{+\infty} \Gamma(t, x-y)\varphi(y)dy$ for all $t > 0$, $x \in \mathbb{R}$, $\varphi \in \mathbb{X}$.

By the properties of Γ, we can find an $N_0 > 0$ such that $\int_{|y|\ge N_0} \Gamma(b_1, y)dy \le \epsilon$. Since $\frac{\partial \Gamma(t,y)}{\partial t} > 0$ for all $t > 0$ and $y^2 > 2t$, we have $\int_{|y|\ge N_1} \Gamma(t,y)dy \le \epsilon$ for all $t \in [a_1, b_1]$, where $N_1 = \max\{N_0, \sqrt{2b_1}\}$. Moreover, since $\int_{-N_1}^{N_1} \Gamma(t,y)dy$ is continuous in $t \in [a_1, b_1]$, there is a $\delta_1 > 0$ such that $|\int_{-N_1}^{N_1} (\Gamma(t_1,y) - \Gamma(t_2,y))dy| < \epsilon$ provided that

$t_1, t_2 \in [a_1, b_1]$ and $|t_1 - t_2| < \delta_1$. Therefore, for any $t_1, t_2 \in [a_1, b_1]$ and $|t_1 - t_2| < \delta_1$, $\psi \in \mathbb{X}_{\widehat{L}}, x \in I$,

$$\begin{aligned}
&|(U_0(t_1)\psi)(x) - (U_0(t_2)\psi)(x)| \\
&= \left| \int_{\mathbb{R}} \Gamma(t_1, x-y)\psi(y)dy - \int_{\mathbb{R}} \Gamma(t_2, x-y)\psi(y)dy \right| \\
&= \left| \int_{\mathbb{R}} (\Gamma(t_1, y) - \Gamma(t_2, y))\psi(x-y)dy \right| \\
&\le \left| \int_{|y|\le N_1} (\Gamma(t_1, y) - \Gamma(t_2, y))\psi(x-y)dy \right| + \left| \int_{|y|\ge N_1} (\Gamma(t_1, y) - \Gamma(t_2, y))\psi(x-y)dy \right| \\
&< 2\epsilon L.
\end{aligned}$$

It follows from the continuity of $\eta(t)$ in $t \in \mathbb{R}_+$ and definitions of $U_0(t)$ and $U(t,s)$ that there exists $\delta_2 > 0$ such that

$$|(U(t_1, 0)\varphi(0, \cdot))(x) - (U(t_2, 0)\varphi(0, \cdot))(x)| < 2\epsilon L$$

for all $x \in I$, $\varphi \in C_{\widehat{L}}$, provided that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta_2$. Let $\bar{\delta} \in (0, \min\{\epsilon, \delta_2\})$. Then for $x \in I$, $\varphi \in C_{\widehat{L}}$, $t_1, t_2 \in [a, b]$, and $|t_1 - t_2| < \bar{\delta}$, we have

$$\begin{aligned}
&|u(t_1, x, \varphi) - u(t_2, x, \varphi)| \\
&\le |(U(t_1, 0)\varphi(0, \cdot))(x) - (U(t_2, 0)\varphi(0, \cdot))(x)| \\
&\quad + \left| \int_0^{t_1} (U(t_1, s)B(s, u_s))(x)ds - \int_0^{t_2} (U(t_2, s)B(s, u_s))(x)ds \right| \\
&\le 2L\epsilon + 2N \cdot 2^K \bar{\delta} \\
&\le 2(L + 2^K N)\epsilon,
\end{aligned} \tag{2.13}$$

where N was defined in the former paragraph of this proof. This implies that $u(t, x, \varphi)$ is equicontinuous in $t \in [a, b]$ for $x \in I$ and $\varphi \in C_{\widehat{L}}$.

Consequently, by (2.12) and (2.13), for any $\varphi \in C_{\widehat{L}}$, $\theta_1, \theta_2 \in [-\tau, 0]$, $x_1, x_2 \in I$ with $|\theta_1 - \theta_2| < \bar{\delta}$ and $|x_1 - x_2| < \delta$, we have

$$\begin{aligned}
&|u_{t_0}(\varphi)(\theta_1, x_1) - u_{t_0}(\varphi)(\theta_2, x_2)| \\
&= |u(t_0 + \theta_1, x_1, \varphi) - u(t_0 + \theta_2, x_2, \varphi)| \\
&\le |u(t_0 + \theta_1, x_1, \varphi) - u(t_0 + \theta_1, x_2, \varphi)| + |u(t_0 + \theta_1, x_2, \varphi) - u(t_0 + \theta_2, x_2, \varphi)| \\
&\le (2L + 2^{K+1}N + 1)\epsilon,
\end{aligned}$$

which indicates that $\{u_{t_0}(\varphi) : \varphi \in C_{\widehat{L}}\}$ is equicontinuous for $(\theta, x) \in [-\tau, 0] \times I$. Therefore, $\{u_{t_0}(\varphi) : \varphi \in C_{\widehat{L}}\}$ is precompact in $C_{\widehat{L}}$ and (A6)(a) follows from $Q_{t_0}(C_{\widehat{L}}) = \{u_{t_0}(\varphi) : \varphi \in C_{\widehat{L}}\}$ for $t_0 > \tau$.

Finally, we show that Q_t satisfies (A6)(b′) with $b^* = \widehat{L}$ for $0 < t \le \tau$. Fix $t_1 \in (0, \tau]$ and define

$$S[\varphi](\theta, x) := \begin{cases} \varphi(0, x), & -\tau \le \theta < -t_1, \\ Q_{t_1}(\varphi)(\theta, x), & -t_1 \le \theta \le 0 \end{cases}$$

for any $\varphi \in C_{\widehat{L}}$. By the above analysis, we know that $\{u(t,\varphi) : a \le t \le b, \varphi \in C_{\widehat{L}}\}$ is precompact in $\mathbb{X}$ for any $0 < a \le b$. In particular, fixing $a = b = t_1$, we can easily see that $\{u_{t_1}(\varphi)(0,\cdot), \varphi \in C_{\widehat{L}}\} = \{u(t_1,\cdot,\varphi), \varphi \in C_{\widehat{L}}\}$ is precompact in $\mathbb{X}$, that is, $Q_{t_1}[C_{\widehat{L}}](0,\cdot)$ is precompact in $\mathbb{X}$.

Since Q_t is an ω-periodic semiflow, it is easy to see that $S[\varphi]$ is continuous on $C_{\widehat{L}}$. Let D be a T-invariant subset of $C_{\widehat{L}}$ (i.e., $T_y D = D$ for all $y \in \mathbb{R}$) with $D(0,\cdot)$ being precompact in $\mathbb{X}$. Now we show that for any given compact interval $I \subseteq \mathbb{R}$, $S[D]$ is equicontinuous on $[-\tau, 0] \times I$, that is, for any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|S[\varphi](\theta_1, x_1) - S[\varphi](\theta_2, x_2)| < \epsilon$ for any $\varphi \in D$ if $\theta_1, \theta_2 \in [-\tau, 0], x_1, x_2 \in I$, and $|\theta_1 - \theta_2| < \delta_1, |x_1 - x_2| < \delta_2$.

Since $S[\varphi](\theta, x) = \varphi(0, x)$ for all $\varphi \in D$, $\theta \in [-\tau, -t_1]$, $x \in I$, and $D(0,\cdot)$ is precompact in $\mathbb{X}$, it is obvious that $S[D]$ is equicontinuous on $[-\tau, -t_1] \times I$.

Note that there exists $N > 0$ such that $\| U(t,s)B(s,u_s) \|_{\mathbb{X}} \le N$ for all $t \in [0, t_1], s \in [0, t], \varphi \in C_{\widehat{L}}$. Let $\delta_0 = \min\{\epsilon/(2^K N), t_1\}$. Then for any $t < \delta_0$, $x \in I$, and $\varphi \in D$, we have

$$\left| \int_0^t U(t,s)B(s,u_s)(x)ds \right| < 2^K N \delta_0 = \epsilon. \tag{2.14}$$

Let $\mathcal{F}(t,\psi) := U(t,0)\psi$ for $(t,\psi) \in [0,\delta_0] \times D(0,\cdot)$. Then $\mathcal{F}$ is continuous on $[0,\delta_0] \times D(0,\cdot)$ and $\mathcal{F}([0,\delta_0] \times D(0,\cdot))$ is precompact in $\mathbb{X}$. Thus, for the above I, there exists $\delta_2 > 0$ such that for $x_1, x_2 \in I$ and $|x_1 - x_2| < \delta_2$, we have

$$|U(t,0)\psi(x_1) - U(t,0)\psi(x_2)| < \epsilon \quad \forall t \in [0,\delta_0], \psi \in D(0,\cdot). \tag{2.15}$$

Moreover, since $\mathcal{F}$ is uniformly continuous on $[0,\delta_0] \times D(0,\cdot)$, there exists $\delta_1 > 0, \delta_3 > 0$ such that $\| \mathcal{F}(\bar{t}_1, \psi_1) - \mathcal{F}(\bar{t}_2, \psi_2) \|_{\mathbb{X}} < \epsilon/2^K$ if $\bar{t}_1, \bar{t}_2 \in [0,\delta_0]$, $\psi_1, \psi_2 \in D(0,\cdot)$, and $|\bar{t}_1 - \bar{t}_2| < \delta_1$, $||\psi_1 - \psi_2||_{\mathbb{X}} < \delta_3$. In particular, we have $\| U(\bar{t}_1, 0)\psi - U(\bar{t}_2, 0)\psi \|_{\mathbb{X}} < \epsilon/2^K$ if $\bar{t}_1, \bar{t}_2 \in [0,\delta_0]$, $\psi \in D(0,\cdot)$, and $|\bar{t}_1 - \bar{t}_2| < \delta_1$. Then

$$|U(\bar{t}_1,0)\psi(x) - U(\bar{t}_2,0)\psi(x)| < \epsilon \quad \forall \psi \in D(0,\cdot), x \in I, \bar{t}_1, \bar{t}_2 \in [0,\delta_0], \text{ and } |\bar{t}_1 - \bar{t}_2| < \delta_1. \tag{2.16}$$

By (2.14)–(2.16), we can easily obtain that if $\theta_1, \theta_2 \in [-t_1, \delta_0 - t_1], x_1, x_2 \in I$ and $|\theta_1 - \theta_2| < \delta_1, |x_1 - x_2| < \delta_2$, then for any $\varphi \in D$,

$$\begin{aligned}
&|S[\varphi](\theta_1, x_1) - S[\varphi](\theta_2, x_2)| \\
&= |Q_{t_1}[\varphi](\theta_1, x_1) - Q_{t_1}[\varphi](\theta_2, x_2)| \\
&= |u(t_1 + \theta_1, x_1, \varphi) - u(t_1 + \theta_2, x_2, \varphi)| \\
&\le |(U(t_1 + \theta_1, 0)\varphi(0,\cdot))(x_1) - (U(t_1 + \theta_2, 0)\varphi(0,\cdot))(x_2)| \\
&\quad + \left| \int_0^{t_1+\theta_1} U(t_1 + \theta_1, s)B(s,u_s)(x_1)ds - \int_0^{t_1+\theta_2} U(t_1 + \theta_2, s)B(s,u_s)(x_2)ds \right| \\
&\le |(U(t_1 + \theta_1, 0)\varphi(0,\cdot))(x_1) - (U(t_1 + \theta_1, 0)\varphi(0,\cdot))(x_2)| \\
&\quad + |(U(t_1 + \theta_1, 0)\varphi(0,\cdot))(x_2) - (U(t_1 + \theta_2, 0)\varphi(0,\cdot))(x_2)| + 2\epsilon \\
&< 4\epsilon,
\end{aligned}$$

which implies that $S[D]$ is equicontinuous on $[-t_1, \delta_0 - t_1] \times I$.

By a similar argument as for (A6)(a), it is easy to see that S[D] is equicontinuous on $[\delta_0 - t_1, 0] \times I$.

Therefore, $S[D]$ is equicontinuous on $[-\tau, 0] \times I$, and hence, $S[D]$ is precompact in $C_{\hat{L}}$. Thus, (A6)(b′) is valid for Q_t, $t \in (0, \tau]$. ∎

It then follows from Lemma 2.6 and [11, Theorems 2.11 and 2.15] that Q_ω has an asymptotic speed of spread $c^*_\omega > 0$.

Consider the linearized system of (2.1) at the zero solution:

(2.17)

$$\begin{cases} \partial_t u(t,x) = d(t)\Delta u - g_u(t,0)u(t,x) + b(t)\partial_u f_{-\tau}(t,0)\displaystyle\int_{\mathbb{R}} \Gamma(a(t), x-y)u(t-\tau, y)dy, \\ u(t,x) = \phi(t,x), \quad t \in [-\tau, 0], \; x \in \mathbb{R}. \end{cases}$$

For $\alpha > 0$, let $u(t,x) = e^{-\alpha x}v(t)$. Substituting $u(t,x)$ into (2.17) yields

$$\begin{aligned} &e^{-\alpha x}v'(t) \\ &= d(t)\alpha^2 e^{-\alpha x}v(t) - g_u(t,0)v(t)e^{-\alpha x} + b(t)\partial_u f_{-\tau}(t,0)v(t-\tau)\textstyle\int_{\mathbb{R}} \Gamma(a(t), y)e^{-\alpha(x-y)}dy. \end{aligned}$$

Since $\Gamma(t,x)$ is even in x and by [18, Proposition 4.2], we obtain

(2.18)

$$\begin{aligned} v'(t) &= d(t)\alpha^2 v(t) - g_u(t,0)v(t) + b(t)\partial_u f_{-\tau}(t,0)v(t-\tau)\int_{\mathbb{R}} \Gamma(a(t), y)e^{\alpha y}dy, \\ &= d(t)\alpha^2 v(t) - g_u(t,0)v(t) + b(t)\partial_u f_{-\tau}(t,0)v(t-\tau)\int_{\mathbb{R}} \Gamma(a(t), y)e^{-\alpha y}dy, \\ &= d(t)\alpha^2 v(t) - g_u(t,0)v(t) + b(t)\partial_u f_{-\tau}(t,0)v(t-\tau)e^{\alpha^2 a(t)}. \end{aligned}$$

Then $u(t,x) = e^{-\alpha x}v(t)$ satisfies (2.17) with $\phi(s,x) = e^{-\alpha x}v(s)$ for $s \in [-\tau, 0]$ and $x \in \mathbb{R}$ if $v(t)$ satisfies (2.18) for $t \geq 0$.

Let M_t be the linear solution map defined by (2.17) and let $v(t, v_0)$ be the solution of (2.18) with $v(s, v_0) = v_0(s)$ for $s \in [-\tau, 0]$, $v_0 \in \mathbb{Y}$. Define $B^t_\alpha(v_0) := M_t(v_0 e^{-\alpha x})(0)$. It is not difficult to see that $B^t_\alpha(v_0) = v(t, v_0)$, and hence, B^t_α is the solution map associated with (2.18) on $\mathbb{Y}$.

Let $\gamma(\alpha)$ be the spectral radius of the Poincaré map associated with (2.18), and [21, Proposition 2.1] implies that $\gamma(\alpha) > 0$. It follows from the proof of [21, Proposition 2.1] that there exists a positive ω-periodic function $w(t)$ such that $v(t) = e^{\lambda(\alpha)t}w(t)$ is a solution of (2.18), where $\lambda(\alpha) = \frac{\ln \gamma(\alpha)}{\omega}$. Define $\psi \in \mathbb{Y}$ by $\psi(\theta) = e^{\lambda(\alpha)\theta}w(\theta)$ for all $\theta \in [-\tau, 0]$. Clearly, $v(t, \psi) = e^{\lambda(\alpha)t}w(t)$ for all $t \geq 0$. Then we have

$$B^t_\alpha(\psi)(\theta) = v(t+\theta, \psi) = e^{\lambda(\alpha)t}e^{\lambda(\alpha)\theta}w(t+\theta) \;\forall \theta \in [-\tau, 0], t \geq 0.$$

By the ω-periodicity of $w(t)$, it follows that

$$B^\omega_\alpha(\psi)(\theta) = e^{\lambda(\alpha)\omega}e^{\lambda(\alpha)\theta}w(\theta) = e^{\lambda(\alpha)\omega}\psi(\theta) \;\forall \theta \in [-\tau, 0],$$

that is, $B^\omega_\alpha(\psi) = e^{\lambda(\alpha)\omega}\psi$. This implies that $e^{\lambda(\alpha)\omega}$ is the principle eigenvalue of B^ω_α with positive eigenfunction ψ.

Let $\Phi(\alpha) := \frac{1}{\alpha}\ln e^{\lambda(\alpha)\omega} = \frac{\lambda(\alpha)\omega}{\alpha} = \frac{\ln \gamma(\alpha)}{\alpha}$. Then we have the following result.

PROPOSITION 2.7. *Assume that* (H1)–(H4) *hold. Let* c^*_ω *be the asymptotic speed of spread of* Q_ω. *Then* $c^*_\omega = \inf_{\alpha>0}\Phi(\alpha) = \inf_{\alpha>0}\frac{\ln\gamma(\alpha)}{\alpha}$.

Proof. When $\alpha = 0$, (2.18) becomes (2.3). It follows from (H4) that $\gamma(0) > 1$, and hence (C7) in [11] is satisfied. Now we prove that $\Phi(\infty) = \infty$. By (2.18), we have

$$v'(t) \geq [\alpha^2 d(t) - g_u(t,0)]v(t)\ \forall t \geq 0,$$

and hence,

$$\frac{w'(t)}{w(t)} \geq \alpha^2 d(t) - g_u(t,0) - \lambda(\alpha).$$

Then

$$0 = \int_0^\omega \frac{w'(t)}{w(t)}dt \geq \int_0^\omega (\alpha^2 d(t) - g_u(t,0))dt - \lambda(\alpha)\omega,$$

which implies that

$$\lambda(\alpha)\omega \geq \alpha^2 \int_0^\omega d(t)dt - \int_0^\omega g_u(t,0)dt.$$

Therefore,

$$\Phi(\alpha) = \frac{\lambda(\alpha)\omega}{\alpha} \geq \alpha\int_0^\omega d(t)dt - \frac{\int_0^\omega g_u(t,0)dt}{\alpha}.$$

Letting $\alpha \to \infty$, we can easily obtain $\Phi(\infty) = \infty$.

Since $G(t,\cdot,\cdot)$ is subhomogeneous in (u,v), it follows from [23, Lemma 2.3.2] that $G(t,u,v) \leq G_u(t,0,0)u + G_v(t,0,0)v$, that is,

$$-g(t,u) + b(t)f_{-\tau}(t,v) \leq -g_u(t,0)u + b(t)\partial_u f_{-\tau}(t,0)v,$$

and hence, we have

$$\begin{aligned}&-g(t,u(t,x)) + b(t)\int_{\mathbb{R}}\Gamma(a(t),x-y)f_{-\tau}(t,u(t-\tau,y))dy\\ &\leq -g_u(t,0)u(t,x) + b(t)\partial_u f_{-\tau}(t,0)\int_{\mathbb{R}}\Gamma(a(t),x-y)u(t-\tau,y)dy.\end{aligned}$$

By the comparison principle, we have $Q_\omega(\varphi) \leq M_\omega(\varphi)$ for any $\varphi \in C_{\beta_0^*}$. Thus, [11, Theorem 3.10] implies that $c^*_\omega \leq \inf_{\alpha>0}\Phi(\alpha)$.

Let $K > 0$ such that $K - g_u(t,0) > 0$ for all $t \in [0,\omega]$. Set $\bar{G}(t,u,v) = Ku + G(t,u,v)$. Then $\bar{G}_u(t,0,0) > 0$, $\bar{G}_v(t,0,0) > 0$ for all $t \in [0,\omega]$. It is easy to see that for any $\epsilon \in (0,1)$, there exists $\delta = \delta(\epsilon) \in (0,L)$ such that

$$\bar{G}(t,u,v) \geq (1-\epsilon)\bar{G}_u(t,0,0)u + (1-\epsilon)\bar{G}_v(t,0,0)v\ \forall (u,v) \in [0,\delta]^2,$$

and hence, for any $(u,v) \in [0,\delta]^2$,

$$G(t,u,v) = -Ku + \bar{G}(t,u,v) \geq [(1-\epsilon)G_u(t,0,0) - \epsilon K]u + (1-\epsilon)G_v(t,0,0)v.$$

Moreover, there exists $\xi = \xi(\delta) > 0$ such that for any $\varphi \in C_{\hat{\xi}}$, we have

$$0 \le u(t, x, \varphi) \le u(t, x, \hat{\xi}) < \delta \quad \forall x \in \mathbb{R}, t \in [0, \omega].$$

Thus, for any $\varphi \in C_{\hat{\xi}}$, $u(t, x, \varphi)$ satisfies

$$\begin{aligned}\partial_t u(t,x) \ge\, & d(t)\Delta u(t,x) + [(1-\epsilon)g_u(t,0) - \epsilon K]u(t,x) \\ & + (1-\epsilon)b(t)\partial_u f_{-\tau}(t,0)\int_{\mathbb{R}} \Gamma(a(t), x-y)u(t-\tau, y)dy \quad \forall t \in [0,\omega].\end{aligned}$$

Let $M_t^\epsilon, t \ge 0$, be the solution maps associated with the linear system

$$\begin{aligned}\partial_t u(t,x) =\, & d(t)\Delta u(t,x) + [(1-\epsilon)g_u(t,0) - \epsilon K]u(t,x) \\ & + (1-\epsilon)b(t)\partial_u f_{-\tau}(t,0)\int_{\mathbb{R}} \Gamma(a(t), x-y)u(t-\tau, y)dy \quad \forall t \in [0,\omega].\end{aligned}$$

The comparison principle implies that $M_t^\epsilon(\varphi) \le Q_t(\varphi)$ for all $\varphi \in C_{\hat{\xi}}, t \in [0,\omega]$. In particular, $M_\omega^\epsilon(\varphi) \le Q_\omega(\varphi)$ for all $\varphi \in C_{\hat{\xi}}$. By a similar analysis for M_t^ϵ as for M_t, it follows from [11, Theorem 3.10] that $\inf_{\alpha>0}\Phi_\epsilon(\alpha) \le c_\omega^*$.

Therefore, $\inf_{\alpha>0}\Phi_\epsilon(\alpha) \le c_\omega^* \le \inf_{\alpha>0}\Phi(\alpha)$ for all $\epsilon \in (0,1)$. Letting $\epsilon \to 0$, we have $c_\omega^* = \inf_{\alpha>0}\Phi(\alpha)$. ◻

Let $c^* = \frac{c_\omega^*}{\omega} = \frac{1}{\omega}\inf_{\alpha>0}\Phi(\alpha) = \frac{1}{\omega}\inf_{\alpha>0}\frac{\ln\gamma(\alpha)}{\alpha}$. The following result shows that c^* is the spreading speed of solutions of (2.1) with initial functions having compact support.

THEOREM 2.8. *Assume that* (H1)–(H4) *hold and let* $c^* = c_\omega^*/\omega$. *Then the following statements are valid.*

(1) *For any* $c > c^*$, *if* $\varphi \in C_{\beta_0^*}$ *with* $0 \le \varphi \ll \beta_0^*$ *and* $\varphi(\cdot, x) = 0$ *for* x *outside a bounded interval, then*

$$\lim_{t\to\infty, |x|\ge ct} u(t,x,\varphi) = 0.$$

(2) *For any* $c < c^*$, *if* $\varphi \in C_{\beta_0^*}$ *with* $\varphi \not\equiv 0$, *then*

$$\lim_{t\to\infty, |x|\le ct} (u(t,x,\varphi) - \beta^*(t)) = 0.$$

Proof. Conclusion (1) follows from [10, Theorem 2.1]. By Lemma 2.4 and [10, Theorem 2.1], for any $c < c^*$, there is a positive number σ such that, if $\varphi \in C_{\beta_0^*}$ with $\varphi(\cdot, x) > 0$ for x on an interval of length 2σ, then $\lim_{t\to\infty,|x|\le ct}(u(t,x,\varphi) - \beta^*(t)) = 0$. It follows from Lemma 2.5 that for any $\varphi \in C_{\beta_0^*}$ with $\varphi \not\equiv 0$, $Q_t(\varphi) \gg 0$ for all $t > 2\tau$. We can fix $t_0 > 2\tau$ and take $Q_{t_0}(\varphi)$ as a new initial value for $u(t,x,\varphi)$. Then by the above analysis, conclusion (2) is valid. ◻

We say $u(t,x) = \mathcal{U}(t, x - ct)$ is an ω-periodic traveling wave of (2.1) connecting $\beta^*(t)$ to 0 if it is a solution of (2.1), $\mathcal{U}(t,\xi)$ is ω-periodic in t, and $\mathcal{U}(t,-\infty) = \beta^*(t)$ and $\mathcal{U}(t,\infty) = 0$ uniformly for $t \in [0,\omega]$. By [10, Theorems 2.2 and 2.3], we have the following result about traveling waves of (2.1).

THEOREM 2.9. *Assume that* (H1)–(H4) *hold. Let* c^* *be defined as* $c^* = c_\omega^*/\omega$. *Then for any* $c \ge c^*$, (2.1) *has an* ω*-periodic traveling wave solution* $\mathcal{U}(t, x - ct)$ *connecting*

$\beta^*(t)$ to 0 such that $\mathcal{U}(t,s)$ is continuous and nonincreasing in s. Moreover, for any $c < c^*$, (2.1) has no ω-periodic traveling wave $\mathcal{U}(t, x-ct)$ connecting $\beta^*(t)$ to 0.

3. Dynamics in a bounded domain. In this section, we consider (1.6) in a bounded spatial domain

$$(3.1)\ \begin{cases} \partial_t u(t,x) = d(t)\Delta u - g(t,u) + b(t)\displaystyle\int_\Omega \Gamma(a(t), x-y) f_{-\tau}(t, u(t-\tau, y))dy, \\ \qquad\qquad (t,x) \in (0,\infty)\times\Omega, \\ Bu(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ u(t,x) = \phi(t,x), \quad t\in[-\tau,0],\ x\in\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class $C^{1+\theta}(0<\theta\le 1)$, the boundary condition is either $Bu = u$ (Dirichlet boundary condition) or $Bu = (\partial u/\partial\nu) + \alpha(x)u$ (Robin type boundary condition) for some nonnegative function $\alpha \in C^{1+\theta}(\partial\Omega,\mathbb{R})$, and $\partial u/\partial\nu$ denotes the differentiation in the direction of outward normal ν to $\partial\Omega$.

Let $p\in(1,\infty)$ be fixed. For each $\beta \in (\frac{1}{2}+\frac{1}{2p}, 1)$, let $\mathbb{X}_\beta$ be the fractional power space of $L^p(\Omega)$ with respect to $-\Delta$ and the boundary condition $Bu = 0$ (see, e.g., [8]). Then $\mathbb{X}_\beta$ is an ordered Banach space with the positive cone $\mathbb{X}_\beta^+$ consisting of all nonnegative functions in $\mathbb{X}_\beta$, and $\mathbb{X}_\beta^+$ has nonempty interior $\mathrm{int}(\mathbb{X}_\beta^+)$. Moreover, $\mathbb{X}_\beta \subseteq C^{1+\nu}(\bar\Omega)$ with continuous inclusion for $\nu \in [0, 2\beta-1-\frac{1}{p})$. Denote the norm on $\mathbb{X}_\beta$ by $\|\cdot\|_\beta$. Then there exists a constant $k_\beta > 0$ such that $\|\phi\|_\infty := \max_{x\in\bar\Omega}|\phi(x)| \le k_\beta \|\phi\|_\beta$ for all $\phi\in\mathbb{X}_\beta$.

Let $\bar C = C([-\tau,0],\mathbb{X}_\beta)$ and $\bar C^+ = C([-\tau,0],\mathbb{X}_\beta^+)$. For convenience, we identify an element $\phi\in\bar C$ as a function from $[-\tau,0]\times\bar\Omega$ to $\mathbb{R}$ defined by $\phi(s,x)=\phi(s)(x)$. For any $N\ge L$, let $\bar C_{\widehat N} = \{\phi\in\bar C : 0\le\phi(s,x)\le N, (s,x)\in[-\tau,0]\times\bar\Omega\}$. For any function $y(\cdot):[-\tau,b)\to\mathbb{X}_\beta$, where $b>0$, define $y_t\in\bar C$, by $y_t(s) = y(t+s)$ for all $s\in[-\tau,0]$, $t\in[0,b)$.

Note that the differential operator Δ generates an analytic semigroup $\bar U_0(t)$ on $L^p(\Omega)$ and that the standard parabolic maximum principle (see, e.g., [15, Corollary 7.2.3]) implies that the semigroup $\bar U_0(t):\mathbb{X}_\beta\to\mathbb{X}_\beta$ is strongly positive in the sense that $\bar U_0(t)(\mathbb{X}_\beta^+\setminus\{0\})\subseteq \mathrm{int}(\mathbb{X}_\beta^+)$ for all $t>0$. By a similar analysis as in section 2, we can write (3.1) as an integral equation (2.7) with $u_0=\phi\in\bar C^+$. It then follows from [13, Corollary 5] that, for any $\phi\in\bar C_{\widehat L}$, (3.1) has a unique mild solution $u(t,x,\phi)$ with $u_0(\cdot,\cdot,\phi)=\phi$ and $u_t(\cdot,\cdot,\phi)\in\bar C_{\widehat L}$ for all $t\ge0$. Moreover, $u(t,x,\phi)$ is a classic solution when $t>\tau$ and the comparison theorem holds for (3.1).

Define a family of operators $\{Q_t\}_{t\ge0}$ on $\bar C^+$ by

$$Q_t(\phi)(s,x) = u(t+s,x,\phi) \quad \forall\phi\in\bar C^+, x\in\bar\Omega, \quad t\ge0, \quad s\in[-\tau,0].$$

Similarly as in section 2, we can show that $\{Q_t\}_{t\ge0}$ is a monotone ω-periodic semiflow on $\bar C^+$; $u(t,x,\phi)>0$ for $t>\tau$, $x\in\bar\Omega$, $\phi\in\bar C^+$ with $\phi\not\equiv0$, and hence, Q_t is strongly positive for $t>2\tau$; moreover, Q_t is compact on $\bar C^+$ for all $t>\tau$. Let $n_1=\min\{n\in\mathbb{N}, n\omega>2\tau\}$. Then $Q_{n_1\omega}$ is compact and strongly positive on $\bar C^+$. We can further show that the periodic semiflow $\{Q_t\}_{t\geq 0}$ is point dissipative on $\bar{C}^+$. By [23, Theorem 1.1.3], we have the following result.

LEMMA 3.1. *Let* (H1)–(H3) *hold. Then* $Q_{n_1\omega}$ *admits a global attractor on* $\bar{C}^+$.

Consider the linearized system of (3.1) at the zero solution

$$\begin{cases} \partial_t \tilde{u}(t,x) = d(t)\Delta\tilde{u} - g_u(t,0)\tilde{u}(t,x) + b(t)\partial_u f_{-\tau}(t,0)\displaystyle\int_\Omega \Gamma(a(t),x-y)\tilde{u}(t-\tau,y)dy, \\ \qquad\qquad t>0, x\in\Omega, \\ B\tilde{u}(t,x)=0, \quad t>0, x\in\partial\Omega, \\ \tilde{u}(s,x)=\phi(s,x), \quad s\in[-\tau,0],\ x\in\Omega, \phi\in\bar{C}. \end{cases} \tag{3.2}$$

Similarly as in Theorem 2.3, we can show that the comparison principle holds for (3.2), and hence, the solution map $\tilde{u}_t$ of (3.2) is monotone increasing for all $t\geq 0$.

Now we consider (3.1) and (3.2) as $n_1\omega$-periodic systems. Define the Poincaré map of (3.2) $P_1:\bar{C}\to\bar{C}$ by $P_1(\phi)=\tilde{u}_{n_1\omega}(\phi)$ for all $\phi\in\bar{C}$, where $\tilde{u}_{n_1\omega}(\phi)(s,x)=\tilde{u}(n_1\omega+s,x,\phi)$ for all $(s,x)\in[-\tau,0]\times\bar{\Omega}$, and $\tilde{u}(t,x,\phi)$ is the solution of (3.2) with $\tilde{u}(s,x)=\phi(s,x)$ for all $(s,x)\in[-\tau,0]\times\bar{\Omega}$. Similarly as in section 2, we can obtain that P_1 is also compact and strongly positive. Let $r_1=r(P_1)$ be the spectral radius of P_1. By the Krein–Rutman theorem (see, e.g., [8, Theorem 7.2]), $r_1>0$ and P_1 has a positive eigenfunction $\bar{\phi}\in\operatorname{int}(\bar{C}^+)$ corresponding to r_1.

LEMMA 3.2. *Let* $\mu=-\frac{1}{n_1\omega}\ln r_1$. *Then there exists a positive* $n_1\omega$*-periodic function* $v(t,x)$ *such that* $e^{-\mu t}v(t,x)$ *is a solution of* (3.2).

Proof. By the definitions of r_1 and $\bar{\phi}$, we have $P_1\bar{\phi}=r_1\bar{\phi}$. Let $\tilde{u}(t,x,\bar{\phi})$ be the solution of (3.2) with $\tilde{u}(s,x)=\bar{\phi}(s,x)$ for all $s\in[-\tau,0], x\in\Omega$. Since $\bar{\phi}\gg 0$, it is not difficult to see that $\tilde{u}(\cdot,\cdot,\bar{\phi})\gg 0$. Let $\mu=-\frac{1}{n_1\omega}\ln r_1$ and $v(t,x)=e^{\mu t}\tilde{u}(t,x,\bar{\phi})$ for all $t\geq-\tau, x\in\Omega$. Then $r_1=e^{-n_1\omega\mu}$ and $v(t,x)>0$ for all $t\in[-\tau,\infty), x\in\Omega$. Moreover,

$$\begin{aligned} &v_t(t,x)\\ &= e^{\mu t}\tilde{u}_t(t,x,\bar{\phi}) + \mu e^{\mu t}\tilde{u}(t,x,\bar{\phi})\\ &= e^{\mu t}[d(t)\Delta\tilde{u} - g_u(t,0)\tilde{u}(t,x,\bar{\phi})\\ &+ b(t)\partial_u f_{-\tau}(t,0)\int_\Omega \Gamma(a(t),x-y)\tilde{u}(t-\tau,y,\bar{\phi})dy] + \mu v\\ &= d(t)\Delta v - g_u(t,0)v(t,x) + e^{\mu\tau}b(t)\partial_u f_{-\tau}(t,0)\int_\Omega \Gamma(a(t),x-y)v(t-\tau,y)dy + \mu v \end{aligned} \tag{3.3}$$

for all $(t,x)\in(0,\infty)\times\Omega$. Thus, $v(t,x)$ is a solution of $n_1\omega$-periodic equation (3.3) with $Bv=0$ on $(0,\infty)\times\partial\Omega$ and $v(s,x)=e^{\mu s}\bar{\phi}(s,x)$ for all $s\in[-\tau,0], x\in\Omega$.

For any $\theta\in[-\tau,0]$, $x\in\Omega$, we have

$$v(n_1\omega+\theta,x) = e^{\mu(n_1\omega+\theta)}\cdot P_1(\bar{\phi})(\theta,x) = e^{\mu(n_1\omega+\theta)}\cdot r_1\bar{\phi}(\theta,x) = e^{\mu\theta}\cdot\tilde{u}(\theta,x,\bar{\phi}) = v(\theta,x).$$

Therefore, $v_0(\theta,\cdot)=v_{n_1\omega}(\theta,\cdot)$ for all $\theta\in[-\tau,0]$, and hence, the existence and uniqueness of solutions of (3.3) imply that

$$v(t,x)=v(t+n_1\omega,x) \quad \forall t\geq-\tau, x\in\Omega,$$

that is, $v(t,x)$ is an $n_1\omega$-periodic solution of (3.3). Clearly, $e^{-\mu t}v(t,x)$ is a solution of (3.2). $\square$

Define $P_0 : \bar{C} \to \bar{C}$ by $P_0(\phi) = \tilde{u}_\omega(\phi)$ for all $\phi \in \bar{C}$, where $\tilde{u}(t,x,\phi)$ is the solution of (3.2) with $\tilde{u}(s,x) = \phi(s,x)$ for all $s \in [-\tau,0]$, $x \in \Omega$. Let $r_0 = r(P_0)$ be the spectral radius of P_0.

THEOREM 3.3. *Let* (H1)–(H3) *hold. For any* $\phi \in \bar{C}^+$*, denote by* $u(t,x,\phi)$ *the solution of* (3.1) *with* $u(s,x) = \phi(s,x)$ *for all* $(t,x) \in [-\tau,0]\times\Omega$*. Then the following two statements are valid.*

(i) *If* $r_0 < 1$*, then* $\lim_{t\to\infty} \| u(t,\cdot,\phi) \|_\beta = 0$ *for every* $\phi \in \bar{C}^+$*.*

(ii) *If* $r_0 > 1$*, then* (3.1) *admits a unique positive* ω*-periodic solution* $u^*(t,x)$ *and* $\lim_{t\to\infty} \| u(t,\cdot,\phi) - u^*(t,\cdot) \|_\beta = 0$ *for all* $\phi \in \bar{C}^+ \setminus \{0\}$*.*

Proof. Since $P_1 = \tilde{u}_{n_1\omega}$, $P_0 = \tilde{u}_\omega$, and $\tilde{u}_{n_1\omega} = \tilde{u}_\omega^{n_1}$, where $\tilde{u}_t$ is the solution map of (3.2), by the properties of spectral radius of linear operators, we know that $r(P_1) = (r(P_0))^{n_1}$, i.e., $r_1 = (r_0)^{n_1}$. Note that the qualitative solutions of (3.1) and (3.2) do not change whether we consider them as $n_1\omega$-periodic systems or ω-periodic systems. The conditions in Theorem 3.3 can be replaced by $r_1 < 1$ and $r_1 > 1$, respectively. In the following, we will consider (3.1) and (3.2) as $n_1\omega$-periodic systems and prove the theorem under the conditions of $r_1 < 1$ and $r_1 > 1$.

In the case where $r_1 < 1$, we have $\mu = -\frac{1}{n_1\omega}\ln r_1 > 0$. By Lemma 3.2, (3.2) has a solution $\tilde{u}(t,x) := \tilde{u}(t,x,\bar{\phi}) = e^{-\mu t}v(t,x)$ with $\tilde{u}(s,x) = \bar{\phi}(s,x)$ for all $(s,x) \in [-\tau,0]\times\Omega$, where $\bar{\phi} \in \mathrm{int}(\bar{C}^+)$ is the positive eigenfunction of P_1 corresponding to r_1 and $v(t,x)$ is $n_1\omega$-periodic in $t \geq -\tau$. Then v is bounded on $[-\tau,\infty)\times\bar{\Omega}$, and hence, there exists $\rho > 0$ such that $\| v(t,\cdot) \|_\infty \leq \rho$ for all $t \geq -\tau$. Thus, $\lim_{t\to\infty} \| \tilde{u}(t,\cdot) \|_\infty = 0$. By the basic analysis of solutions of (3.2), it follows that $\lim_{t\to\infty} \| \tilde{u}(t,\cdot) \|_\beta = 0$.

Given $\phi \in \bar{C}^+$, since $\lim_{\delta\to 0^+}(\bar{\phi} - \delta\phi) = \bar{\phi} \in \mathrm{int}(\bar{C}^+)$ for any $\epsilon > 0$, there exists $\delta_\phi > 0$, such that $\bar{\phi} - \delta\phi \in B_\epsilon(\bar{\phi}) \subseteq \bar{C}^+$ for $0 < \delta \leq \delta_\phi$, where $B_\epsilon(\bar{\phi})$ is an open ball in $\bar{C}^+$ centered at $\bar{\phi}$ with radius ϵ. Therefore, $\bar{\phi} \geq \delta_\phi\phi$ in $\bar{C}^+$. It then follows from the comparison principle that $\tilde{u}(t,x) \geq \delta_\phi\tilde{u}(t,x,\phi)$ for all $t \geq -\tau, x \in \bar{\Omega}$, where $\tilde{u}(t,\cdot,\phi)$ is the solution of (3.2) with $\tilde{u}(s,x) = \phi(s,x)$ for all $(s,x) \in [-\tau,0]\times\Omega$. Thus, $\lim_{t\to\infty} \| \tilde{u}(t,\cdot,\phi) \|_\infty = 0$, and hence, $\lim_{t\to\infty} \| \tilde{u}(t,\cdot,\phi) \|_\beta = 0$ for any $\phi \in \bar{C}^+$.

Note that a solution of (3.1) satisfies

$$\partial_t u(t,x) \leq d(t)\Delta u - g_u(t,0)u(t,x) + b(t)\partial_u f_{-\tau}(t,0)\int_\Omega \Gamma(a(t),x-y)u(t-\tau,y)dy$$

for any $t > 0$, $x \in \Omega$. Similarly to the proof of Theorem 2.3, we can show that the comparison theorem for abstract functional differential equations [13, Proposition 3] can be applied to (3.1) and (3.2). Therefore, for any $\phi \in \bar{C}^+$, $u(t,\cdot,\phi) \leq \tilde{u}(t,\cdot,\phi)$ for all $t \geq -\tau$, where $u(t,\cdot,\phi)$ and $\tilde{u}(t,\cdot,\phi)$ are solutions of (3.1) and (3.2), respectively. It then follows that solutions of (3.1) satisfy $\lim_{t\to\infty} \| u(t,\cdot,\phi) \|_\beta = 0$ for all $\phi \in \bar{C}^+$.

In the case where $r_1 > 1$, we have $\mu < 0$. Let $\bar{C}_0 = \{\phi \in \bar{C}^+ : \phi \not\equiv 0\}$, $\partial\bar{C}_0 = \bar{C}^+ \setminus \bar{C}_0 = \{0\}$. Similarly to the proof of Lemma 2.5, we can show that for any $\phi \in \bar{C}_0$, the solution $u(t,x,\phi)$ of (3.1) satisfies $u(t,x,\phi) > 0$ for all $t > \tau$, $x \in \Omega$. It follows that $Q_t(\bar{C}_0) \subseteq int(\bar{C}^+)$ for all $t > 2\tau$. Clearly, $Q_t(0) = 0$ for all $t \geq 0$. We now have the following claim.

Claim. Zero is a uniform weak repeller for $\bar{C}_0$ in the sense that there exists $\delta_0 > 0$ such that $\lim_{t\to\infty}\sup \| Q_t(\phi) \|_\beta \geq \delta_0$ for all $\phi \in \bar{C}_0$.

Indeed, we consider the following system:

$$(3.4)\quad \begin{cases} \partial_t u^\varepsilon(t,x) = d(t)\Delta u^\varepsilon - (g_u(t,0)+\varepsilon)u^\varepsilon(t,x) \\ \qquad\qquad + b(t)(\partial_u f_{-\tau}(t,0)-\varepsilon)\displaystyle\int_\Omega \Gamma(a(t),x-y)u^\varepsilon(t-\tau,y)dy, \\ Bu^\varepsilon(t,x)=0, \quad t>0, x\in\partial\Omega, \\ u^\varepsilon(s,x)=\phi(s,x), \quad s\in[-\tau,0],\ x\in\Omega, \phi\in\bar{C}. \end{cases}$$

Define the Poincaré map of (3.4) $P_\varepsilon : \bar{C}\to\bar{C}$ by

$$P_\varepsilon(\phi) = u^\varepsilon_{n_1\omega}(\phi) \quad \forall \phi\in\bar{C},$$

where

$$u^\varepsilon_{n_1\omega}(\phi)(s,x) = u^\varepsilon(n_1\omega+s,x,\phi) \quad \forall (s,x)\in[-\tau,0]\times\bar{\Omega}$$

and $u^\varepsilon(t,x,\phi)$ is the solution of (3.4) with $u^\varepsilon(s,x)=\phi(s,x)$ for all $s\in[-\tau,0], x\in\Omega$. Let $r_\varepsilon = r(P_\varepsilon)$ be the spectral radius of P_ε. Since $r_1 = r(P_1) > 1$, there exists a sufficiently small positive number ε_1 such that $r_\varepsilon > 1$ for all $\varepsilon\in[0,\varepsilon_1)$. We fix an $\varepsilon\in(0,\varepsilon_1)$. Since $\lim_{u\to 0^+}\frac{g(t,u)}{u} = g_u(t,0)$ and $\lim_{u\to 0^+}\frac{f_{-\tau}(t,u)}{u} = \partial_u f_{-\tau}(t,0)$ uniformly for $t\in[0,n_1\omega]$, there exists $\delta_\varepsilon>0$ such that $g(t,u) < (g_u(t,0)+\varepsilon)u$ and $f_{-\tau}(t,u) > (\partial_u f_{-\tau}(t,0)-\varepsilon)u$ for $u\in(0,\delta_\varepsilon)$, $t\in[0,n_1\omega]$. Let $\delta_0 = \delta_\varepsilon/k_\beta$. Suppose, by contradiction, that there exists $\phi_0\in\bar{C}_0$ such that $\lim_{t\to\infty}\sup \parallel Q_t(\phi)\parallel_\beta < \delta_0$. Then there exists $t_0>\tau$ such that $\parallel u(t,\cdot,\phi_0)\parallel_\infty \le k_\beta \parallel u(t,\cdot,\phi_0)\parallel_\beta < \delta_\varepsilon$ for all $t\ge t_0$. Therefore, $u(t,x,\phi_0)$ satisfies

$$(3.5)\quad \partial_t u(t,x) > d(t)\Delta u - (g_u(t,0)+\varepsilon)u(t,x) + b(t)(\partial_u f_{-\tau}(t,0)-\varepsilon)\int_\Omega \Gamma(a(t),x-y)u(t-\tau,y)dy$$

for $t\ge t_0, x\in\Omega$. Let $\bar{\phi}_\varepsilon$ be the positive eigenfunction of P_ε associated with r_ε and $\mu_\varepsilon = -\frac{1}{n_1\omega}\ln r_\varepsilon$. Then by Lemma 3.2, the solution $u^\varepsilon(t,x,\bar{\phi}_\varepsilon)$ of (3.4) with $u^\varepsilon(s,x) = \bar{\phi}_\varepsilon(s,x)$ for all $s\in[-\tau,0]$, $x\in\Omega$, satisfies $u^\varepsilon(t,x,\bar{\phi}_\varepsilon) = e^{-\mu_\varepsilon t}v_\varepsilon(t,x)$, where $v_\varepsilon(t,x)$ is a positive $n_1\omega$-periodic function in $t\ge -\tau$. Since $u(t,x,\phi_0)>0$ for all $t\ge\tau$, $x\in\Omega$, there exists $\zeta>0$ such that

$$u(t_0+s,x,\phi_0) \ge \zeta u^\varepsilon(s,x,\bar{\phi}_\varepsilon) = \zeta\bar{\phi}_\varepsilon(s,x) \quad \forall s\in[-\tau,0], x\in\bar{\Omega}.$$

By (3.5) and the comparison theorem, we have

$$u(t,x,\phi_0) \ge \zeta u^\varepsilon(t-t_0,x,\bar{\phi}_\varepsilon) = \zeta e^{-\mu_\varepsilon(t-t_0)}v_\varepsilon(t,x) \quad \forall t\ge t_0, x\in\bar{\Omega}.$$

Since $\mu_\varepsilon<0$, it follows that $u(t,x,\phi_0)$ is unbounded, a contradiction. Thus, the claim is true.

By the claim above, $Q_{n_1\omega}$ is weakly uniformly persistent with respect to $(\bar{C}_0, \partial\bar{C}_0)$. Since $Q_{n_1\omega}$ admits a global attractor on $\bar{C}^+$, it follows from [23, Theorem 1.3.3] that $Q_{n_1\omega}$ is uniformly persistent with respect to $(\bar{C}_0,\partial\bar{C}_0)$ in the sense that there exists $\delta_1>0$ such that $\lim_{n\to\infty}\inf \parallel Q^n_{n_1\omega}(\phi)\parallel_\beta \ge \delta_1$ for all $\phi\in\bar{C}_0$.

Note that $Q_{n_1\omega}$ is compact, point dissipative, and uniformly persistent. It follows from [23, Theorem 1.3.6] that $Q_{n_1\omega} : \bar{C}_0\to\bar{C}_0$ admits a global attractor A_0 and has a fixed point $\hat{\phi}$ in A_0. Similarly as in the proof of Lemma 2.4, we can show that $Q_{n_1\omega}$ is strictly subhomogeneous. Then [22, Lemma 1] implies that $Q_{n_1\omega}$ has at most one fixed point. Thus, $Q_{n_1\omega}$ has a unique equilibrium $\hat{\phi}$ in $\bar{C}_0$. Clearly, by the strong monotonicity of $Q_{n_1\omega}$, we have $\hat{\phi} \in \mathrm{int}(\bar{C}^+)$. Moreover, it follows from [23, Theorem 2.3.2] that $A_0 = \{\hat{\phi}\}$ since $Q_{n_1\omega}$ is strongly monotone and strictly subhomogeneous. Thus, $\hat{\phi}$ is globally attractive in $\bar{C}_0$ for $Q_{n_1\omega}$.

Let $u(t,x,\hat{\phi})$ be the solution of (3.1) with $u(s,x) = \hat{\phi}(s,x)$ for all $(s,x) \in [-\tau, 0] \times \Omega$. Since $\hat{\phi}$ is a fixed point of $Q_{n_1\omega}$ and is globally attractive in $\bar{C}_0$, $u(t,x,\hat{\phi})$ is an $n_1\omega$-periodic solution of (3.1) which attracts all solutions of (3.1) in $\bar{C}^+ \setminus \{0\}$. That is,

$$\lim_{t\to\infty} \| u(t,\cdot,\phi) - u(t,\cdot,\hat{\phi}) \|_\beta = 0 \ \forall \ \phi \in \bar{C}_0.$$

Now we show that $u(t,x,\hat{\phi})$ is also ω-periodic. Since $Q_{n_1\omega}(\hat{\phi}) = \hat{\phi}$, we have $Q_\omega(Q_{n_1\omega}(\hat{\phi})) = Q_\omega(\hat{\phi})$, i.e., $Q_{n_1\omega}(Q_\omega(\hat{\phi})) = Q_\omega(\hat{\phi})$, which implies that $Q_\omega(\hat{\phi})$ is also a fixed point of $Q_{n_1\omega}$. By the fact that $\hat{\phi} \gg 0$ and the fact that Q_ω is monotone, it follows that $Q_\omega(\hat{\phi}) \gg 0$. Note that $Q_{n_1\omega}$ has a unique fixed point in $\mathrm{int}(\bar{C}^+)$. Then $Q_\omega(\hat{\phi}) = \hat{\phi}$, that is, $\hat{\phi}$ is a fixed point of Q_ω, and hence, $u(t,x,\hat{\phi})$ is an ω-periodic solution of (3.1). Thus, $u^*(t,x) := u(t,x,\hat{\phi})$ for all $(t,x) \in [-\tau,\infty) \times \bar{\Omega}$, is the desired ω-periodic solution. □

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