Classifying and Orienting Quadric Surfaces By Algebraic Inspection

The key to classifying and graphing quadric surfaces is to combine a geometric and an algebraic view. We begin by assuming that the equation for the surface is given in a coordinate system that is convenient for that surface. Thus, any axis of symmetry is a coordinate axis, any plane of symmetry is a coordinate plane, and so on. This assumption eliminates terms that are products of variables and also lower order terms (such as x when the equation includes x^2). Thus, we consider only those surfaces that can be written as

$$Ax^m + By^n + Cz^p = K, \qquad m, n, p \in \{1, 2\}, \quad \max(m, n, p) = 2,$$

where A, B, C, and K are any numbers, including 0. When faced with an equation of this form, one plan is to begin by finding the correct one of the four general cases below. Once the correct case is determined, you can determine the correct orientation and subcase.

- Note on notation: I will use lower case letters to indicate parameters that must be positive and capital letters for parameters not so restricted.

1. Cylinders: one variable is missing

Equations of the form

$$Ax^m + By^n = K, \qquad AB \neq 0, \quad \max(m, n) = 2 \tag{1}$$

represent cylinders that consist of a curve in the xy plane extended into the third dimension. (If $m = n = 1$, then we have a straight line in the xy plane, corresponding to a plane in three dimensions.) Similarly, equations with x missing or y missing are cylinders extending into the dimension corresponding to the missing variable.

1. $y = Ax^2$ with $A \neq 0$ and $x = By^2$ with $B \neq 0$ are parabolas in the plane and parabolic cylinders in space.
2. $ax^2 + by^2 = r^2$, where $a, b, r > 0$, are ellipses in the plane and elliptic cylinders in space.
3. $ax^2 - by^2 = r^2$ and $by^2 - ax^2 = r^2$, where $a, b, r > 0$, are hyperbolas in the plane and hyperbolic cylinders in space. Note that the location of the minus sign matters. In $ax^2 - by^2 = r^2$, there are points for all values of y, but there are no points for $x^2 < r^2/a$. Similarly, the graph of $by^2 - ax^2 = r^2$ has points for all values of x, but not for all values of y.

2. Ellipsoids: all variables to the second power, all coefficients of the same sign

Equations of the form

$$a^2x^2 + b^2y^2 + c^2z^2 = r^2, \qquad a, b, c, r > 0 \tag{2}$$

represent ellipsoids centered at the origin. The axis intercepts are $x = \pm r/a$, $y = \pm r/b$, and $z = \pm r/c$.

3. Hyperboloids: all variables to the second power, one coefficient of different sign

Equations of the form

$$z^2 = ax^2 + by^2 + K, \qquad a, b > 0 \tag{3}$$

represent hyperboloids symmetric about the z axis. Equations of the forms $x^2 = by^2 + cz^2 + K$ and $y^2 = ax^2 + cz^2 + K$ are similar, but symmetric about the x and y axes, respectively. There are three distinct subcases of hyperboloids, depending on the sign of K. In the examples below, note particularly the $x = 0$ traces.

subcase:	$K = -r^2 < 0$	$K = 0$	$K = r^2 > 0$
name:	hyperboloid of one sheet	elliptic cone	hyperboloid of two sheets
example:	$z^2 = x^2 + y^2 - 1$	$z^2 = x^2 + y^2$	$z^2 = x^2 + y^2 + 1$
$x = 0$ trace:	$z^2 = y^2 - 1$	$z^2 = y^2$	$z^2 = y^2 + 1$

Table 1: The Three Types of Hyperboloids.

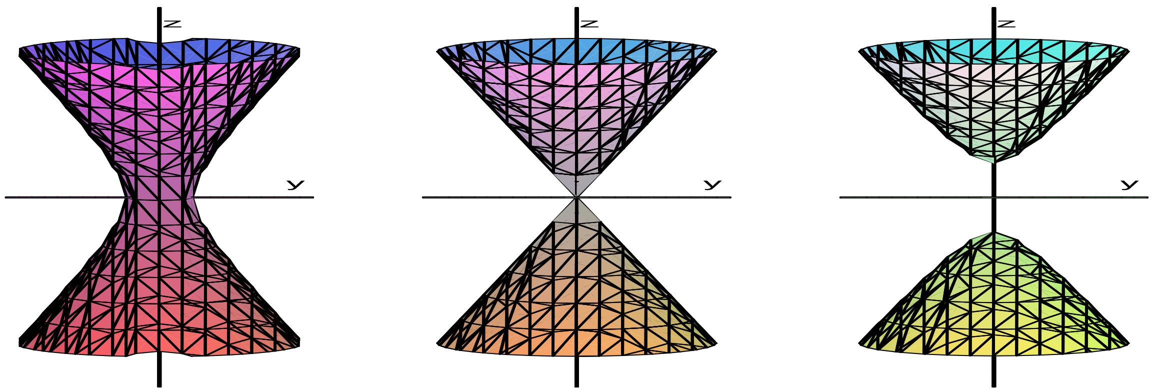

Figure 1: The hyperboloids from Table 1, ordered from left to right. In each case, the x axis is pointing toward the reader and slightly downward.

4. Paraboloids: one variable to the first power, two variables to the second power

Equations of the form

$$z = Ax^2 + By^2, \qquad AB \neq 0 \tag{4}$$

represent paraboloids symmetric about the z axis. There are two cases, depending on whether the signs of A and B are the same or different.

subcase:	$AB > 0$	$AB < 0$
name:	elliptic paraboloid	hyperbolic paraboloid
example:	$z = x^2 + y^2$	$z = y^2 - x^2$
$x = 0$ trace:	$z = y^2$	$z = y^2$
$y = 0$ trace:	$z = x^2$	$z = -x^2$

Table 2: The Two Types of Paraboloids.

Hyperbolic paraboloids are difficult to visualize; however, they are seldom used as surfaces in multivariable calculus problems.

Note:

Equations with two variables to the first power and one variable to the second are rotated cylinders. For example, if we have

$$2x + 3y - z^2 = 0,$$

we could use

$$u = 2x + 3y, \qquad v = 3x - 2y$$

instead of x and y. In the new coordinate system, the graph is the parabolic cylinder $u - z^2 = 0$