The Definite Integral Concept

A multivariable calculus course spends at least half of the time on definite integrals. There are a variety of definite integrals, with details that differ according to the geometry. Nevertheless, there is just one basic definite integral concept. This concept provides a unified framework that can be used to organize the computation of any definite integral.

Integrals in One Variable

We begin with integrals in one variable. These include the integrals over intervals in time or length, familiar from one-variable calculus, and also an integral over a curve in higher-dimensional space.

Accumulation in Time

Anyone who has seen my desk knows that material accumulates on it over time. Consider the basic question: "How much stuff has accumulated from time a to a later time b?" Our goal is to describe a procedure that could be used systematically to answer the question.

Imagine that we set up a video camera to continuously film my desk from time a to time b. At time a, we throw a sheet over the desk to cover up what is already there. When time b is reached, we examine the video recording. A stack of papers got added to the pile at time t_1, and a book got added at time t_2, with $a < t_1 < t_2 < b$. The amount of stuff that was added is the sum of the stack of papers and the book.

So much for discrete additions to the pile on the desk. In theory, a similar procedure could be worked out for stuff added continuously to the desk. Suppose a machine drops papers onto my desk at a rate $f(t)$. During an interval of infinitesimal duration dt, the desk accumulates an amount $f(t)\,dt$. The total amount of stuff that accumulates over the whole time interval is the sum of all the infinitesimal amounts $f(t)\,dt$. We denote a sum of infinitely-many infinitesimal amounts using a definite integral sign; hence, the total amount of stuff is $\int_a^b f(t)\,dt$. More generally,

$$\text{total stuff} = \int_{t=a}^{b} \frac{\text{stuff}}{\text{time}}\,dt.$$

This formula works for any kind of "stuff." We can calculate the accumulated heat energy in some region if we know the rate at which heat energy accumulates. We can calculate the change in x coordinate of a moving object if we know the rate (velocity) at which changes in x accumulate.

Aggregation Over a Line

Now consider a metal wire laid along the x axis from $x = a$ to $x = b$. Suppose the wire has linear density $f(x)$ grams per unit length. How much material is in the wire? In theory, we can cut the wire into infinitely many bits of length dx. Each bit has mass $f(x)\,dx$. The sum of the infinitely-many infinitesimal bits is given by a definite integral: $m = \int_a^b f(x)\,dx$. In general, this is almost the same setup as for accumulation in time:

$$\text{total stuff} = \int_{x=a}^{b} \frac{\text{stuff}}{\text{length}}\,dx.$$

As with accumulation, it doesn't matter what kind of "stuff" we have. As a special case, suppose we want to calculate the length of the wire. Then

$$\text{total length} = \int_{x=a}^{b} \frac{\text{length}}{\text{length}}\, dx = \int_a^b 1\, dx.$$

Aggregation Over a Curve C

Now suppose our wire is laid along a curve C, which could be in the plane or in three-dimensional space. We can still write down an expression for the total amount of stuff in the wire. The only difference is that we no longer have a straight line coordinate x that we can use to measure the length of a small bit of wire or to indicate the location of the first and last bits. In principle, we can use ds to denote the little bit of length, measured along the wire, and we can simply indicate the curve C rather than specific limits of integration. Thus,

$$\text{total stuff} = \int_C \frac{\text{stuff}}{\text{length}}\, ds.$$

In particular,

$$\text{total length} = \int_C \frac{\text{length}}{\text{length}}\, ds = \int_C 1\, ds.$$

Conceptually, an integral over a curve is the same as an integral over a line.

The big difference between integrals over a curve C and integrals over an interval $a < x < b$ is that we usually cannot calculate integrals over a curve directly. In most cases, we calculate integrals over curves using a two-step method:

1. Rewrite the curve integral as a standard one-variable integral over an interval, using some parameter as the integration variable.

2. Calculate the integral using the fundamental theorem or numerical approximation.

The details of step 1 depend on the particular curve and must be deferred to the study of vector calculus later in the course.

Integrals in Two or Three Variables

In one-variable calculus, we can only do integrals over multi-dimensional regions if we can write the integral using a single variable, as for example in computing volumes of revolution. This is a limitation we need to overcome with multi-variable integration. Here we begin with a quick look at the calculation of appropriately-expressed integrals in two or three variables, then we survey the kinds of integrals that we might encounter.

Aggregation Over a Plane Region R

Suppose we want to know how much stuff is on my desk, but we don't have a video recording of the accumulation process. In theory, we could partition the desk into infinitely-many infinitesimal bits of area with size dA, each located at its own coordinates (x, y). If $f(x, y)$ is the density of stuff per unit area, then $f(x, y)\, dA$ is the amount of stuff in a bit of area located at (x, y).

The total amount of stuff is obtained by adding up all the little bits within the plane region R of the desk. Without worrying about how to do the calculation, we can write the result as

$$\text{total stuff} = \iint_R \frac{\text{stuff}}{\text{area}}\, dA.$$

Since the area per unit area is 1, we have the special case

$$\text{area} = \iint_R 1\, dA.$$

In all cases, the procedure for evaluating integrals over a plane region R requires two steps:

1. Rewrite the integral as an iterated integral in some coordinate system.

2. Calculate the iterated integral using the fundamental theorem or numerical approximation.

In principle, the concept is very simple, although in practice both of these steps can be problematic.

Aggregation Over a Curved Surface S

What if we want to know how much stuff is aggregated over a nonplanar surface S? In theory, this is only slightly different from aggregation over a planar region R. We partition the surface into infinitely-many infinitesimal bits of area dS and add up all the stuff on the different bits:

$$\text{total stuff} = \iint_S \frac{\text{stuff}}{\text{area}}\, dS.$$

Since the area per unit area is 1, we have the special case

$$\text{area} = \iint_S 1\, dS.$$

The procedure for evaluating these integrals consists of the same steps as the procedure for integrals over a plane region, except that the details are much more complicated.

Aggregation Over a 3D Solid Region D

Aggregation over a three-dimensional region D follows the same pattern as the other cases. The amount of stuff in a bit of volume dV is the product of the stuff per unit volume with the amount of volume, and the total is obtained by adding up all the little bits in the region. Thus,

$$\text{total stuff} = \iiint_D \frac{\text{stuff}}{\text{volume}}\, dV.$$

Since the area per unit area is 1, we have the special case

$$\text{volume} = \iiint_D 1\, dV.$$

These integrals are evaluated by the same two-step method as integrals over plane regions, the only difference being that the corresponding iterated integral will have three layers rather than two.

Summary

All integrals of scalar functions over spatial regions in n dimensions fit a common pattern.

$$\text{total stuff} = \int_Q \frac{\text{stuff}}{\text{unit size}} \, dQ,$$

where Q is a region in n dimensions and dQ is a little bit of length, area, or volume for that region, as appropriate. As a special case

$$\text{total size of } Q = \int_Q 1 \, dQ.$$

We evaluate all such integrals (thinking of a one-dimensional integral as a special case of iterated integral), but first we have to rewrite the integral in iterated form. This requires formulas for quantities such as dA and dS in a variety of coordinate systems along with a procedure for determining the integration limits for iterated integrals in various geometries.