Iterated Integrals

All definite integrals have three components:

1. A region for the integration;
2. An integrand function;
3. A differential.

To evaluate a 2-dimensional integral in multivariable calculus, we need to rewrite it in iterated form, which consists of a one-variable integral inside another one-variable integral. Three-dimensional integrals are similar. Once an integral has been written in iterated form, then it is merely a matter of evaluating the integrals.

Evaluating Interated Integrals

A typical iterated integral is the quantity

$$B = \int_0^1 \int_x^1 3xy^2 \, dy \, dx.$$

This expression has two integral signs, each with limits of integration, and two differentials. Think of the integral as though it were written with an extra set of brackets:

$$B = \int_{x=0}^1 \left[\int_{y=x}^1 3xy^2 \, dy \right] \, dx.$$

The inside part is

$$\int_{y=x}^1 3xy^2 \, dy.$$

This is a one-variable integral with differential dy, so it is an integral over the interval $x < y < 1$, where x is some unknown quantity that is independent of y. We can perform the integration by remembering that x is a constant. Just as $\pi y^2/2$ is an antiderivative of πy, $xy^2/2$ is an antiderivate *with respect to* y of xy. The result of the integration will be a function of x; thus, we can interpret the iterated integral as

$$B = \int_0^1 g(x) \, dx, \qquad \text{where} \qquad g(x) = \int_x^1 xy \, dy.$$

The integral is evaluated in stages, from the inside out, being careful to treat any variable that is not the current integration variable as a constant. Thus,

$$g(x) = xy^3 \Big|_{y=x}^{y=1} = x - x^4$$

and

$$B = \int_0^1 \left(x - x^4 \right) \, dx = \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Bigg|_{x=0}^{x=1} = \left(\frac{1}{2} - \frac{1}{5} \right) - (0 - 0) = 0.3.$$

Note that each variable disappears after it is used as the integration variable.

Now consider the expressions

$$C = \int_0^{\pi} \int_{\sin\theta}^{1+\sin\theta} \int_{r\cos\theta}^{r} r\cos\theta \, dz \, dr \, d\theta, \qquad D = \int_0^x \int_0^y xy \, dx \, dy.$$

The first is a legitimate iterated integral, which we can calculate in three steps:

$$g_1(r,\theta) = \int_{r\cos\theta}^{r} r\cos\theta \, dz, \qquad g_2(\theta) = \int_{\sin\theta}^{1+\sin\theta} g_1(r,\theta)\, dr, \qquad C = \int_0^{\pi} g_2(\theta)\, d\theta.$$

The latter is not a legitimate iterated integral because the interpretation

$$g(y) = \int_0^y xy\, dx, \qquad D = \int_0^x g(y)\, dy$$

does not make sense. The inside integral should remove x from the expression, so x cannot return in the outside integral. Stated another way, if y is the outside variable and x the inside variable, then the lmits of the inside integral can depend on y, but the limits of the outside integral cannot depend on x.

Setting Up Iterated Integrals

Once a multidimensional integral has been set up as an iterated integral, it is easy to evaluate it, as shown above. Setting up iterated integrals is a skill that requires more practice. The details are different from one problem to another, but the general plan is always the same.

1. Choose two integration variables for a 2D integral or three integration variables for a 3D integral. The choice of integration variables determines the form needed for the integrand and the differential.

2. For a 2D integral, choose one of the integration variables to mark slices and the other to mark points on a slice. Then identify the range of these variables, taking care to note that the range of the slicing variable must be constant, while the range of the variable that marks points on a slice could be different for different slices.

3. For a 3D integral, choose a coordinate surface for two of the variables, such as the xy or yz planes in rectangular coordinates. Set up the two variables in that surface as in the 2D case, and then give the range of the third variable in terms of the first two variables.

The choice of integration variables depends on the ease of describing the region and evaluating the integrals. Circular regions suggest polar coordinates rather than rectangular coordinates. Cones are generally best done with cylindrical coordinates. Portions of spheres are often best done with spherical coordinates.

The integrand must be written in terms of the integration variables; for example if one wants to use polar coordinates for $\iint_R f(x,y)\, dA$, then it is necessary to replace x and y in the integrand with $r\cos\theta$ and $r\sin\theta$, respectively.

Each coordinate system has a specific formula for the differential:

Name	Definitions	dA, dV
2D rectangular		$dx\, dy$
polar	$x = r\cos\theta,\ \ y = r\sin\theta$	$r\, dr\, d\theta$
3D rectangular		$dx\, dy\, dz$
cylindrical	$x = r\cos\theta,\ \ y = r\sin\theta$	$r\, dr\, dz\, d\theta$
spherical	$r = \rho\sin\phi,\ \ z = \rho\cos\phi$	$\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

The order of factors in the differential depends on the assignments of the variables. The last factor in the differential must correspond to the first variable in the description, and so on.

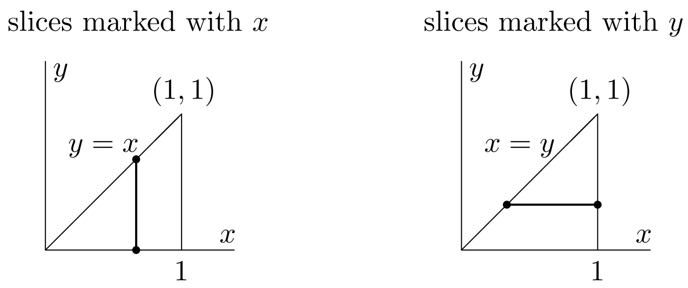

Figure 1: The region bounded by $y = x$, $x = 1$, and $y = 0$, showing an arbitrary slice for each choice of slicing variable.

As an example, consider an integral to determine the mass of a flat metal plate located inside the triangle defined by the equations $y = x$, $x = 1$, and $y = 0$ and having mass per unit area of $f(x, y)$. The mass at a particular point is $f(x, y)\, dA$, so the total mass is

$$m = \iint_R f(x, y)\, dA.$$

Figure 1 shows the region R with different choices for the slicing variable.

Slices marked by x With slices marked by x, we first identify the range of slices as $0 \leq x \leq 1$. Along a slice, the range of y values usually depends on the value of x for that slice. For R, the range of y values along a slice is $0 \leq y \leq x$. Hence, the integral is

$$m = \int_{x=0}^{1} \int_{y=0}^{x} f(x, y)\, dy\, dx.$$

Note that the differential for the first variable goes last so that we have an inside integral with y as the integration variable and x as an independent variable, along with an outside integral in x.

Slices marked by x With slices marked by y, we first identify the range of slices as $0 \leq y \leq 1$. Along a slice, the range of x values usually depends on the value of y for that slice. For R, the range of x values along a slice is $y \leq x \leq 1$. Hence, the integral is

$$m = \int_{y=0}^{1} \int_{x=y}^{1} f(x, y)\, dx\, dy.$$

As before, the differential for the first variable goes last. We have an inside integral with x as the integration variable and y as an independent variable, along with an outside integral in y.