Notes on Flux Integrals

Surface integrals arise when we need to find the total of a quantity that is distributed on a surface or that flows across a surface. The standard integral with respect to area for functions of x and y is a special case, where the surface is given by $z = 0$. Other surfaces can lead to much more complicated integrals.

Types of Surface Integrals

Consider a surface S in three-dimensional space. Let dS represent the area of a little bit of the surface and let $\vec{\mathbf{n}}$ be the unit normal to the surface. If the surface is the boundary of a three-dimensional region, then $\vec{\mathbf{n}}$ is by definition the outward normal. If S is an open surface, then either of the two normal vectors can be used for $\vec{\mathbf{n}}$. Given the surface area and normal, we can define an area vector at a point by $d\vec{\mathbf{A}} = \vec{\mathbf{n}}\, dS$.

Let $f(\vec{\mathbf{r}})$ be a measure of some scalar quantity per unit area at the point indicated by the position vector $\vec{\mathbf{r}}$. Then $f\, dS$ is the amount of that scalar quantity contained in the bit of surface, and the total amount of the scalar quantity is

$$\iint_S f\, dS.$$

In particular, $\iint_S 1\, dS$ is the area of the surface.

Aside from surface area, the Hughes-Hallett text does not do surface integrals of scalar functions, so we do not include them here.

Now let $\vec{\mathbf{v}}(x, y, z)$ be a vector field, which we can think of as the velocity of a fluid in the region. Then $\vec{\mathbf{v}} \cdot \vec{\mathbf{n}}$ is the velocity component normal to the surface. With dS the area of a bit of surface, the flow (volume per time) through that bit of area is $\vec{\mathbf{v}} \cdot \vec{\mathbf{n}}\, dS$. The total flow through the surface can be written as

$$\text{flux} = \iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}}\, dS.$$

While the notation $\displaystyle\iint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$ is a nice way to interpret flux integrals, it has two disadvantages. First, it is nonstandard; students who encounter flux integrals in science and engineering will find them written as $\displaystyle\iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}}\, dS$. Second, some computational methods rely on separate formulas for $\vec{\mathbf{n}}$ and dS, and the notation $d\vec{\mathbf{A}}$ makes it harder to remember these formulas. Hence, it is best to write flux integrals as $\displaystyle\iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}}\, dS$.

Note the dimensions: $\vec{\mathbf{v}}$ is length per time, $\vec{\mathbf{n}}$ is dimensionless, and dS is length squared. Thus, the flux has dimensions of volume per unit time. If we want to calculate the flux of any kind of "stuff", we need to replace $\vec{\mathbf{v}}$ with a vector that measures the flow rate of stuff per unit time per unit area, in the same way velocity can be interpreted as flow rate of volume per unit time per unit area.

Evaluating Flux Integrals

For regions that lie in the xy plane, we have $S = R$ and $dS = dA$, and we resolve the integral by choosing a two-dimensional coordinate system (xy or $r\theta$) and writing the region R, the integrand $\vec{\mathbf{v}} \cdot \vec{\mathbf{k}}$, and the differential dA in terms of the two coordinates. Integrals on other surfaces use the same idea, but with an important difference. We have to denote the three coordinates of surface points using just two parameters. Then we write the region S, the function $\vec{\mathbf{v}}$, and the differential $\vec{\mathbf{n}}\, dS$

in terms of the two parameters. Usually, the two parameters are chosen from the most convenient coordinate system, which means rectangular or polar for most surfaces (including cones), cylindrical or spherical for cylinders and spheres.

There are four common special cases to consider:

1. S is parallel to a coordinate plane, such as the surface $y = 1$;
2. S is not parallel to a coordinate plane, but can be projected onto a coordinate plane[1];
3. S is a portion of a circular cylinder; and
4. S is a portion of a sphere.

Only rarely does one see a surface that does not fit one of these categories.

Our goal for each case is to identify the appropriate parameters and the formulas for the differentials.

Case 1 — S is parallel to a coordinate plane

For surfaces $z = c$, with upward normal, we can just substitute c for z and think of the integral as over the region R that sits below or above S in the xy plane. Similarly, we can use a region R in the yz plane for a surface $x = c$ with normal $\vec{\mathbf{i}}$ and a region R in the zx plane for a surface $y = c$ with normal $\vec{\mathbf{j}}$. (We use the zx plane rather than the xz plane to be consistent with the right-hand rule.) Thus,

$$\vec{\mathbf{n}}\,dS = \pm\vec{\mathbf{k}}\,dx\,dy, \qquad \vec{\mathbf{n}}\,dS = \pm\vec{\mathbf{j}}\,dx\,dz, \qquad \vec{\mathbf{n}}\,dS = \pm\vec{\mathbf{i}}\,dy\,dz, \tag{1}$$

for planes $z = c$, $y = c$, and $x = c$, respectively, taking care to choose the correct sign.

Case 2a — S projects onto the xy plane

The formula for $\vec{\mathbf{n}}\,dS$ is surprisingly simple, but it takes a bit of work to derive it. Suppose the surface is given as $g(x, y, z) = c$. Then we can identify $\vec{\nabla} g$ and $-\vec{\nabla} g$ as normals to the surface, one pointing in either direction. Since $\vec{\mathbf{n}}$ must be a unit vector, we have

$$\vec{\mathbf{n}} = \frac{\pm\vec{\nabla} g}{\|\vec{\nabla} g\|}.$$

The differential dS must measure the surface area on the graph of $g = c$ that corresponds to the region R in the xy-plane. This quantity is therefore related to the differential dA by

$$dA = dS\,|\cos\theta|,$$

where θ is the angle between the normal vector for the surface and the normal vector to the xy-plane, which is $\vec{\mathbf{k}}$. (For example, $dA = dS$ when the surface is horizontal and $dA = 0$ if the surface is vertical.) Using the dot product, we have

$$\cos\theta = \frac{\vec{\nabla} g \cdot \vec{\mathbf{k}}}{\|\vec{\nabla} g\|};$$

[1] By this, we mean that the process of squashing the surface onto the coordinate plane does not cause points on the surface to merge with each other, as would happen with a sphere. Algebraically, we can project a surface onto a coordinate plane if we can solve the equation of the surface uniquely for the third variable.

thus,

$$dS = \frac{dA}{|\cos\theta|} = \frac{\|\vec{\nabla} g\|}{|\vec{\nabla} g \cdot \vec{\mathbf{k}}|}\, dA.$$

Rather than calculating $\vec{\mathbf{n}}$ and dS separately, we can avoid computing $\|\vec{\nabla} g\|$ by combining the factors before doing the calculation. The result is

$$\vec{\mathbf{n}}\, dS = \frac{\pm\vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{k}}|}\, dx\, dy. \tag{2}$$

This formula provides the simplest way to compute a surface integral for any surface that can be projected onto the xy-plane, aside from portions of cylinders and spheres. Note that the correct choice of sign in the numerator of (2) depends on the orientation of the surface and should be determined from a sketch.

Cases 2b,c — S projects onto the yz or zx plane

These cases are analogous to Case 2a. Taking $\vec{\mathbf{n}}$ to be the normal and noting that the normals to the yz and zx planes are $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$ respectively, the formulas for the differentials are

$$\vec{\mathbf{n}}\, dS = \frac{\pm\vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{j}}|}\, dx\, dz \tag{3}$$

for projection onto the zx plane and

$$\vec{\mathbf{n}}\, dS = \frac{\pm\vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{i}}|}\, dy\, dz \tag{4}$$

for projection onto the yz plane. Note that there is a simple connection between the three formulas of case 2. The unit vector in the dot product has to be the one that is normal to the plane whose coordinates are given as factors of dA. As in (2), the choice of sign in (3) and (4) depends on the direction of the appropriate surface normal and should be determined from a sketch.

Case 3 — S is a portion of a circular cylinder of radius a

For a surface $r = a > 0$, we can use θ and z as the parameters. The vector $\langle x, y, 0\rangle$ is an outward normal, so we can use it (after normalization) for $\vec{\mathbf{n}}$. The necessary formulas are

$$\vec{\mathbf{n}} = \frac{\langle x, y, 0\rangle}{a} = \langle \cos\theta, \sin\theta, 0\rangle, \qquad dS = a\, dz\, d\theta. \tag{5}$$

The flux integral is then

$$\iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}}\, dS = \int_{\theta_1}^{\theta_2} \int_{z_1}^{z_2} \vec{\mathbf{v}} \cdot \langle \cos\theta, \sin\theta, 0\rangle\; a\, dz\, d\theta,$$

where any spatial variables in $\vec{\mathbf{v}}$ must be written in terms of the parameters z and θ.

Case 4 — S is a portion of a sphere of radius R

For a surface $\rho = R > 0$, we can use θ and ϕ as the parameters. The position vector $\vec{\mathbf{r}}$ is an outward normal, so we can use it (after normalization) for $\vec{\mathbf{n}}$. The necessary formulas are

$$\vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}}{R}, \qquad dS = R^2 \sin\phi\, d\phi\, d\theta. \tag{6}$$

The flux integral is then

$$\iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}} \, dS = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} (\vec{\mathbf{v}} \cdot \vec{\mathbf{r}})(\theta, \phi) \; R \sin\phi \, d\phi \, d\theta,$$

where any spatial variables in $\vec{\mathbf{v}}$ must be written in terms of the parameters θ and ϕ. (Note that the differential for the flux integral has only one factor of R because the position vector $\vec{\mathbf{r}}$ is also a length. Taken together, the product of $\vec{\mathbf{v}}$, $\vec{\mathbf{r}}$, and R has the dimension volume per time, as it should.)

Summary

Flux integrals are integrals of a vector field $\vec{\mathbf{v}}$ *across* a surface S:

$$\text{flux} = \iint_S \vec{\mathbf{v}} \cdot \vec{\mathbf{n}} \, dS.$$

They are evaluated by choosing a coordinate system and identifying two of the coordinates in the system as the integration variables. All other variables and functions must then be written in terms of the two integration variables. We need consider just four cases:

1. S is parallel to a coordinate plane, such as the surface $y = 1$.

   The region R in the coordinate plane is the same as the surface S, and the differential is

   $$\vec{\mathbf{n}} \, dS = \pm \vec{\mathbf{k}} \, dx \, dy, \qquad \vec{\mathbf{n}} \, dS = \pm \vec{\mathbf{j}} \, dx \, dz, \qquad \vec{\mathbf{n}} \, dS = \pm \vec{\mathbf{i}} \, dy \, dz, \tag{1}$$

   for planes $z = c$, $y = c$, and $x = c$, respectively, taking care to choose the correct sign for the normal.

2. S is not parallel to a coordinate plane, but can be projected onto a coordinate plane.

   Write the surface as $g(x, y, z) = 0$. Choose the coordinate plane and then R is the projection of S onto that plane. The differential is

   $$\vec{\mathbf{n}} \, dS = \frac{\pm \vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{k}}|} \, dx \, dy, \quad \vec{\mathbf{n}} \, dS = \frac{\pm \vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{j}}|} \, dx \, dz, \quad \vec{\mathbf{n}} \, dS = \frac{\pm \vec{\nabla} g}{|\vec{\nabla} g \cdot \vec{\mathbf{i}}|} \, dy \, dz \tag{2}$$

   for projection onto the xy, zx, and yz planes, respectively, taking care to choose the correct sign for the normal.

3. S is a portion of a circular cylinder $r = a$.

   Use θ and z as the integration variables, with

   $$\vec{\mathbf{n}} = \frac{\langle x, y, 0 \rangle}{a} = \langle \cos\theta, \sin\theta, 0 \rangle, \qquad dS = a \, dz \, d\theta. \tag{3}$$

4. S is a portion of a sphere $\rho = R$.

   Use θ and ρ as the integration variables, with

   $$\vec{\mathbf{n}} = \frac{\vec{\mathbf{r}}}{R}, \qquad dS = R^2 \sin\phi \, d\phi \, d\theta. \tag{4}$$