Introduction to Asymptotics

1. A Model Problem

The following problem is a simplified version of a problem that arose in a research project I did in 2016. We'll eventually do the modeling that led to this problem.

Problem 1 *Given parameters* $0 < P < 1$ *and* $\mu > 1$*, find* $x_0 > 0$ *such that*

$$x' = -\mu x + 2\mu y - 2P\mu e^{-t}, \qquad x(0) = x_0, \quad x(1/2) = x_0; \tag{1}$$

$$y' = x - y, \qquad y(0) = 1, \quad y(1/2) = 1. \tag{2}$$

Our primary interest in Problem 1 is not in finding the value of x_0, but rather in identifying conditions that must be satisfied for the existence of a solution. The system of differential equations has a 2-parameter family of solutions, and x_0 is an additional unknown; however, there are four auxiliary conditions. This suggests that a solution is only possible if P and μ satisfy some relationship to be determined. Thus, the actual problem of interest is

Problem 2 *Given parameter* $\mu > 1$*, find any* P *values for which Problem 1 has a solution.*

2. Exact Solution

With μ given, the differential equations will have a 3-parameter family of solution, with P and two integration constants as the parameters. These parameters will have to satisfy a system of three algebraic equations:

$$x(1/2) = x(0), \qquad y(1/2) = y(0), \qquad y(0) = 1. \tag{3}$$

The differential equations can be written in vector form as

$$u' = Au + f, \tag{4}$$

where

$$A = \begin{pmatrix} -\mu & 2\mu \\ 1 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} -2P\mu e^{-t} \\ 0 \end{pmatrix} \tag{5}$$

You can find nonhomogeneous linear systems such as this in most differential equations books. The main idea is that the problem can be broken into smaller parts:

1. Find one solution u_p of the system.
   (a) For this case, there will be a solution $u_p(t) = v_p e^{-t}$, where v_p is a constant vector.
2. Find the 2-parameter family of solutions u_c of the simpler system $u' = Au$.
   (a) First find two specific solutions of the form $u_j(t) = v_j e^{\lambda_j t}$.
      i. The eigenvalues λ_j are the solutions of the equation $\det(A - \lambda I) = 0$.
      ii. The eigenvector v for any particular λ is any solution of the matrix equation $(A - \lambda I)v = 0$.
   (b) Then $u_c(t) = c_1 u_1(t) + c_2 u_2(t)$ (given that the system has two components).
3. The general solution is $u(t) = u_c(t) + u_p(t)$.

Following this procedure yields the eigenvalues

$$\lambda_1 = \frac{-(\mu+1)+\sqrt{\mu^2+6\mu+1}}{2}, \qquad \lambda_2 = \frac{-(\mu+1)-\sqrt{\mu^2+6\mu+1}}{2}, \tag{6}$$

corresponding eigenvectors

$$v_1 = \begin{pmatrix} \lambda_1+1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \lambda_2+1 \\ 1 \end{pmatrix}, \tag{7}$$

and solutions

$$x(t) = c_1(\lambda_1+1)e^{\lambda_1 t} + c_2(\lambda_2+1)e^{\lambda_2 t}, \tag{8}$$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + Pe^{-t}, \tag{9}$$

Substituting these solutions into (3) yields the algebraic system

$$\left(1-e^{\frac{\lambda_1}{2}}\right)(\lambda_1+1)c_1 + \left(1-e^{\frac{\lambda_2}{2}}\right)(\lambda_2+1)c_2 = 0, \tag{10}$$

$$\left(1-e^{\frac{\lambda_1}{2}}\right)c_1 + \left(1-e^{\frac{\lambda_2}{2}}\right)c_2 + \left(1-e^{-\frac{1}{2}}\right)P = 0, \tag{11}$$

$$c_1 + c_2 + P = 1. \tag{12}$$

This system can be solved using a computer algebra system or by hand, with the solution

$$P = \frac{\left(1-e^{\frac{\lambda_1}{2}}\right)\left(1-e^{\frac{\lambda_2}{2}}\right)(\lambda_1-\lambda_2)}{\left(1-e^{\frac{\lambda_1}{2}}\right)\left(e^{-\frac{1}{2}}-e^{\frac{\lambda_2}{2}}\right)(\lambda_1-\lambda_2)+\left(1-e^{-\frac{1}{2}}\right)\left(e^{\frac{\lambda_1}{2}}-e^{\frac{\lambda_2}{2}}\right)(\lambda_2+1)}. \tag{13}$$

3. Asymptotic Analysis of the Solution (13)

So far, we have done a lot of messy differential equation solving and algebra, and we've got an answer that doesn't look particularly interesting. What is interesting is that the value of P actually changes very little with μ: on the interval $1 < \mu < 40$, we have P decreasing from 0.635 to 0.626. Our messy function of μ is only slightly different from a constant. Given that, we ought to be able to approximate P as a constant plus a simple small-amplitude function of μ. In this section, we'll obtain such an approximation by applying asymptotic expansion to the solution (13). Later, we'll get the same answer by applying a *regular* perturbation method to the algebraic system (3). We'll improve that method by applying a regular perturbation method to the eigenvalue equation rather than using the asymptotic expansion of the exact solution for the eigenvalues. We can also apply a regular perturbation method directly to Problem 1; however, the method will not work in that case, as Problem 1 is actually a *singular* perturbation problem. The goals of this analysis are to provide illustrations of asymptotic expansion and regular perturbation, showing how they are applied at different stages of a problem, and to see what happens in an example where regular perturbation does not work.

A Taylor Series Result Using a Perturbation Method

We need a computational formula,

$$\sqrt{1+\epsilon} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O(\epsilon^3), \qquad \epsilon \to 0. \tag{14}$$

This formula is simply the first three terms of the Taylor series for the function $f(\epsilon) = \sqrt{1+\epsilon}$ with series center at 0. We could get it by applying the Taylor coefficient formula, with the results following after we have computed $f'(0)$ and $f''(0)$. This is a lot of unnecessary work. Instead, we look for a power series solution with undetermined coefficients:

$$\sqrt{1+\epsilon} = 1 + a\epsilon + b\epsilon^2 + O(\epsilon^3), \qquad \epsilon \to 0.$$

All we need to do to find a and b is square both sides, being careful to collect terms with like powers of ϵ:

$$\begin{aligned} 1+\epsilon &= (1 + a\epsilon + b\epsilon^2)(1 + a\epsilon + b\epsilon^2) + O(\epsilon^3) \\ &= 1 + 2a\epsilon + (a^2 + 2b)\epsilon^2 + O(\epsilon^3). \end{aligned}$$

The results $a = 1/2$ and $b = -1/8$ follow from the requirement that the two functions in the equation must be the same polynomial in ϵ.

Our derivation of (14) is an example of a perturbation method: We pre-identify the structure of a solution as a series in terms of powers of a small parameter and then reduce the original problem to a sequence of problems that determine each successive term in the series. Regular perturbation methods are most commonly used for linear differential equations with nonconstant coefficients, but they can be used for a variety of problems from algebra to partial differential equations. We'll see that not all perturbation problems are regular, but for now think of the method as being general.

Leading Order Approximation as $\mu \to \infty$

If we want to get an approximation for large μ, the asymptotic approach is to think of $\mu \to \infty$. This gives us

$$\sqrt{\mu^2 + 6\mu + 1} = \mu + O(1),$$

which results in the approximations

$$\lambda_1 = O(1), \quad \lambda_2 = -\mu + O(1).$$

Note that the cancelation of the μ terms in λ_1 does not tell us that the approximation is 0; it tells us that we need the $O(1)$ terms to get an actual leading order approximation. From

$$\sqrt{\mu^2 + 6\mu + 1} = \mu\sqrt{1 + \frac{6}{\mu} + O\left(\frac{1}{\mu^2}\right)},$$

we can use (14) with $\epsilon = 6/\mu$ to obtain

$$\lambda_1 = 1 + O\left(\frac{1}{\mu}\right),$$

which will suffice. Thus,

$$e^{\frac{\lambda_1}{2}} = \sqrt{e} + O\left(\frac{1}{\mu}\right), \quad \lambda_1 - \lambda_2 = \mu + O(1),$$

and so on. The quantity $e^{\frac{\lambda_2}{2}} \approx e^{-\mu}$ is a special case. Where $1/\mu$ is algebraically small, $e^{-\mu}$ is exponentially small, as if it were $1/\mu^{\infty}$. This means that it never counts when added to a term that is only algebraically small, and we can ignore it altogether. Substituting these leading order approximations into the solution (13) yields

$$P \sim \frac{(1-\sqrt{e})\,\mu}{(1-\sqrt{e})\left(e^{-\frac{1}{2}}\right)\mu + \left(1-e^{-\frac{1}{2}}\right)(\sqrt{e})\,(-\mu)} = \frac{1-\sqrt{e}}{(1-\sqrt{e})\left(e^{-\frac{1}{2}}\right) + (1-\sqrt{e})} = \frac{\sqrt{e}}{\sqrt{e}+1},$$

where the notation "$\sim$" (read "is asymptotic to") must be used rather than an equal sign because we omitted the big oh terms. (It is better to use $\sim$ even when including big oh terms. We'll have a mathematicial definition of this notation later.)

Two-Term Approximation as $\mu \to \infty$

We can get a higher order approximation from (13) using the same procedure we used to get the leading order term. We just have to be careful to keep track of which terms count and which are absorbed into big oh terms. For example,

$$\sqrt{\mu^2+6\mu+1} \sim \mu\sqrt{1+\left(\frac{6}{\mu}+\frac{1}{\mu^2}\right)} \sim \mu\left[1+\frac{1}{2}\left(\frac{6}{\mu}+\frac{1}{\mu^2}\right) - \frac{1}{8}\left(\frac{6}{\mu}+\frac{1}{\mu^2}\right)^2 + O\left(\frac{1}{\mu^3}\right)\right]$$

$$\sim \mu\left[1+\left(\frac{3}{\mu}+\frac{1}{2\mu^2}\right) - \left(\frac{36}{8\mu^2}\right) + O\left(\frac{1}{\mu^3}\right)\right] \sim \mu + 3 - \frac{4}{\mu} + O\left(\frac{1}{\mu^2}\right).$$

From this result, we have

$$\lambda_1 \sim 1 - \frac{2}{\mu} + O\left(\frac{1}{\mu^2}\right), \quad \lambda_2 \sim -\mu - 2 + O\left(\frac{1}{\mu}\right). \tag{15}$$

Tracking everything through very carefully, we eventually get a final result

$$P \sim \frac{\sqrt{e}}{\sqrt{e}+1}\left(1 + \frac{2e-3\sqrt{e}}{e-1}\cdot\frac{1}{\mu}\right) + O\left(\frac{1}{\mu^2}\right). \tag{16}$$

4. Perturbation Solution from the Linear System (3)

The first principle of asymptotics is

- Methods that make use of the known structure of a problem are almost always better than methods that don't.

In the derivation of the solution for P in (13), we failed to make use of the insight that the solution for P should have an asymptotic expansion in powers of $1/\mu$. Instead, we could go back to the problem from which we obtained (13)—the system of linear algebraic equations (3) with solutions given by a slightly modified version of (7)–(9).

We start with the asymptotic expansions (15) for the eigenvalues. These lead to eigenvectors

$$v_1 = \begin{pmatrix} 2 - \frac{2}{\mu} + O\left(\frac{1}{\mu^2}\right) \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\mu - 1 + O\left(\frac{1}{\mu}\right) \\ 1 \end{pmatrix}.$$

Using the eigenvectors in this form violates the second principle of asymptotics, which is

- Always work with small rather than large.

Eigenvectors can be rescaled by multiplying by constants, and we should rescale v_2 to eliminate the large term in the first component:

$$v_1 = \begin{pmatrix} 2 - \frac{2}{\mu} + O\left(\frac{1}{\mu^2}\right) \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 - \frac{1}{\mu} + O\left(\frac{1}{\mu^2}\right) \\ \frac{1}{\mu} \end{pmatrix}. \tag{17}$$

With these eigenvectors, the solution formulas for the differential equations are

$$x(t) \sim c_1 \left(2 - \frac{2}{\mu}\right) e^{\lambda_1 t} + c_2 \left(-1 - \frac{1}{\mu}\right) e^{\lambda_2 t} + O\left(\frac{1}{\mu^2}\right), \tag{18}$$

$$y(t) = c_1 e^{\lambda_1 t} + \frac{1}{\mu} c_2 e^{\lambda_2 t} + P e^{-t}, \tag{19}$$

The seemingly minor difference in how we write v_2 is actually critical, because it shows us that the y component of v_2 is small, and that in turn means that the c_2 term in the solution for y is small.[1] To leading order, the two algebraic conditions on y will be enough to determine c_1 and P. We will only need the condition on x to get the leading order result for c_2.

Leading Order Approximation as $\mu \to \infty$

Using asymptotic expansion, we can calculate

$$y\left(\frac{1}{2}\right) \sim e^{\frac{\lambda_1}{2}} c_1 + e^{-\frac{1}{2}} P \sim \left(\sqrt{e} - \frac{\sqrt{e}}{\mu}\right) c_1 + \frac{1}{\sqrt{e}} P. \tag{20}$$

To leading order, the periodicity and initial conditions on y are

$$\begin{aligned} (\sqrt{e} - 1)c_1 + \left(\frac{1}{\sqrt{e}} - 1\right) P &= 0, \\ c_1 \qquad\qquad + \quad P &= 1. \end{aligned}$$

This system is easily solved by hand, with the results

$$c_1 \sim \frac{1}{\sqrt{e} + 1}, \quad P \sim \frac{\sqrt{e}}{\sqrt{e} + 1}. \tag{21}$$

We care only about P, so we do not need to consider the periodicity condition on x at all.

Compare the amount of work needed for this perturbation solution with that needed for the calculation of the exact solution and subsequent expansion. This is a good illustration of the first principle of asymptotics.

[1] Using the original choice of v_2, the smallness of the second term in (19) is incorporated into c_2 rather than begin explicitly clear. We would not know that c_2 is small until after solving the 3-component system.

Two-Term Approximation as $\mu \to \infty$

With the addition of the $O(1/\mu)$ terms, the two conditions on y yield the equations

$$\left(\sqrt{e} - 1 - \frac{\sqrt{e}}{\mu}\right) c_1 - \frac{1}{\mu} c_2 + \left(\frac{1}{\sqrt{e}} - 1\right) P = O\left(\frac{1}{\mu^2}\right), \tag{22}$$

$$c_1 + \frac{1}{\mu} c_2 + P = 1. \tag{23}$$

Clearly we can simplify this system by replacing (22) by the sum of the two equations (also multiplying by $\sqrt{e}$ for convenience):

$$\left(e - \frac{e}{\mu}\right) c_1 + P = \sqrt{e} + O\left(\frac{1}{\mu^2}\right), \tag{24}$$

To leading order, we have the system

$$c_1 + P = 1, \quad ec_1 + P = \sqrt{e},$$

with solution (21). For higher order terms, we begin by writing the known structure of the asymptotic approximation as

$$P \sim \frac{\sqrt{e}}{\sqrt{e}+1}\left(1 + \frac{P_1}{\mu}\right) + O\left(\frac{1}{\mu^2}\right), \quad c_1 \sim \frac{1}{\sqrt{e}+1}\left(1 + \frac{c_{11}}{\mu}\right) + O\left(\frac{1}{\mu^2}\right), \quad c_2 \sim c_{20} + O\left(\frac{1}{\mu}\right). \tag{25}$$

We did not need the $O(1)$ solution for c_2 to get the $O(1)$ solution for P, so we also won't need the $O(1/\mu)$ term for c_2 to get the corresponding term for P.

Substituting the structure (25) into the equations (23)–(24), we obtain

$$\frac{1}{\sqrt{e}+1}\left(1 + \frac{c_{11}}{\mu}\right) + \frac{c_{20}}{\mu} + \frac{\sqrt{e}}{\sqrt{e}+1}\left(1 + \frac{P_1}{\mu}\right) = 1 + O\left(\frac{1}{\mu^2}\right),$$

$$\left(1 - \frac{1}{\mu}\right) \frac{e}{\sqrt{e}+1}\left(1 + \frac{c_{11}}{\mu}\right) + \frac{\sqrt{e}}{\sqrt{e}+1}\left(1 + \frac{P_1}{\mu}\right) = \sqrt{e} + O\left(\frac{1}{\mu^2}\right).$$

Of course the leading order terms cancel, so we can collect the $O(1/\mu)$ terms to get (multiplying by $\sqrt{e}+1$ and simplifying)

$$c_{11} + (\sqrt{e}+1)\, c_{20} + \sqrt{e}\, P_1 = 0, \tag{26}$$

$$\sqrt{e}\, c_{11} + P_1 = \sqrt{e}. \tag{27}$$

To complete the system, we need to get c_{20}, which comes from the leading order periodicity condition for x. From (3) and (18), this quickly reduces to

$$c_2 \sim 2c_1\left(1 - e^{\frac{\lambda_1}{2}}\right) \sim -2\frac{\sqrt{e}-1}{\sqrt{e}+1} = c_{20}. \tag{28}$$

Substituting this result into (26) and then solving (26)–(27) yields the final result (16), but with far less calculation than was required in finding the exact solution and then applying asymptotic expansion.

5. Perturbation Analysis of the Eigenvalue Problem

We can improve on the method of Section 4. Notice that we began the method with two-term approximations (15) of the eigenvalues (6). We got those results by doing an asymptotic expansion of the exact solution of the eigenvalue equation. Would it be easier to get them by doing a regular perturbation solution of the eigenvalue equation? Yes!

The eigenvalues were found from the polynomial equation $\det(A - \lambda I) = 0$, which is

$$\lambda^2 + (\mu + 1)\lambda - \mu = 0. \tag{29}$$

The second and third terms have factors of μ, so it seems reasonable to assume that those terms are the largest ones. We can rearrange the equation as

$$\mu\lambda - \mu = -\lambda^2 - \lambda, \tag{30}$$

with the expectation that the quantity on the right is less than $O(\mu)$ For this to work, we must have $\lambda \sim 1$; otherwise the two sides will have different orders. From this, it seems reasonable to expect

$$\lambda_1 \sim 1 + \frac{\alpha}{\mu}.$$

Substituting this into the rearranged equation, and keeping only the most important terms on each side, yields

$$\alpha = -1 - 1 = -2;$$

hence, we have found the two-term approximation for λ_1 given in (15), but without having to solve the quadratic exactly and then do an asymptotic expansion.

Clearly the equation must have a second solution, and it must not come from the case where the second and third terms are the largest. Perhaps it should be the first and second terms. To explore this, we rewrite the equation as

$$\lambda^2 + \mu\lambda = \mu - \lambda. \tag{31}$$

With each term on the left larger than the terms on the right, we must conclude $\lambda \sim -\mu$. This result needs to be checked against the assumptions. We would have λ^2 and $\mu\lambda$ be $O(\mu^2)$, while each term on the right is only $O(\mu)$. This is consistent, and it leads to the expectation

$$\lambda_2 \sim -\mu + \beta.$$

Before substituting into the rearranged equation (31), we should factor the equation as

$$\lambda(\lambda + \mu) = \mu - \lambda.$$

Now it is easy to substitute in only the most important term for each factor:

$$(-\mu)(\beta) = \mu - (-\mu),$$

which yields $\beta = -2$, reproducing the expansion we found in (15).

There can't be more solutions for the quadratic, but there is an ordering assumption we haven't tried yet. Suppose we assume the first and third terms are the largest ones. We then write the equation as

$$\lambda^2 - \mu = -(\mu + 1)\lambda. \tag{32}$$

If the terms on the right are relatively small, then we get two solutions, $\lambda \sim \pm\sqrt{\mu}$. However, this result makes the right side $O(\mu^{3/2})$, which is larger than the two terms on the left, contradicting the assumption. This particular guess for the largest terms is inconsistent.

6. Perturbation Analysis of the Original Problem

It was better to work from the solutions of the differential equations rather than the exact solution. Perhaps it would be better to work from the differential equations themselves. Normally this is the case, but here we can identify a problem that can arise when solving a problem using a regular perturbation method. The difficulty is immediately apparent from a leading order analysis of (1). Since μ is large, the equation simplifies to

$$x \sim 2y - 2Pe^{-t}. \tag{33}$$

But the auxiliary conditions require x and y to be periodic and e^{-t} is not. The only escape is to conclude $P = 0$, which we know is wrong. (Had this been our initial attempt at analysis, we would have at this point concluded $P = 0$, which would have led to a contradiction in the other conditions.)

It is instructive to attempt to do the analysis without noting the failure of periodicity. Substituting (33) into (2) yields

$$y' - y \sim -2Pe^{-t},$$

with solutions

$$y \sim c_1 e^t + Pe^{-t}, \quad x \sim 2c_1 e^t. \tag{34}$$

The initial and terminal conditions on y are then

$$1 = c_1 + P = \sqrt{e}\, c_1 + \frac{P}{\sqrt{e}}.$$

Our leading order result (21) is actually the unique solution of this system. So the only condition that cannot be satisfied by the regular perturbation method is the periodicity condition $x(1/2) = x(0)$. The failure of the regular perturbation solution to satisfy this condition means that none of the results obtained in this calculation can be trusted. While some are actually correct, they were derived from an incorrect assumption about the structure of the solution.

What we have seen here is what happens when you try to apply a regular perturbation method to a singular perturbation problem. Everything goes just fine for a while, but eventually there is a condition that cannot be satisfied and the whole chain of reasoning collapses. Much of the regular perturbation solution will be used as part of a singular perturbation solution, but we'll need to learn how to do that.