Steady Flow Across a Flat Plate

Basic Fluid Mechanics

The basic idea of fluid mechanics is to model a fluid as a continuous medium and write equations to keep track of important properties. If there is no energy exchange, the unknowns are the velocity vector $\mathbf{u}$, the density ρ, and the pressure p. These are related by the continuity equation, representing conservation of mass, the momentum equation for the velocity vector, and an equation of state.

Equation of State

Thermodynamics is concerned with the relationships of state variables in a fluid, including density, pressure, temperature, entropy, enthalpy, and free energy. Any two of these can be considered as independent, and then the rest are determined by thermodynamic laws such as the ideal gas law. In our limited setting, the temperature is taken to be fixed, so we can think of density as the second independent variable. This means that there is a thermodynamic equation of state that can be written generically as

$$p = F(\rho).$$

The derivation of the acoustics equations from the basic conservation laws of fluid mechanics ultimately connects the speed of sound in the medium to the function F:

$$c = \sqrt{F'(\rho)}.$$

We all know that air can be compressed, but this can only be done by confining it to a space of fixed volume. If you try to compress air by pushing on it, the air will simply move out of the way. We'll see that the extent to which a fluid can be compressed by movement at a speed significantly less than the speed of sound is very small. For now, we assume

$$\rho \sim \rho_0(1 + \epsilon S), \tag{1}$$

where ρ_0 is the standard density and ϵ is some small parameter to be determined through scaling. We can then obtain an asymptotic expansion for the pressure:

$$p \sim F(\rho_0 + \epsilon\rho_0 S) \sim F(\rho_0) + \epsilon\rho_0 S F'(\rho_0) = p_0 + \epsilon c^2 \rho_0 S. \tag{2}$$

Continuity

The continuity equation is a standard conservation law for mass. Since ρ is the density of mass and $\rho\mathbf{u}$ is the mass flux per unit area, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0. \tag{3}$$

For near-incompressible flow, we may take $\rho \sim \rho_0$ to leading order, which reduces the continuity equation in two dimensions to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4}$$

Momentum

The momentum equation is a continuous analog of Newton's second law of motion $mv' = F$. We can think of it as a conservation law with a known quantity of momentum being created by the external force of pressure and the internal force of viscosity-induced stress.[1] In vector notation, this is

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \rho \nabla^2 \mathbf{u}.$$

The quantity ν is the kinematic viscosity, which is related to the dynamic viscosity μ by the equation $\mu = \rho\nu$. Note that the quantity in the parentheses works out to a scalar of two terms, and this scalar must be applied to each of the velocity components. In scalar form, we have the momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{6}$$

Flow Across a Flat Plate

We seek to solve the system (2,4,5,6) in a simple setting in which ambient flow of velocity $u = u_\infty$ and $v = 0$ passes over a flat plate located at $y = 0$, $x > 0$, and $-\infty < z < \infty$, where y is the vertical coordinate. The domain for the differential equations is then $x > 0$, $y > 0$, and there are auxiliary conditions at the two boundaries and also at $y = \infty$. The ambient flow provides boundary conditions

$$u = u_\infty, \quad v = 0, \quad \rho = \rho_0 \qquad \text{at} \qquad y = \infty, \quad x = 0.$$

No-flow conditions at the plate are

$$u = 0, \quad v = 0 \qquad \text{at} \qquad y = 0.$$

There is no boundary condition for ρ at the plate because the differential equations are only first-order in p. For a well-posed problem, there should be boundary conditions on the velocity at $x = \infty$, but these do not play any role in the problem.

The Outer Region

Observe that the ambient flow satisfies all requirements except the no-flow condition on u at the plate. Thus, we have a boundary layer at $y = 0$ and need only consider the layer itself.

[1] This is what is called a Newtonian fluid. Some fluids are more complicated and solids are always so.

Scaling in the Layer

We need to proceed with nondimensionalization before we can choose the correct scales. There is no obvious length scale in the problem, so we assume that the correct scale in the outer region is some quantity L. This will then be the correct scale in the inner layer for x, while the scale for y in the layer will be δL, where δ is a small as-yet-unknown dimensionless combination formed from L, u_∞, c, and ν. Similarly, u_∞ is the obvious scale for u, but we clearly need a smaller scale αu_∞ for v. We can scale time using L/u_∞, and we are already scaling density by ρ_0 and density changes by $\epsilon\rho_0$. We will need to determine the correct choices for α, δ, and ϵ in terms of the parameters L, u_∞, c, and ν. It seems reasonable that c will only enter into the scaling for ϵ, as it arises in the equation of state.

It is helpful at this point to identify two important parameters in fluid dynamics, the Reynolds number and the Mach number:

$$\mathrm{Re} = \frac{u_\infty L}{\nu}, \qquad \mathrm{Ma} = \frac{u_\infty}{c} \ll 1.$$

The Reynolds number is the ratio of inertial forces to viscous forces and is generally on the order of 1000 to 10000, although it can be small for highly viscous fluids or very small reference lengths. The Mach number is the ratio of fluid speed to sound speed, and is generally small.

With the substitutions

$$x = LX, \quad y = \delta LY, \quad u = u_\infty U, \quad v = \alpha u_\infty V, \quad t = \frac{L\tau}{u_\infty},$$

the continuity equation becomes

$$\frac{\partial U}{\partial X} + \frac{\alpha}{\delta}\frac{\partial V}{\partial Y} = O(\epsilon).$$

If the equation is to be properly scaled, there must be a dominant balance between two or more terms, so the only possibility is to take $\alpha = \delta$. Thus, we have

$$U_X + V_Y = 0. \tag{7}$$

The momentum equations then become

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + \frac{\epsilon}{\mathrm{Ma}^2}\frac{\partial S}{\partial X} = \frac{1}{\mathrm{Re}}\frac{\partial^2 U}{\partial X^2} + \frac{1}{\delta^2\mathrm{Re}}\frac{\partial^2 U}{\partial Y^2}$$

and

$$\frac{\partial V}{\partial \tau} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} + \frac{\epsilon}{\delta^2\mathrm{Ma}^2}\frac{\partial S}{\partial Y} = \frac{1}{\mathrm{Re}}\frac{\partial^2 V}{\partial X^2} + \frac{1}{\delta^2\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2}.$$

The scales can now be determined by dominant balance arguments.

1. Second derivatives with respect to Y must be important in the inner layer.
2. The scale ϵ must be chosen so that S enters into one of the equations at leading order.

Note that the second dervatves in X are less important than the second derivatives in Y because of small δ. The latter will be brought into the dominant balances of both equations if we take $\delta^2 \mathrm{Re} = 1$. Given flow with a large Reynolds number, this does make δ small as required.

Similarly, the extra factor of δ^2 makes the gradient of S more important in the V equation than the U equation. Including that term in the dominant balance of the former requires $\epsilon = \delta^2 \mathrm{Ma}^2$, which means that density perturbations are extremely small. With these changes, the steady-state leading order momentum equations are

$$UU_X + VU_Y = U_{YY} \tag{8}$$

and

$$UV_X + VV_Y + S_Y = V_{YY}. \tag{9}$$

Equations (7) and (8), along with the boundary conditions, are a well-posed problem for the velocity components U and V, and then (9) serves to define the density and pressure perturbations. The usual treatment of the problem usually considers only the flow itself, so we neglect the additional equation for S. Some problems have a pressure gradient imposed from outside the domain of the problem, in which case these terms are significant; here, the pressure gradient induced by the movement of the fluid is too small to affect the flow.

Stream Function

Observe that the continuity equation (7) can be solved by assuming the existence of a stream function ψ such that

$$U = \psi_Y, \qquad V = -\psi_X. \tag{10}$$

Of course we do not know if there is such a function, but we will attempt to find one that satisfies the momentum equation (8) and the boundary conditions. Substituting from (10) into (8) yields the differential equation that the stream function must satisfy:

$$\psi_Y \psi_{XY} - \psi_X \psi_{YY} = \psi_{YYY}. \tag{11}$$

The boundary conditions become

$$\psi_Y = 1, \quad \psi_X = 0 \qquad \text{at} \qquad Y = \infty, \quad X = 0, \tag{12}$$

$$\psi_Y = 0, \quad \psi_X = 0 \qquad \text{at} \qquad Y = 0. \tag{13}$$

Observe that the introduction of a stream function decreases the number of dependent variables at a cost of increasing the order of the equation.

Similarity Transformation

Although it is not obvious, there is a similarity transformation that reduces the stream function problem (11-13) to a problem with just one independent variable. Since the correct similarity transformation cannot be determined by dimensional analysis, we begin by assuming a general form

$$\psi = \beta X^q f(\eta), \qquad \eta = \frac{Y}{\delta X^p}, \tag{14}$$

where p and q must be determined and the coefficients α and β can be chosen for convenience. Differentiating (14) with respect to Y and employing the nonhomogeneous boundary condition in (12) fixes one relationship between p and q. Computation of the remaining derivatives of ψ and substitution into (11) fixes a second relationship between p and q, thereby determining the correct similarity structure. A convenient choice of α and β then yields the problem

$$f''' + ff'' = 0, \qquad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1 \tag{15}$$

with $U = f'$.

Solution by Shooting

Let $g = f'$ and $h = f''$, thereby recasting (15) as a system

$$f' = g, \qquad g' = h, \qquad h' = -fh, \tag{16}$$

with boundary conditions

$$f(0) = 0, \qquad g(0) = 0, \qquad g(\infty) = 1. \tag{17}$$

This system is difficult to solve numerically because of the boundary condition at ∞. However, we can instead consider the related family of initial value problems using the same differential equations and the initial conditions

$$f(0) = 0, \qquad g(0) = 0, \qquad h(0) = \sigma. \tag{18}$$

The initial value problem (16), (18) can be solved numerically for any given σ to define a function $\gamma(\sigma)$ by $\gamma(\sigma) = g(R)$, where R must be large enough so that $h(R)$ is close to 0.[2] The solution will match that of the boundary value problem (16), (17) if σ can be chosen to satisfy the equation $\gamma(\sigma) = 1$. This equation can be solved by an iterative technique that does not require derivatives, such as the secant method. Each step in the secant method calls for the determination of $\gamma(\sigma_n)$, which is done by the numerical differential equation solver. Few steps will be required because γ is clearly a monotone increasing function of σ.

[2] The differential equations have asymptotic behavior as $t \to \infty$ in which $h \to 0$, $g \to g_\infty$, and f approaches a linear function with slope g_∞.