Dominant Balance Example Problem

- Find the asymptotic behavior as $x \to \infty$, including all non-vanishing terms, for the graph of the equation
$$x^2 + xy - y^3 = 0.$$

The idea of a dominant balance argument is that no one term in an equation can dominate all of the others. While it is possible that more than two could be comparable, it is much more likely that there is a dominant balance between two terms and the rest are small. We therefore have to consider three cases:

1. Suppose $y^3 \ll x^2, xy$ as $x \to \infty$. Then $xy \sim -x^2$, so $y \sim -x$. Then $y^3 = O(x^3)$, which means $y^3 \gg x^2$ as $x \to \infty$, which contradicts the initial assumption. Thus, we cannot have a dominant balance as $x \to \infty$ that excludes y^3.

2. Suppose $x^2 \ll xy, y^3$ as $x \to \infty$. Then $y^3 \sim xy$, so $y \sim \pm\sqrt{x}$. Then $xy = O(x^{3/2})$, which means $x^2 \gg xy$, contradicting the initial assumption and forcing the conclusion that x^2 cannot be omitted from the dominant balance.

3. Suppose $xy \ll x^2, y^3$ as $x \to \infty$. Then $y^3 \sim x^2$, so $y \sim x^{2/3}$. So $xy = O(x^{5/3})$, which is consistent with the initial assumption. There is no contradiction, so the behavior $y \sim x^{2/3}$ is correct as $x \to \infty$.

Once we have one or more correct dominant balances, we can peel off the asymptotic behavior to get additional terms. In these additional calculations, we can use what we already know about the orders of terms to restrict the dominant balance alternatives. Assume[1]

$$y = x^{2/3} + y_1(x), \qquad y_1 \ll x^{2/3}. \tag{1}$$

Then
$$y^3 = x^2 + 3x^{4/3}y_1 + 3x^{2/3}y_1^2 + y_1^3.$$

Grouping the two dominant terms yields

$$\left[x^2 + 3x^{4/3}y_1 + 3x^{2/3}y_1^2 + y_1^3\right] - x^2 = x\left[x^{2/3} + y_1\right],$$

or
$$3x^{4/3}y_1 + 3x^{2/3}y_1^2 + y_1^3 = x^{5/3} + xy_1. \tag{2}$$

Equation (2) is now a problem to be solved for the leading order behavior of y_1. Because of the way we did the bookkeeping, we know in advance that the terms on each side of the equation are sorted into the proper asymptotic order, so we do not need to do a dominant balance argument and can immediately get
$$3x^{4/3}y_1 \sim x^{5/3},$$

or

$$y_1 \sim \frac{1}{3}x^{1/3}.$$

[1]The equation is merely a definition of y_1, so there is no assumption needed for it. The assumption is on the ordering of y_1, but that assumption is known to be consistent by our earlier argument that $y \sim x^{2/3}$.

It appears likely that the next term in the asymptotic behavior is a constant, so we need to look for it in order to be sure of having all non-vanishing terms. To do that, we need to assume

$$y_1 = \frac{1}{3}x^{1/3} + y_2(x), \qquad y_2 \ll x^{1/3}. \tag{3}$$

Rather than writing down every term, we can truncate each expansion so that we take at most two terms from the first term on each side of (2), one term from the second terms, and none from any subsequent terms. Thus,

$$3x^{4/3}\left[\frac{1}{3}x^{1/3} + y_2 + o(y_2)\right] + 3x^{2/3}\left[\frac{1}{9}x^{2/3} + O\left(x^{1/3}y_2\right)\right] + [O(x)] = x^{5/3} + x\left[\frac{1}{3}x^{1/3} + O\left(y_2\right)\right].$$

At this point, we expect the first term on each side to cancel because that follows from (3), but we actually get an unexpected second cancelation as well, leaving us with

$$3x^{4/3}y_2 + o(x^{4/3}y_2) + O(xy_2) + O(x) = O(xy_2).$$

Clearly we needed to take an additional term from at least some of the expansions. Nevertheless, we can still do a dominant balance. The terms of order xy_2 are clearly smaller than the term $3x^{4/3}y_2$, so we may conclude

$$3x^{4/3}y_2 = O(x).$$

We could track down the coefficient of the order x term, but it is enough to note that we have shown

$$y_2 = O\left(x^{-1/3}\right) \to 0.$$

Hence, the nonvanishing portion of the asymptotic behavior is

$$y \sim x^{2/3} + \frac{1}{3}x^{1/3}. \tag{4}$$