SOLVING FIRST-ORDER LINEAR EQUATIONS

In this section, we will discover a general method that can be used in principle to solve any first-order linear equation.[1]

1. An Illustrative Example

Suppose we want to solve the initial value problem

$$\frac{dy}{dx} + 2y = 4x, \qquad y(0) = 1. \tag{1}$$

This equation is linear, but not separable, so we need a new method called *variation of parameters*. The idea is that the linear equation (1), which is nonhomogeneous, is closely related to the homogeneous equation

$$\frac{dy}{dx} + 2y = 0, \tag{2}$$

which is separable as well as linear. Equation (2) has the general solution

$$y = Cy_1(x), \qquad y_1(x) = e^{-2x}.$$

The idea of variation of parameters is to look for a solution of a nonhomogeneous equation like (1) that is similar to the solution of the associated homogeneous equation, with the parameter C replaced by a function $u(x)$. So we assume that the solution of (1) has the form

$$y(x) = u(x)y_1(x) = e^{-2x}u(x), \tag{3}$$

where u is a function to be determined. In effect, we are converting the original equation with unknown $y(x)$ into a new equation with unknown $u(x)$. To do this, we first take the derivative of (3), getting

$$y' = e^{-2x}u'(x) - 2e^{-2x}u(x) \tag{4}$$

from the product rule. Then we substitute (3) and (4) into (1) to get a differential equation for u:

$$[e^{-2x}u'(x) - 2e^{-2x}u(x)] + 2[e^{-2x}u(x)] = 4x.$$

The u terms cancel, leaving

$$e^{-2x}u'(x) = 4x,$$

or

$$u'(x) = 4xe^{2x}. \tag{5}$$

At this point it is good to step back and review what we did. We started with a linear differential equation (1) for y. We arbitrarily wrote the unknown solution y in terms of an unknown function u in Equation (3). We obtained the differential equation (5) for u by substituting (3) into (1). In effect, Equation (3) defines u by $u = e^{2x}y$. If the y equation has a solution, then so does the u equation, and vice versa. So replacing y with u does no harm. Does it do any good? To answer this, compare the two equations (1) and (5). Equation (1) is a differential equation we can't solve using separation of variables. Equation (5) is a mere calculus problem because the function u does not appear in the equation except as u'. So the new problem is MUCH better than the original problem.

[1] "In theory, there is no difference between theory and practice. In practice, there is." — Yogi Berra

We can find u using integration by parts, which we should think of as

$$\int r(x)s'(x)\,dx = r(x)s(x) - \int r'(x)s(x)\,dx. \tag{6}$$

We can write the integration by parts formula using any symbols for the two functions that we like, except for symbols that have a specific meaning in our problem. So we can't use the usual notation with functions u and v. Taking $r = x$ and $s' = 4e^{2x}$, we have $r' = 1$ and $s = 2e^{2x}$. So

$$u = \int 4xe^{2x}\,dx = 2xe^{2x} - \int 2e^{2x}\,dx = 2xe^{2x} - e^{2x} + C = (2x-1)e^{2x} + C. \tag{7}$$

Combining this result with Equation (3) gives us the solution family[2]

$$y = 2x - 1 + Ce^{-2x}. \tag{8}$$

The last step is to apply the initial condition to determine the constant C. Substituting $x = 0$ and $y = 1$ into the general solution (8) yields the equation $1 = -1 + C$, so $C = 2$. The solution of the initial value problem is

$$y = 2x - 1 + 2e^{-2x}. \tag{9}$$

2. Summary of the Method

To solve a differential equation of the general form

$$a_1(x)y'(x) + a_0(x)y(x) = g(x), \tag{10}$$

we employ the method of variation of parameters:

1. Use separation of variables to find a solution y_1 of the associated homogeneous equation

$$a_1(x)y'(x) + a_0(x)y(x) = 0. \tag{11}$$

2. Substitute the form

$$y(x) = u(x)y_1(x) \tag{12}$$

into the original differential equation (10), thereby obtaining a new differential equation of the form

$$u'(x) = U(x), \tag{13}$$

where U is whatever appears on the right side of the new equation.

3. Integrate U to obtain a formula for $u(x)$. It is okay to omit the integration constant from this formula.

4. Write the solution of the original problem as

$$y(x) = Cy_1(x) + u(x)y_1(x). \tag{14}$$

[2]Note that the homogeneous solution Cy_1 is part of the solution of the full problem. Taking $C = 0$ for convenience gives us a particular solution $y = 2x - 1$. You can easily check that this function solves the original problem. Instead of having the integration constant C in the formula for u, we could have left it out and then written our final solution as $y = Ce^{-2x} + ue^{-2x}$.

As before, let's step back and take stock of what we have. The four steps in the procedure worked fine for the illustrative example. Will this procedure always work? The answer is a qualified yes because complications can arise.

1. In theory, step 1 should always work (provided the requirements of the existence theorem are met) because (11) can always be separated as
$$\frac{y'}{y} = -\frac{a_0(x)}{a_1(x)}$$
and then
$$\ln y = -\int \frac{a_0(x)}{a_1(x)}\,dx.$$
In practice, we might not be able to find an elementary antiderivative for $a_0(x)/a_1(x)$. This difficulty can be overcome by using a definite integral to define y_1, but that makes the problem much harder.

2. Step 2 always works. When (12) is substituted into (10), the function u always disappears, giving an equation that can be solved algebraically for u'.

3. Step 3 has the same problem as step 1. If we can't find an elementary antiderivative for U, then we have to use a definite integral to define u.

4. Step 4 always works.

3. More Examples

Example 1
Solve
$$xy' - 4y = x^6 e^x.$$

1. We start by separating variables to solve
$$xy' - 4y = 0.$$
Separation results in the integral form
$$\int \frac{1}{y}\,dy = \int \frac{4}{x}\,dx.$$
Since any solution can serve as y_1, we can ignore absolute values and integration constants, so we get
$$\ln y = 4\ln x = \ln x^4,$$
with the final result
$$y_1 = x^4.$$

2. Next we assume
$$y = x^4 u(x),$$
so then
$$y' = x^4 u' + 4x^3 u.$$
Substituting this into the original equation and canceling the u terms yields
$$x^5 u' = x^6 e^x,$$
which means
$$u' = xe^x.$$

3. We can integrate this to get an antiderivative

$$u = (x-1)e^x.$$

4. Substituting back into $y = x^4 u$ gives us a particular solution $y = (x^5 - x^4)e^x$. The solution of the original problem is then

$$y = Cx^4 + (x^5 - x^4)e^x.$$

◇

Example 2
Solve

$$y' + y = f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}, \qquad y(0) = 0.$$

This problem must be handled carefully because it consists of two different differential equations, one for $x < 1$ and one for $x > 1$. They have to be solved separately and then patched together.

For $x > 1$ we have the homogeneous equation

$$y' + y = 0,$$

which has solution

$$y = C_2 e^{-x}, \qquad x \ge 1. \tag{15}$$

Note that we are naming the constant C_2. We'll use C_1 for the constant in the other solution. These constants usually have different values. Note also that we do not have an initial condition for this solution. Our initial condition is at $x = 0$, but that is not part of the interval of definition for this solution. We will have to use the solution on $0 \le x \le 1$ to obtain $y(1)$ before we can find C_2.

Equation (15) gives us the homogeneous solution of the equation for $x < 1$, so we can assume

$$y = e^{-x}u(x), \qquad 0 \le x \le 1. \tag{16}$$

Substituting the formula into $y' + y = 1$ eventually gets us to $e^{-x}u' = 1$, or

$$u' = e^x.$$

This has a solution $u = e^x$, so (16) gives us the particular solution $y = e^{-x}e^x = 1$. A quick check shows that we might have been able to guess that to be a solution of $y' + y = 1$. Of course it is not all the solutions of that equation. All of them are given by adding the homogeneous solution:

$$y = 1 + C_1 e^{-x}, \qquad 0 \le x \le 1. \tag{17}$$

Taken together, (15) and (17) are the solutions of the differential equation. The initial condition must be satisfied by (17). With $x = 0$ and $y = 0$, we get $C_1 = -1$. Thus, the solution for $x \le 1$ is

$$y = 1 - e^{-x}, \qquad 0 \le x \le 1. \tag{18}$$

From this solution, we specifically obtain

$$y(1) = 1 - e^{-1},$$

and this serves as the initial condition needed to find C_2, with the result $C_2 = e - 1$. The full solution is

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ (e-1)e^{-x}, & x > 1 \end{cases}, \qquad y(0) = 0.$$

◇