USING THE FIRST TRANSLATION THEOREM

We compute derivatives by combining specific formulas, such as $(e^{at})' = ae^{at}$ with general rules like the product rule. Something similar happens with Laplace transforms, although the general rules are not as general. There is no general product rule, but there is a rule, called the first translation theorem, that functions as a product rule when one of the factors is an exponential function.

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a). \tag{1}$$

This rule can be used to compute both Laplace transforms and inverse Laplace transforms. Think of it as a translation of one quantity in t space with a corresponding quantity in s space. Normal rules of algebra apply separately in t and s spaces, but translations between the two can only follow a limited set of rules.

Using the theorem to compute Laplace transforms

Example 1

Say we want to find the Laplace transform of t^3e^{-2t}. This function has the structure needed for the first translation theorem. We first identify the quantities a and f that appear on the left side of (1). These are $a = -2$ and $f(t) = t^3$. Since there is not a separate Laplace transform formula for every power of t, it is better to think of $f(t) = t^3$ as a combination of formulas that match the table and assignment of letters to quantities. Thus, the full unpacking of t^3e^{-2t} is

$$a = -2, \qquad f(t) = t^n, \quad n = 3.$$

The next step is to us the identifications from the left side to identify the quantities that appear on the right side of (1); that is, F and a. We already have a. To find F, we use the appropriate table entry for the transform of t^n with $n = 3$; thus,

$$F(s) = \frac{3!}{s^4} = \frac{6}{s^4}.$$

Finally, we just need to put the pieces together, using the first translation theorem to convert between t and s spaces.

$$\mathcal{L}\{t^3e^{-2t}\} = \mathcal{L}\{e^{-2t}f(t)\} = F(s+2) = \frac{6}{(s+2)^4}.$$

◇

To summarize the procedure for computing the transform of a function whose structure is $g(t) = e^{at}f(t)$ for some constant a and function f:

1. Identify the components that match the left side of (1): the constant a and the function f, including both the general form of f and any specific parameter values.

2. Use the identifications from the left side to compute the components needed for the right side.

3. Write the results of the computation.

It is particularly helpful to write the computation results in a way that directly quotes the translation theorem using any of the numbers specific to the problem, but leaving the generic function names. Thus, "$e^{-2t}f(t)$" and "$F(s+2)$." This is important in any contexts where the same letter is used in two generic formulas, as frequently happens with the second translation theorem.

Using the theorem to compute inverse Laplace transforms

The first translation theorem is equally useful for computing inverse Laplace transforms, but it is much less obvious how to do it. When you get a result like

$$Y(s) = \frac{s}{s^2 + 2s + 5},$$

you can't just read off the value of a and the function F. The procedure is similar, but with more nuance:

To invert a tranform $Y(s)$ using the first translation theorem:

1. Identify the components that match the right side of (1): the constant a and the function F.
   (a) Manipulate the function Y so that the choice of a becomes clear.
   (b) Solve the equation $F(s - a) = Y(s)$ to determine the function $F(s)$.
2. Use the identifications from the right side to compute the components needed for the left side.
3. Write the results of the computation.

Example 2
To compute $y(t)$ for

$$Y(s) = \frac{1}{s^2 + 2s + 1},$$

we first have to identify the kind of quadratic function in the denominator of Y. If it has real roots, then we can factor it and use partial fraction decomposition. If not, then we might have a repeated root or no real roots. This one has a repeated root:

$$Y(s) = \frac{1}{(s+1)^2}.$$

This function might or might not correspond to a table entry, depending on how large your table is. We can invert it without a special table entry by using the first translation theorem. To do this, we need to think of the right side as a function of $s + 1$ rather than s; that is, we take $a = -1$ and define a new function F by

$$F(s+1) = Y(s) = \frac{1}{(s+1)^2}. \tag{2}$$

This is progress, but we can only use transform inversion formulas for functions $F(s)$. To obtain $F(s)$ from $F(s+1)$, we need to do a substitution in which we replace $s+1$ with s. This can be very confusing, so let's make it more formal. Define a new variable $\bar{s}$ by

$$\bar{s} = s + 1.$$

Applying this substitution formula to (2), we have

$$F(\bar{s}) = \frac{1}{\bar{s}^2}.$$

The symbol used in a function equation can be anything, so we can drop the bars. Thus, the result for step 1 of the procedure is

$$a = -1, \qquad F(s) = \frac{1}{s^2}.$$

For step 2, we already have a and invert the function F to get

$$f(t) = t.$$

Then we can finish with

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s+1)\} = e^{-t} f(t) = te^{-t}.$$

◇

Example 3
To compute $y(t)$ for

$$Y(s) = \frac{s}{s^2 + 2s + 5},$$

we note that the denominator has no real roots. This means that we need to complete the square:

$$s^2 + 2s + 5 = (s^2 + 2s + 1) + 4 = (s + 1)^2 + 4,$$

so

$$Y(s) = \frac{s}{(s + 1)^2 + 4}.$$

We can invert this without a special table entry by using the first translation theorem. To do this, we need to think of the right side as a function of $s + 1$ rather than s; that is, we take $a = -1$ and define a new function F by

$$F(s + 1) = Y(s) = \frac{s}{(s + 1)^2 + 4}. \tag{3}$$

As in Example 2, define a new variable $\bar{s}$ by

$$\bar{s} = s + 1.$$

Note that we could equivalently write

$$s = \bar{s} - 1.$$

Applying each of these substitution formulas to (3) where it is most convenient, we have

$$F(\bar{s}) = \frac{\bar{s} - 1}{\bar{s}^2 + 4}.$$

Thus, the result for step 1 of the procedure is

$$a = -1, \qquad F(s) = \frac{s - 1}{s^2 + 4}.$$

For step 2, we already have a and invert the function F to get

$$f(t) = \cos 2t - \frac{1}{2} \sin 2t.$$

Then we can finish with

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s + 1)\} = e^{-t} f(t) = e^{-t} \left(\cos 2t - \frac{1}{2} \sin 2t \right).$$

$\diamond$