# Laplace's Method

## The Basic Method

Suppose we want to approximate the integral

$$I(x) = \int_0^\infty \frac{1}{1+t}\, e^{-xt}\, dt$$

as $x \to \infty$. Laplace's method is a very powerful tool for problems such as this. The idea is that $e^{-xt}$ is a *controlling factor* for the integrand, which means that its changes are exponentially fast compared to any other factors that may be present. If we plot $e^{-xt}$ against $t$ for large $x$, we get a graph that is finite at $t = 0$, but rapidly becomes exponentially small. The importance of this can be seen by dividing the integral up into two parts:

$$I = I_1 + I_2, \qquad I_1 = \int_0^\epsilon \frac{1}{1+t}\, e^{-xt}\, dt, \quad I_2 = \int_\epsilon^\infty \frac{1}{1+t}\, e^{-xt}\, dt,$$

where for the moment we will think of $\epsilon$ as a fixed small constant. Now suppose we change variables in $I_2$ using $u = t - \epsilon$. Then

$$I_2 = \int_0^\infty \frac{1}{1+\epsilon+u}\, e^{-xu-\epsilon x}\, dt = e^{-\epsilon x} \int_0^\infty \frac{1}{1+\epsilon+u}\, e^{-xu}\, du.$$

Clearly the integral factor is asymptotic to the original integral $I$, so we have

$$I_2 \sim e^{-\epsilon x} I.$$

This simple result is very powerful for two reasons.

1. Suppose we get an asymptotic expansion for $I_1$ that goes in powers of $x$ (with negative powers, of course, since the limit as $x \to \infty$ of the integral is clearly 0). Then *all* of the terms in that expansion dominate the exponentially small integral $I_2$. Hence, $I_2$ can be completely ignored in any further approximations. We signify this by writing

   $$I \sim \int_0^\epsilon \frac{1}{1+t}\, e^{-xt}\, dt + \text{EST}, \quad x \to \infty, \quad \epsilon = O(1),$$

   where EST stands for "exponentially small terms."

2. Although we have assumed $\epsilon$ to be finite, this is not actually needed to justify the labeling of $I_2$ as exponentially small. All that is necessary is $\epsilon x \gg 1$, which allows for the possibility that $\epsilon$ is actually small as $x \to \infty$, as long as it isn't too small.

Taking everything together, we can conclude

$$I \sim \int_0^\epsilon \frac{1}{1+t}\, e^{-xt}\, dt + \text{EST}, \quad x \to \infty, \quad \epsilon \gg x^{-1}. \tag{1}$$

In requiring $\epsilon \gg x^{-1}$, we have left open the possibility $\epsilon \ll 1$, which we now assume. This allows us to use the asymptotic expansion

$$\frac{1}{1+t} \sim 1 - t + t^2 - t^3 + \cdots,$$

which yields

$$I \sim \int_0^{\epsilon} (1 - t + t^2 - t^3 + \cdots)\, e^{-xt}\, dt, \quad x \to \infty, \quad x^{-1} \ll \epsilon \ll 1. \tag{2}$$

The next step in the approximation is the rescaling of the integral. As it currently stands, we have exponentially rapid decay over a vanishingly small interval. If we stretch the integration variable, we can get algebraic decay on a non-vanishing interval. We simply choose the substitution

$$u = xt,$$

which changes the approximation to

$$I \sim \frac{1}{x} \int_0^{\epsilon x} \left(1 - \frac{u}{x} + \frac{u^2}{x^2} - \frac{u^3}{x^3} + \cdots\right) e^{-u}\, du,$$

or

$$I \sim \frac{1}{x} \int_0^{\epsilon x} e^{-u}\, du - \frac{1}{x^2} \int_0^{\epsilon x} u\, e^{-u}\, du + \frac{1}{x^3} \int_0^{\epsilon x} u^2\, e^{-u}\, du - \cdots. \tag{3}$$

We can now extend the upper limits of these integrals from $\epsilon x$, which is already large, to infinity. Doing so adds only exponentially small terms, so it makes no difference to the asymptotic expansion. Thus,

$$I \sim \frac{1}{x} \int_0^{\infty} e^{-u}\, du - \frac{1}{x^2} \int_0^{\infty} u\, e^{-u}\, du + \frac{1}{x^3} \int_0^{\infty} u^2\, e^{-u}\, du - \cdots. \tag{4}$$

The remaining integrals can all be calculated using the general formula

$$\int_0^{\infty} u^n e^{-u}\, du = n!,$$

with the result

$$I \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots, \quad x \to \infty, \quad x^{-1} \ll \epsilon \ll 1. \tag{5}$$

## More Generality

As a second example, consider

$$I(x) = \int_0^1 (\cosh t)\, e^{-xt^2}\, dt.$$

The exponential function is a controlling factor, so we can isolate the integral in the vicinity of the maximum at $t = 0$ by making sure we only go out to $\epsilon$ with $\epsilon^2 x \gg 1$:

$$I(x) \sim \int_0^{\epsilon} (\cosh t)\, e^{-xt^2}\, dt + \text{EST}, \quad x \to \infty, \quad x^{-1/2} \ll \epsilon \ll 1.$$

Using the asymptotic series

$$\cosh t = 1 + \frac{t^2}{2} + \frac{t^4}{4!} + \cdots$$

yields

$$I(x) \sim \int_0^{\epsilon} \left(1 + \frac{t^2}{2} + \frac{t^4}{4!} + \cdots\right) e^{-xt^2}\, dt, \quad x \to \infty, \quad x^{-1/2} \ll \epsilon \ll 1.$$

Rescaling with $u = xt^2$ yields

$$I(x) \sim \int_0^{\epsilon^2 x} \left(1 + \frac{u}{2x} + \frac{u^2}{24x^2} + \cdots\right) e^{-u} \frac{du}{2\sqrt{xu}},$$

or

$$I(x) \sim \frac{1}{2\sqrt{x}}\left[\int_0^{\epsilon^2 x} u^{-1/2}e^{-u}\,du + \frac{1}{2x}\int_0^{\epsilon^2 x} u^{1/2}e^{-u}\,du + \frac{1}{24x^2}\int_0^{\epsilon^2 x} u^{3/2}e^{-u}\,du + \cdots\right].$$

As before, we can raise the limits of integration to infinity with only an exponentially small error, and this allows us to express the integrals in terms of the Gamma function, which is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\,e^{-t}\,dt, \qquad x > 0,$$

with result

$$I(x) \sim \frac{1}{2\sqrt{x}}\left[\Gamma(1/2) + \frac{\Gamma(3/2)}{2x} + \frac{\Gamma(5/2)}{24x^2} + \cdots\right].$$

The needed Gamma function values are known:

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \qquad \Gamma(5/2) = \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi}, \qquad \Gamma(7/2) = \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi}.$$

Hence we obtain the final expansion

$$I(x) \sim \frac{\sqrt{\pi}}{2\sqrt{x}}\left[1 + \frac{1}{4x} + \frac{1}{32x^2} + \cdots\right].$$

## Summary

We have developed a procedure for doing all integrals of the form

$$I(x) = \int_0^b f(t)\,e^{-cxt^p}\,dt, \qquad b, c, p > 0, \quad x \to \infty \tag{6}$$

1. Change the upper limit to $\epsilon$, which introduces only an exponentially small error if $\epsilon$ is not too small.
2. Expand $f(t)$ about $t = 0$, which is possible as long as $\epsilon \ll 1$.
3. Rescale the integral using $u = cxt^p$.
4. Change the new upper limits to $\infty$, again introducing an exponentially small error as long as $\epsilon$ is not too small.
5. Rewrite the integrals in terms of the Gamma function and evaluate if possible.

## Stirling's Formula

Laplace's method can be extended to much more complicated integrals of the form[1]

$$I(x) = \int_a^b f(t)\,e^{-x\phi(t)}\,dt, \qquad \phi(0) < \phi(t) \text{ for } t \neq 0, \qquad a \leq 0 \tag{7}$$

or even the still more general form

$$I(x) = \int_a^b f(t)\,e^{\psi(t,x)}\,dt. \tag{8}$$

[1] Without loss of generality, we can assume $\phi(0) = 0$.

An example of the latter is the asymptotic expansion of the Gamma function, which is used in Stirling's formula for approximating $n!$ for large $n$. Recall that the Gamma function can be defined as

$$\Gamma(x+1) = \int_0^\infty t^x \, e^{-t} \, dt. \tag{9}$$

The key to doing this expansion is to gradually convert the Gamma function to problems of type (8), then (7), and then (6), after which we can apply Laplace's method with a small modification.

1. The first difficulty with (9) is that the controlling factor requires a combination of $t^x$, which has its maximum as $t \to \infty$, and $e^{-t}$, which is exponentially small as $t \to \infty$. The solution to this difficulty is to rewrite $t^x$ as $e^{x \ln t}$. Thus we have an integral of form (8):

$$\Gamma(x+1) = \int_0^\infty e^{-t + x \ln t} \, dt.$$

2. The integrand has its maximum at the maximum of the exponential function. Since $-t + x \ln t$ approaches $-\infty$ at both ends of the integration interval, its maximum is when the derivative is 0, which is at $t = x$. The substitution $t = xs$ would give us an integral with the maximum at $s = 1$, but it is better to use $t = x(1+s)$ so that the maximum is at $s = 0$. With this substitution, we get

$$\Gamma(x+1) = \int_{-1}^\infty e^{(-x-xs)+x[\ln x + x \ln(1+s)])} x \, ds,$$

or

$$\left(\frac{e}{x}\right)^x \frac{\Gamma(x+1)}{x} = I(x) \equiv \int_{-1}^\infty e^{-x\phi(s)} \, ds, \qquad \phi(s) = s - \ln(1+s). \tag{10}$$

$I(x)$ is now in the form (7), with $\phi$ having its global minimum of 0 at $s = 0$.

3. The difficulty with form (7) as compared to form (6) is that the controlling factor is not a power function. Given that the integral is dominated by the local behavior near $s = 0$, we can obtain a power function controlling factor by using a local approximation for $\phi$. From the basic asymptotic series

$$\ln(1+s) \sim s - \frac{1}{2}s^2 + \frac{1}{3}s^3 - \frac{1}{4}s^4 + \cdots,$$

we obtain

$$\phi(s) \sim \frac{1}{2}s^2 - \frac{1}{3}s^3 + O(s^4).$$

Hence, we have

$$I(x) \sim \int_{-1}^\infty e^{-\frac{1}{2}xs^2 + \frac{1}{3}xs^3 + O(xs^4)} \, ds,$$

which we can write as

$$I(x) \sim \int_{-1}^\infty e^{\frac{1}{3}xs^3 + O(xs^4)} \, e^{-\frac{1}{2}xs^2} \, ds \tag{11}$$

This result is in the form (6), with the first exponential function as $f(s)$ and the second one as the controlling factor.

Equation (11) has the correct form for application of Laplace's method as given in the summary, with a minor correction needed because of the appearance of $x$ in $f(s)$. We want to use the expansion

$$e^{\frac{1}{3}xs^3+O(xs^4)} \sim 1+\left[\frac{1}{3}xs^3+O(xs^4)\right]+O(x^2s^6),$$

which requires that $xs^3 \ll 1$. Hence, we must restrict the interval of the integral to $(-\epsilon,\epsilon)$ where $\epsilon^3 x \ll 1$, rather than $\epsilon \ll 1$ as in the usual case. However, we cannot make $\epsilon$ too small because we want the error in reducing the interval of integration to be exponentially small. This requires $\epsilon^2 x \gg 1$. There is an acceptable range of $\epsilon$ because we found the correct controlling factor. Hence, we have

$$I(x) \sim \int_{-\epsilon}^{\epsilon}\left[1+\frac{1}{3}xs^3+O(xs^4)+O(x^2s^6)\right]e^{-\frac{1}{2}xs^2}\,ds.$$

Having an integration interval that includes negative $s$ requires a little extra care. Note that the first term is even and the second odd; hence, we can double the integral from 0 to $\epsilon$ for the first and set the second to 0:

$$I(x) \sim 2\int_{0}^{\epsilon}\left[1+O(xs^4)+O(x^2s^6)\right]e^{-\frac{1}{2}xs^2}\,ds. \tag{12}$$

Next, we rescale by defining $u=\frac{1}{2}xs^2$, which yields

$$I(x) \sim 2\int_{0}^{\frac{1}{2}x\epsilon^2}\left[1+O\left(\frac{1}{x}\right)\right]e^{-u}\,\frac{du}{\sqrt{2xu}}.$$

Now we can extend the limits of integration to infinity as usual to get

$$I(x) \sim \sqrt{\frac{2}{x}}\int_{-\infty}^{\infty}\left[u^{-1/2}+O\left(\frac{1}{x}\right)\right]e^{-u}\,du \sim \sqrt{\frac{2}{x}}\,\Gamma\left(\frac{1}{2}\right) \sim \sqrt{\frac{2\pi}{x}}.$$

We therefore obtain the leading order approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x. \tag{13}$$

Evaluated at an integer value $x=n$, we obtain Stirling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{14}$$

This result is remarkably accurate, with error less than 1% for $n \geq 9$. With careful expansion to get the second term in the approximation, we can show that the relative error is approximately $1/12n$.