OVERVIEW OF THE LAPLACE TRANSFORM METHOD

The Laplace transform provides an alternate method for solving the same problems that can be solved by undetermined coefficients. It has its advantages and disadvantages. The disadvantage is that it is usually more complicated than the usual method. The advantage is that it is much better for solving problems with discontinuous or impulsive forcing. The following steps are needed to solve a differential equation of the form

$$Ly = g$$

with initial conditions, where L is a linear differential operator with constant coefficients and g is a generalized exponential function; that is, a function of the sort that allow the method of undetermined coefficients. The method relies heavily on the transform formula

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0), \tag{1}$$

which converts derivatives in t into algebraic relations in s.

The method consists of these steps:

1. Define the unknown function $Y(s) = \mathcal{L}\{y(t)\}$. Notation systems are not universal, so it is necessary to define any notation that is not part of the original problem.

2. Apply the Laplace transform to the differential equation. Formula (1) can be applied to second-order derivatives by substituting y' for f, and higher-order derivatives similarly. Because of (1), the initial conditions in t space are incorporated into the transformed equation in s space. The result of applying the transform is a linear algebraic equation for the unknown function $Y(s)$.

3. Solve the algebraic equation for $Y(s)$. One should be flexible in the amount of algebraic simplification. Sometimes it is better to combine terms by adding rational functions, while other times it is better to keep them separate.

4. Recover $y(t)$ from its transform $Y(s)$. The term "recover" is chosen deliberately here. There is a calculation that can be done to compute inverse Laplace transforms, but it requires a contour integral in the complex plane. Instead, the best we can do is manipulate the formula for $Y(s)$ so as to take advantage of entries in a table of Laplace transforms. Often this is done by a partial fraction decomposition.