

# Introduction to Mathematical Models in Epidemiology

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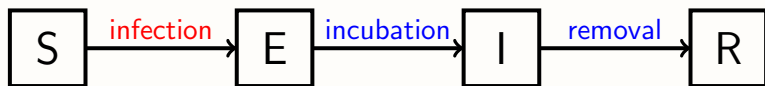
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# 1. Class structure

- ▶ Individuals in a population are divided into classes. These can vary from one model to another. Examples:
  - **S**: *Susceptible* – can be infected
  - **E**: *Exposed* – infected but not infectious
  - **I**: *Infectious* – can transmit the disease to susceptibles
  - **R**: *Removed* – no longer infectious
- ▶ Sometimes the names are misleading.
  - '*Exposed*' should be '*Latent*'
  - *Removed* includes people who are still sick and may include people who are deceased
- ▶ Models are designated by the class structure: SIR, SIS, SEIR, SEAIR, SEAIRHD etc

## 2. Processes

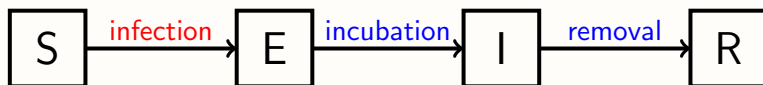
- ▶ Processes move individuals from one class to another.
  - Some models have processes that bring individuals into or out of the system.
- ▶ Example: Basic SEIR model



- Rate of change of  $S$  is  $-\text{infection}$
- Rate of change of  $E$  is  $\text{infection} - \text{incubation}$
- Rate of change of  $I$  is  $\text{incubation} - \text{removal}$
- Rate of change of  $R$  is  $\text{removal}$

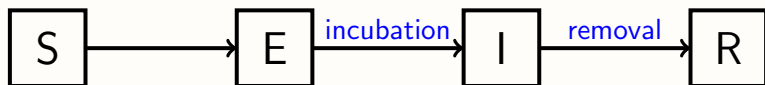
## 2. Processes – Two Types

- ▶ Processes are either **transmissions** or **transitions**.



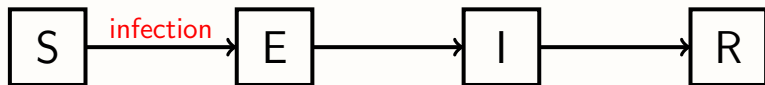
- **Transmissions** require interaction with another class.
  - Susceptibles are infected by Infectives.
- **Transitions** happen without any interaction.
  - Incubation of Latent individuals and removal of Infectious individuals happen spontaneously.

## 2.1 Processes – Transitions



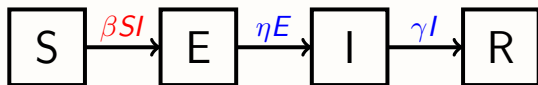
- ▶ Transition rates are (assumed to be) proportional to the **leaving** class
  - incubation rate = constant \*  $E = \eta E$
  - removal rate = constant \*  $I = \gamma I$
- ▶ **Rate constants are reciprocals of average time in class.**
  - Average removal time 10 days  $\rightarrow \gamma = 0.1$

## 2.2 Processes – Transmissions



- ▶ Transmission rates are proportional to the **leaving** class size
  - infection rate = force of infection \*  $S = \lambda S$
- ▶ The force of infection is proportional to the sum of the **transmitting** classes (just I for SEIR)
  - force of infection = constant \*  $I = \beta I$
- ▶ The infection rate is  $\beta I * S = \beta SI$

## 2.3 Summary – SEIR epidemic model



$$S' = -\beta SI$$

$$E' = \beta SI - \eta E$$

$$I' = \eta E - \gamma I$$

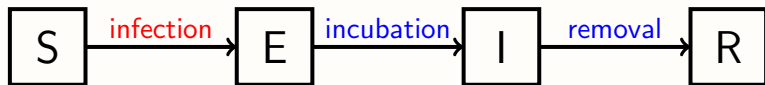
$$R' = \gamma I$$

- ▶ Let  $N = S + E + I + R$ . Then  $N' = 0$ , so  $N$  is constant.
  - The  $R$  equation is not needed because  $R = N - S - E - I$ .



## 3.1 Model Type – Epidemic

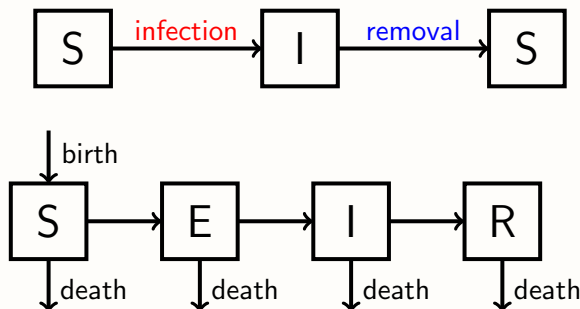
- ▶ **Epidemic** models have no means for replenishment of susceptibles.
  - These do not have births or natural deaths, so they are intended only for short time intervals (up to a few years).



- ▶ Including deceased individuals as 'Removed' makes the total population constant, which simplifies the model.

## 3.2 Model Type – Endemic

- ▶ **Endemic** models have some means for replenishment of susceptibles.
  - The focus of analysis is on determining long term behavior.



## 4. Basic reproductive number

- ▶ **Basic reproductive number  $\mathcal{R}_0$ :**  
the average number of secondary infections caused by one primary infective in a fully susceptible population.
  - $\mathcal{R}_0 > 1$  is needed to start an epidemic.
- ▶ The total number is the average rate times the average time.
- ▶ Calculation of average transmission rate:
  - Recall that the **transmission rate** is  $\beta SI$
  - Transmission rate **per infective**:  $\beta S$
  - Rate per infective in a **fully-susceptible population**:  $\beta N$

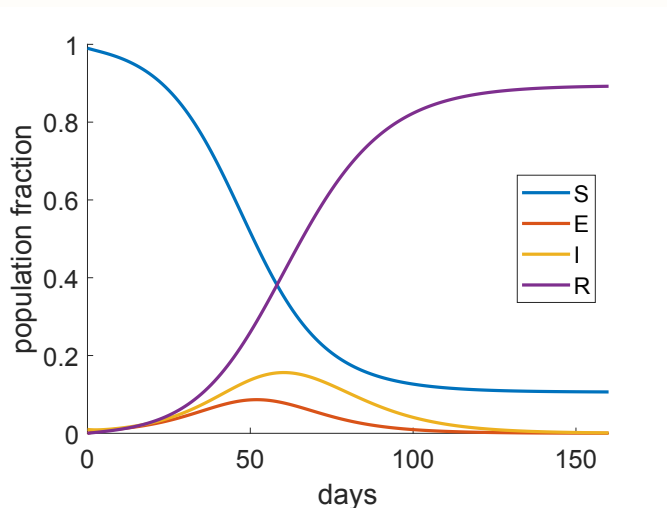
## 4. Basic reproductive number

- ▶ **Basic reproductive number**  $\mathcal{R}_0$ :  
transmission rate per infective in a fully susceptible population multiplied by average time in the Infectious class.
- ▶ Average transmission rate:  $\beta N$
- ▶ Calculation of average time:
  - Recall that the **removal rate** is  $\gamma I$ .
  - The average time is  $1/\gamma$ .

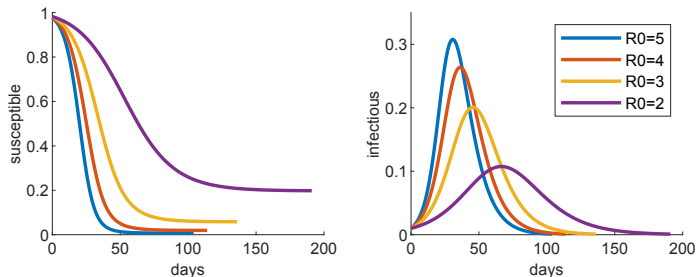
$$\mathcal{R}_0 = \beta N \cdot \frac{1}{\gamma} = \frac{\beta N}{\gamma}.$$

- ▶ Other diseases (like COVID-19) can be more complicated.

## 5 Behavior of Epidemic Models – Typical history



## 5. Behavior of Epidemic Models – Importance of $\mathcal{R}_0$



- ▶ Conjectures based on simulations:
  1. Not everyone gets the disease.
    - Larger  $\mathcal{R}_0$  means fewer escape.
  2. The epidemic ends (in theory) with  $I = 0$ .
    - This is because we ignored births.

## 5.1 Epidemic Models – Not everyone gets the disease

$$\frac{dR}{dS} = \frac{R'}{S'} = \frac{\gamma I}{-\beta SI} = -\frac{\gamma}{\beta} \frac{1}{S}.$$

1. Integrate this equation using the fact that at time 0 we have  $S = S(0)$  and  $R = R(0)$ .
2. Let  $s = S/N$ ,  $r = R/N$ ,  $s_0 = S(0)/N$ ,  $r_0 = R(0)/N$ .  
Rearrange the solution from Question 1 to get

$$\ln \frac{s_0}{s} = \mathcal{R}_0(r - r_0). \quad (1)$$

3. Solve for  $s$  and use the result to show that  $s \geq s_0 e^{-\mathcal{R}_0} > 0$ .
- The fraction of susceptibles is always decreasing, but never 0.

## 5.2 Epidemic Models – The epidemic ends with $I = 0$

4. Rewrite the  $S'$  equation as  $I dt = -\beta^{-1} dS/S$  and integrate from time 0 to time  $\infty$ .
  - Just leave the integral on the left side because you don't have a formula for  $I$
  - Do the integral on the right side. You may assume  $\lim_{t \rightarrow \infty} S = S_\infty > 0$ . (Why?)
5. You have just shown that  $\int_0^\infty I dt$  is a finite number. What can you conclude about  $\lim_{t \rightarrow \infty} I$ ?



## 5.3 Epidemic Models – Final size relation

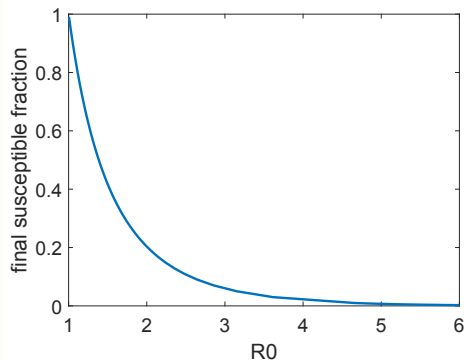
6. You now know that  $I \rightarrow 0$ . Explain why that means  $E \rightarrow 0$  also. Conclude that  $s_\infty + r_\infty = 1$ .
7. Use the result of Question 6 with Equation (1) to get the final size relation

$$\ln \frac{s_0}{s_\infty} = \mathcal{R}_0(1 - r_0 - s_\infty). \quad (2)$$

- This result can be used to estimate  $\mathcal{R}_0$  for an epidemic that is finished (assuming no interventions).
8. Explain why  $1 - r_0 - s_\infty$  is the fraction of people who have the disease at some point in the epidemic.

## 5.3 Epidemic Models – Final size relation

- ▶ Assume no initial immunity.



- ▶  $R_0$  for COVID-19 is about 5.7.