Mathematical Modeling Via Multiple Representations

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Outline

Framework for Modeling via Representations

Resource-Limited Growth: An Example of Mathematical Modeling via Representations

- An Experiential Representation
- A Numerical Representation
- Three Visual Representations
- A Verbal Representation
- A Symbolic Representation
- Computer Implementation

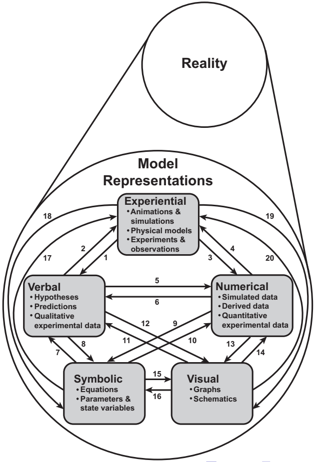
Models and Modeling

Definition

A **model** is a simplified, abstract or concrete representation of relationships and/or processes in the real world, constructed for some purpose.

"Rule-of-Five" Model Representations:

- Verbal
- Visual
- Symbolic
- Numerical
- Experiential



Models and Modeling

Definition

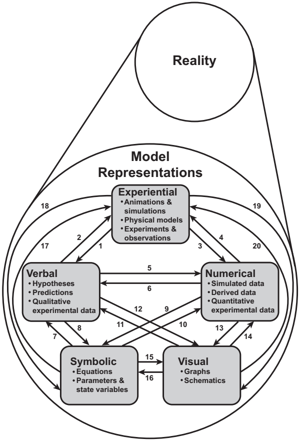
Modeling is the act of moving between representations/models (arrow), checking model with reality and/or revising model.

Modeling activities

- Moving between representations/models (arrow)
- Checking model with reality
- Creating and revising model

Modeling process

- A set of modeling activities from reality to "good enough."
- Reality & experiential are key!
- Defined to include approaches like data science



The Challenge

Yogi Berra:

"**In theory, there is no difference between theory and practice. In practice, there is.**"

- So how do we implement this theory in the classroom?
  - With classroom projects that 'model' modeling with a directed sequence of modeling activities.

Modeling Activity Objectives

1. Illustrate the true nature of science
   - Theory without observation (natural and/or experimental) is mere speculation.
   - Observation without theory is just a collection of data.
   - **Scientific progress is due to the combination of theory and observation.**
2. Provide a rich experience of mathematical modeling
   - Use all five representations and make many connections.
3. Develop a sophisticated view of models in biology
   - **Models are not depictions of reality; they are abstractions that under best circumstances have explanatory value.**
4. Teach the principles of density-dependent growth

Approach

- The real world is complicated.
  - Hard to collect data.
  - Many confounding complications.
  - Demographic stochasticity.
- The nature of science is more easily discovered using real data from an artificial world. (e.g., C.S. Holling, 1959)
  - Easy to collect data.
  - Based on simple mechanisms.
  - Must have demographic stochasticity!

Experiential – Materials and Setup

			○	○			
		○	X	X	○		
		○	X	X	○		
			○	○			

X — square is occupied

○ — square is available

Time	Pop.	Prev.	Incr.
0	4	NA	NA
1		4	
2			
3			
4			
5			
⋮	⋮	⋮	⋮
20			

Experiential – Simulation Rules

		○	○̸	⊙			
	○	X	X	X	○		
		○	X	X	X	○	
			○	X	○		
				○			

X — square is occupied
○ — square is available

1. For each available square:
   a. Roll one die for each adjacent occupied square.
   b. If any die is 5 or 6, mark the square with a slash (/).
2. Change the slashes into X's. Record population.
3. Mark new available squares with a circle (○).

- Stop when nearly all squares are occupied.

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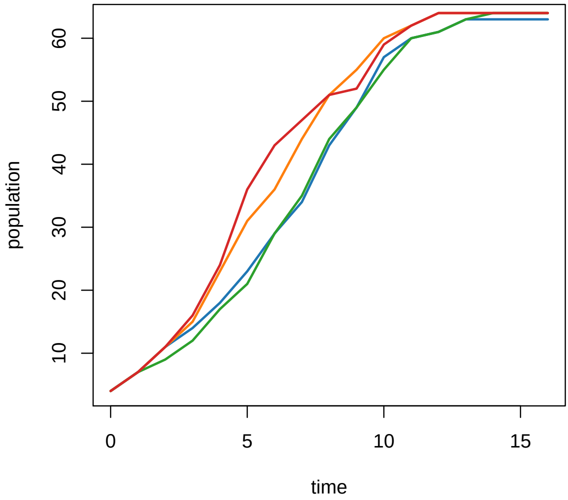
Numerical – Lots of Data

Time	Pop.	Prev.	Incr.
0	4	NA	NA
1	7	4	3
2	11	7	4
3	15	11	4
4	23	15	8
5	31	23	8
6	36	31	5
7	44	36	8
8	51	44	7
9	55	51	4
10	60	55	5
11	62	60	2

Time	Pop.	Prev.	Incr.
0	4	NA	NA
1	7	4	3
2	11	7	4
3	14	11	3
4	18	14	4
5	23	18	5
6	29	23	6
7	34	29	5
8	43	34	9
9	49	43	6
10	57	49	8
11	60	57	3
12	61	60	1

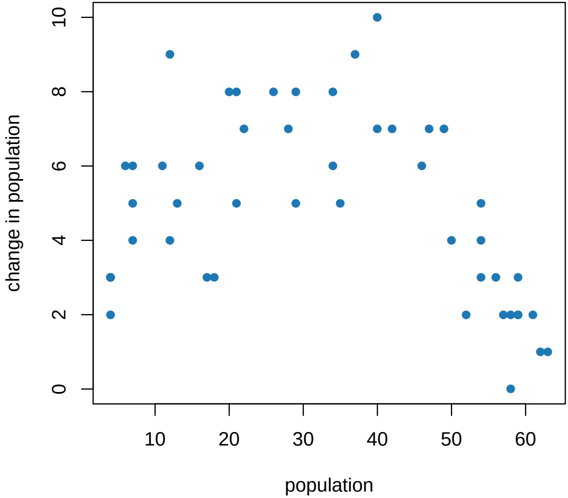
Visual – Population Graphs



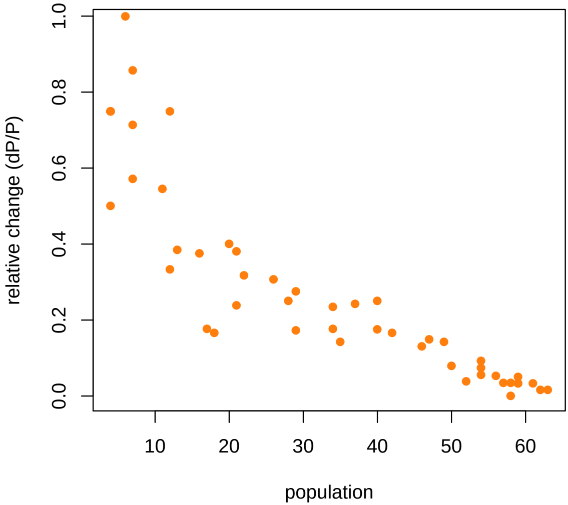
Visual – What Else?

- Can we think of other, possibly better, ways to plot the data?
  - Notice that slopes of the orange and green lines are the same for the same populations?
- How about plotting population change vs population?
- Other ideas?

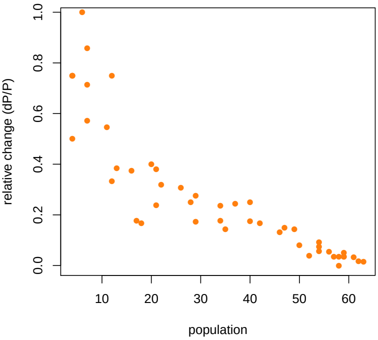
Visual – Change vs Population



Visual – Relative Change vs Population



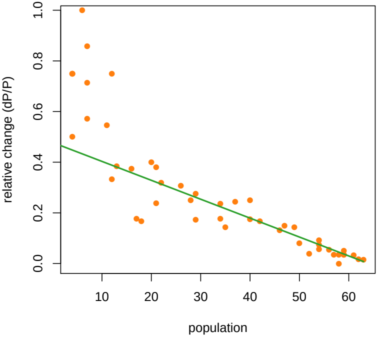
Verbal – An Empirical Hypothesis



- Ignore the demographic stochasticity (scatter).
  - Is there a signal hiding under the noise?

- **Maybe the relative change is a linear function of the population, reaching 0 when the space is full.**

Verbal – An Empirical Hypothesis



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  - Is there a signal hiding under the noise?

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Symbolic – Dynamic Equation

- Discrete

$$\frac{\Delta P}{P} = r\left(1 - \frac{P}{K}\right)$$

or

$$\Delta P = rP\left(1 - \frac{P}{K}\right)$$

- Continuous

$$\frac{dP/dt}{P} = r\left(1 - \frac{P}{K}\right)$$

or

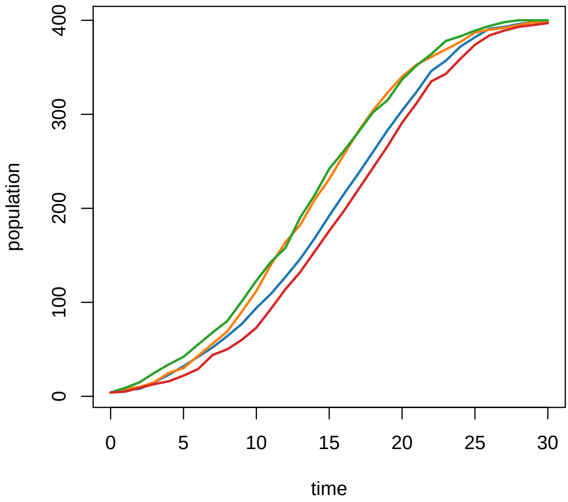
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

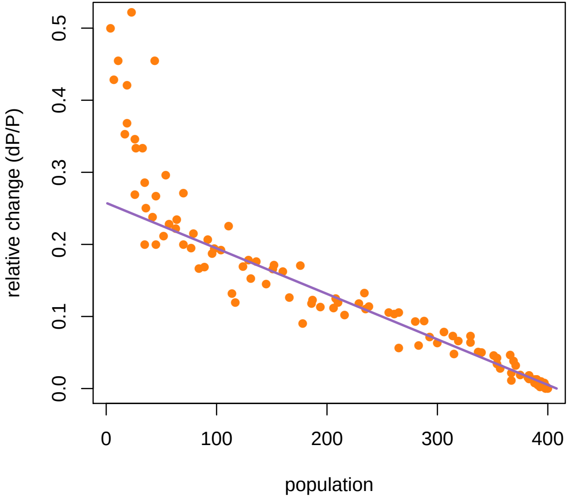
We need numerical implementation of a statistical method to fit r to the data (given K).

(See Ledder, Coll Math J, 47 (109), 2017.)

Improvements

- So far, we're working with very limited data (like real ecologists) and a very simple setting. With a computer simulation, we can add detail and collect much more data quickly.
- PopGrowth.R (GL) and LogGrowth.nlogo (M.D. LaMar)
  - arena size, $8 \leq s \leq 50$, best at about 20
  - birth probability, $0 < b < 1$, best at 0.1 to 0.8
  - death probability, must be $0 \leq d < b$, best at 0 to b/4
  - number of trials, 1 to 4
  - starting setup: center or edge
  - curve fit options: none, r only, r and K


population
0
100
200
300
400
0
5
10
15
20
25
30
time


relative change (dP/P)
0.0
0.1
0.2
0.3
0.4
0.5
0
100
200
300
400
population

Wrapping it Up

Remember our objectives:

3. Develop a sophisticated view of models in biology
   - Models are not depictions of reality; they are abstractions that under best circumstances have explanatory value.

- In this study, we developed a model for a synthetic system.
  - We actually know the true biological processes, which are completely different from the model.
- But the model does a great job of predicting the results.