PARTIAL FRACTION DECOMPOSITION

Partial fraction decomposition is the process of splitting rational functions into individual terms with simple denominators. The method consists of rules for determining the structures of the terms along with techniques for determining the coefficients. Some techniques are more efficient than others, but not all authors present the most efficient techniques.

Determining the Structure

The rules for determining the structure of a partial fraction decomposition are easier to understand if you are aware of a basic principle:

- There must be n terms in the structure corresponding to each factor of degree n in the denominator of the rational function.

All polynomials can in principle be factored into linear and irreducible quadratic factors, so we just need two rules, one to deal with each type.

1. A factor $(s-a)^m$ in the denominator of the rational function gives rise to m terms in the decomposition structure. These are
$$\frac{c_1}{(s-a)}, \quad \frac{c_2}{(s-a)^2}, \quad \cdots \quad \frac{c_m}{(s-a)^m}.$$
2. A factor $(s^2+bs+d)^m$ has a total degree of $2m$; hence, it gives rise to $2m$ terms. These are
$$\frac{c_1 s + c_2}{(s^2+bs+d)}, \quad \frac{c_3 s + c_4}{(s^2+bs+d)^2}, \quad \cdots \quad \frac{c_{2m-1}s + c_{2m}}{(s^2+bs+d)^m}.$$

It is usually best to avoid notation containing characters that provide no information. If we use c_1, c_2, and so on for the coefficients, the information is contained in the subscript and not the base c. It is better to just use different letters for each coefficient and leave out subscripts.

As an example of the partial fraction decomposition structure, suppose

$$Y(s) = \frac{-s^3+s+20}{(s-1)^2(s^2+4)}.$$

The obvious choice for the structure is

$$\frac{-s^3+s+20}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4}.$$

There is one slight change we might prefer if the rational function is a Laplace transform that we wish to invert, as we assume here. If we use the structure

$$Y(s) = \frac{-s^3+s+20}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs}{s^2+4} + \frac{2D}{s^2+4},$$

then we can see from a table of Laplace transforms that the final result will be

$$y(t) = Ae^t + Bte^t + C\cos 2t + D\sin 2t.$$

The extra factor of 2 in the last term of the structure serves the purpose of allowing identification of the coefficients in the decomposition with the coefficients in the inverse transform.

Evaluating the Coefficients

To find the coefficients for $y(t)$, we have to solve the equation

$$\frac{-s^3+s+20}{(s-1)^2(s^2+4)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs}{s^2+4}+\frac{2D}{s^2+4}.$$

The natural thing to do at this point is to add the fractions on the right side; however, it is less work to multiply by the common denominator to clear the fractions:

$$-s^3+s+20=A(s-1)(s^2+4)+B(s^2+4)+Cs(s-1)^2+2D(s-1)^2. \tag{1}$$

Note that each term has to be multiplied by each factor of the denominator. Some of these cancel the denominator factors in the term and others have to be kept; for example, one factor of $s-1$ clears the denominator of the first term, while the other factor of $s-1$ and the factor of s^2+4 are retained.

From here there are a number of paths, but the key idea is that (1) is not to be solved for s; rather, it has to be solved for the coefficients in a way that works for all possible values of s. We could multiply out the right hand side and combine like terms. Then we would have a cubic polynomial on each side of the equation, and their coefficients would have to be equal.

It is more efficient to follow a particular strategy to find the coefficients using multiple ideas. The main idea is to look for values to plug in for s that would make some of the terms disappear. If we evaluate at $s=1$, we have

$$-1+1+20=0+5B+0+0,$$

from which we obtain $B=4$ with very little work. Although there is no number as good as 1, $s=0$ has some advantages. This gives us teh equation

$$20=-4A+4B+2D=-4A+16+2D,$$

which we can write as

$$D=2A+2. \tag{2}$$

After picking convenient values of s, we still need to compare polynomial coefficients, but we don't need to compute all of them. Without actually writing down the polynomial on the right side, it should not be too hard to see that the s^3 coefficient will be $A+C$; hence, comparing s^3 coefficients gives us

$$-1=A+C,$$

or $C=-A-1$. If we are especially alert, we might notice that C and D share a simple relationship inherited from their formulas in terms of A. This makes C a better choice for unknown than A, and we now have

$$D=-2C, \qquad A=-C-1. \tag{3}$$

To finish, there is nothing better than collecting the s^2 coefficients, which takes a little more computation. The result of this calculation is that the s^2 coefficient on the right side is $-A+B-2C+2D$; thus,

$$0=-A+B-2C+2D=-(-C-1)+4-2C+2(-2C)=5-5C.$$

At long last, we now have $C=1$, from which we get $D=-2C=-2$ and $A=-C-1=-2$. Thus, the final result is

$$y=(4t-2)e^t+\cos 2t-2\sin 2t.$$

In summary, the recommended procedure for finding the coefficients is this:

1. Multiply by the common denominator to clear all fractions. Do not multiply out the polynomial from the partial fraction structure.

2. Evaluate the polynomial equation at each of the values that corresponds to a linear factor $(s - a)$. Each of these evaluations gives you a coefficient value.

3. Evaluate the equation at $s = 0$. This does not usually give a coefficient value, but it does give an equation that can be solved for one coefficient in terms of others.

4. Working from the highest power, equate the polynomial coefficients, preferably without multiplying out the whole polynomial. The number of equations needed is given by the total number of coefficients minus the number of s values previously used to make coefficient equations; in the example, we had 4 coefficients and used two values of s, so we needed the first two polynomial coefficients.