PIECEWISE CONTINUOUS FUNCTIONS

All of the problems you have solved with Laplace transforms up to this point would probably have been easier to solve using the direct method of undetermined coefficients. The real value of the Laplace transform method is its ability to manage problems $L = g$ where the forcing function is piecewise continuous. In the direct method, the only way to do such problems is to find separate solutions on each interval of continuity and then find the extra coefficients in the general solutions by connecting the pieces. With the Laplace transform, we can take advantage of a key fact: functions that are piecewise continuous in t space correspond to continuous functions in s space. This means that no matter how many times the forcing function suddenly changes, we will only have to do one Laplace transform solution. Given this knowledge, it is easy to see why the Laplace transform method is standard in electrical engineering applications, where power sources can be turned on an off with switches.

The Heaviside function and switches

If we have a problem with piecewise-continuous forcing, the first step is to write the piecewise continuous function in terms of a single formula. This requires a function called the unit step function ($\mathcal{U}$) by some authors and the Heaviside function (H) by others (after Oliver Heaviside, one of the pioneers of the Laplace transform method). Outside of differential equations textbooks, the Heaviside name and notation are more common, so we choose them here.

The Heaviside function has a very simple definition:

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} . \tag{1}$$

It functions as a switch because multiplying any function by it turns that function on at time 0 while ignoring it for times less than 0.

$$f(t)H(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases} . \tag{2}$$

The Heaviside function can also turn a function off by adding its negative to it starting at time 0.

$$f(t) - f(t)H(t) = \begin{cases} f(t), & t < 0 \\ 0, & t \geq 0 \end{cases} . \tag{3}$$

Finally, we can piece together two different functions by turning one of them off and the other on.

$$f_1(t) + [f_2(t) - f_1(t)]H(t) = \begin{cases} f_1(t), & t < 0 \\ f_2(t), & t \geq 0 \end{cases} . \tag{4}$$

The second term in the sum is absent when $t < 0$ and present, yielding f_2, when $t \geq 0$.

Initial value problems begin at time 0 and are undefined prior to that, so a switch that turns on at time 0 is always 1. We don't need the Heaviside function for that case. But we can easily shift the Heaviside function to switch on at any time $a > 0$ using

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} . \tag{5}$$

Using the definition of the Laplace transform, it is not difficult to determine that the Laplace transform of the Heaviside function is given by

$$\mathcal{L}\left\{H(t-a)\right\} = \frac{e^{-as}}{s}. \tag{6}$$

An initial value problem with a switch

Suppose we want to solve the initial value problem

$$y'' + 4y = \begin{cases} 0, & t < \pi \\ 12, & t \geq \pi \end{cases} = 12H(t-\pi), \qquad y(0) = 1, \quad y'(0) = 0.$$

To solve this problem, we first define $Y = \mathcal{L}\{y\}$. The transforms of the derivatives are

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = sY - 1$$

and

$$\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s^2 Y - s.$$

Hence, the problem in s space is

$$s^2 Y - s + 4Y = 12\frac{e^{-\pi s}}{s}.$$

From here, we can simplify the equation and solve for Y, with the result

$$Y(s) = \frac{s}{s^2+4} + \frac{12}{s(s^2+4)}e^{-\pi s}.$$

This has been very easy so far, but we still need to invert the transform. The first term is just the Laplace transform of the cosine function. We'll have to learn the second translation theorem to get any farther. For now, the best we can say is

$$y(t) = \cos 2t + \mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)}e^{-\pi s}\right\}.$$

Well, we can't expect the whole problem to be easy!