Scaling

Proper scaling is an art. In my research work, I usually change my choices of scales and dimensionless parameters several times. Nevertheless, there are some basic principles that can be used to identify good choices. We will illustrate these using examples.

1. A Continuously-Stirred Tank Reactor

Consider a tank with constant volume V that contains a solution of some chemical reactant with time-dependent concentration (mass/volume) $C(T)$.[1] The chemical undergoes a reaction that decreases the amount at a rate proportional to the concentration, with rate constant k. Meanwhile, fresh solution with concentration C_i enters the reactor at a rate of q (volume/time) and the well-mixed solution is drawn out of the reactor with the same flow rate so as to maintain the constant volume. The net change in reactant mass will be the sum of the rates of change due to each of the processes of inflow, outflow, and reaction. With each term having dimension mass/time, and assuming the initial concentration is the same as that of the input stream, we have

$$V\frac{dC}{dT} = qC_i - qC - VkC, \qquad C(0) = C_i. \tag{1}$$

The full process of scaling consists of three stages, usually in order:

1. Choose the scales.
2. Replace the dimensional variables with the dimensionless variables.
3. Identify a set of dimensionless parameters that emerge from the replacement of variables.

It is sometimes best to use arbitrary scales X_r for a variable X so as to defer making the actual choice until after the nondimensionalization of step 2. This adds a lot of algebraic complexity that is usually unnecessary, so it should be avoided if possible.

Direct scaling for C: The direct method of choosing scales is to identify upper bounds or representative values of a variable. Here, the differential equation shows that C is decreasing when $C = C_i$; hence, C_i is an upper bound for C. This might not be a good choice if C were always far less than C_i, but here C_i also serves as the initial value. We will therefore define a scaled concentration with

$$C = C_i c. \tag{2}$$

Indirect scaling for T: Time has no upper bound, nor is there an obvious representative value. There are actually two characteristic values of time. The mean residence time t_R for the chemical reaction is obtained by approximating the original differential equation through a combination of discretization and a dominant balance in which the flow is considered unimportant. This yields

$$V\frac{\Delta C}{\Delta T} \approx VkC; \tag{3}$$

using reference values we have

$$\frac{C_i}{t_R} = kC_i,$$

[1] I use the convention of distinguishing dimensional and nondimensional quantities using case. Here the dimensionless concentration and time will be c and t, so I am using T for dimensional time.

or

$$t_R = \frac{1}{k}. \tag{4}$$

Similarly, we can determine a mean residence time t_F for flow with the dominant balance

$$V\frac{\Delta C}{\Delta T} \approx q\Delta C, \tag{5}$$

which yields

$$t_F = \frac{V}{q}. \tag{6}$$

Many (all?) written descriptions of scaling start by using dimensional methods to identify $1/k$ and V/q as time scales. This is not very helpful because it fails to ascribe a physical significance to either value. Indirect scaling must be done by calculating scales from simplifying approximations based on physical intuition for the scenario of interest. This example was described as a chemical reactor, so we should assume that the chemical reaction is at least as important as the flow; hence we should choose t_R and write

$$T = \frac{1}{k}t, \qquad \frac{d}{dT} = k\frac{d}{dt}. \tag{7}$$

If we were studying pollution in a lake, we might expect the flow to be more important than the reaction and we should choose t_F.

Dimensionless parameters: The indicated variable changes yield the problem

$$\frac{dc}{dt} = \frac{q}{kV}(1-c) - c, \qquad c(0) = 1. \tag{8}$$

The dimensionless grouping q/kV has emerged from the scaling, so we can define

$$\eta = \frac{q}{kV} = \frac{q/V}{k} = \frac{t_R}{t_F}. \tag{9}$$

Dimensionless parameters can always be thought of as giving the relative importance of two processes or a ratio of two time or length scales. Here, η is the relative importance of flow compared to reaction and also the ratio of the reaction time to the flow time. With the assumption that the flow rate should be less than the reaction rate (or the residence time for flow should be longer than that for reaction), we expect $\eta < 1$. If this is a working chemical reactor, we should expect $\eta \ll 1$ and we could obtain the solution as a regular perturbation expansion (of course this problem can be solved exactly instead).

2. The Projectile Problem

Newton's Second Law of Motion says that the rate of change of momentum of an object mv is equal to the sum of the forces acting on it. If we take M and R to be the mass and radius of a planet, V the initial velocity of a projectile of mass m, and use the Newtonian model for gravitational force, we obtain an initial value problem for the height $Y(T)$ of the projectile:

$$m\frac{d^2Y}{dT^2} = F_g = -\frac{GMm}{(R+Y)^2}, \qquad Y(0) = 0, \quad Y'(0) = V.$$

We can write this equation in a more familiar form by using the symbol g to represent the gravitational force per unit mass at the planet's surface. We can therefore equate the gravitational force at $Y = 0$ with $-mg$, yielding

$$g = \frac{GM}{R^2};$$

hence, we can replace GM by gR^2 and obtain

$$\frac{d^2Y}{dT^2} = -\frac{gR^2}{(R+Y)^2}, \qquad Y(0) = 0, \quad Y'(0) = V. \tag{10}$$

Direct scaling for height and velocity The radius R has the right dimension for a length scale, but is it a representative value for the height Y? This depends on the values of the dimensional parameters. It would certainly be too large if the projectile is a golf ball and possibly too small for a rocket. There is no direct scale for time; but the initial velocity V is a direct scale for velocity. In this model, velocity always decreases, so V is an upper bound as well as a representative value. It does not matter that velocity is not one of the variables we are using in the model. Choosing a velocity scale binds the length scale Y_r and time scale T_r together by the relation

$$\frac{Y_r}{T_r} = V. \tag{11}$$

We can also obtain a direct velocity scale from a partial solution of either the original problem or an approximate problem. In this case, we can employ an energy argument, which consists of multiplying the differential equation by velocity and integrating from the initial time to an arbitrary time. Both integrals can be calculated exactly, with the result

$$\frac{1}{2}\left(\frac{dY}{dT}\right)^2 = \frac{gR^2}{R+Y} + \left(\frac{1}{2}V^2 - gR\right).$$

Now consider the case of a rocket that is just able to escape the earth's gravity. In this case, we must have $Y \to \infty$ as $dY/dT \to 0$. The initial velocity for which this condition is just met is the escape velocity

$$V_e = \sqrt{2gR}.$$

We could use V_e instead of V as a velocity scale, but the factor of $\sqrt{2}$ is unnecessary because scaling is about orders of magnitude rather than finite factors. If we want velocity scaled on the order of escape velocity, we should choose

$$\frac{Y_r}{T_r} = \sqrt{gR}. \tag{12}$$

Indirect scalings In the chemical reactor problem, we obtained scalings using a dominant balance argument. In the projectile problem, it is reasonable to expect that the acceleration and the gravitational force at the surface should be roughly balanced. This gives us

$$\frac{\Delta Y}{(\Delta T)^2} \approx -g,$$

which yields a relationship between the scales:

$$Y_r = gT_r^2. \tag{13}$$

Choosing the scales We have four possible scale relations, given by $Y_r = R$ and (11)–(13). We can only choose two of these, and not both (11) and (12). There are five ways that we can choose two non-contradictory scale relations; however, these yield only three distinct choices of scales. Each of these results in a (common) parameter that is related to the ratio of possible velocity scales, so all resulting models will be written with the parameter

$$\nu = \frac{V}{\sqrt{gR}}. \tag{14}$$

1. If we don't use the initial velocity, then the scales $Y_r = R$ and $T_r = \sqrt{R/g}$ we can simultaneously satisfy the three relations for planet size, escape velocity, and dominant balance, yielding the model
$$y'' = -\frac{1}{(1+y)^2}, \qquad y(0) = 0, \quad y'(0) = \nu. \tag{15}$$
2. If we combine the planet radius and initial velocity, we have $Y_r = R$ and $T_r = R/V$, with the result
$$\nu^2 y'' = -\frac{1}{(1+y)^2}, \qquad y(0) = 0, \quad y'(0) = 1. \tag{16}$$
3. If we combine the initial velocity and dominant balance scales, we get $Y_r = V^2/g$ and $T_r = V/g$, with the result
$$y'' = -\frac{1}{(1+\nu^2 y)^2}, \qquad y(0) = 0, \quad y'(0) = 1. \tag{17}$$

If the parameter ν is neither large nor small, then all of these scaling choices are asymptotically equivalent and we could make the choice based on algebraic convenience. In this case, (15) is best because the parameter is simply an initial value. However, we usually want a model to work for a large variation in parameter values, and restricting ν to be an $O(1)$ parameter is limiting. If ν is small, then (15) has a solution in which y' starts small and decreases. The scaling is not much better if ν is large.

The second set appears at first glance to indicate a singular perturbation problem when ν is small. This is not a correct interpretation of the equations, however. If you assume an initial layer, then the outer problem has no solution, contradicting the assumption. Real singular perturbation problems have an outer approximation showing a dominant balance between two or more terms, whereas (16) only has two terms. The small parameter in front of the derivative merely means that the problem has been scaled incorrectly. This is not the case if ν is large. The model predicts a velocity that decreases only slightly over time, which is just what would happen if the initial velocity is far greater than the escape velocity of the planet. To complete the scaling, we should rewrite the second order equation as a system for height and velocity V. After scaling, we would obtain a dominant balance with the additional assumption $v \sim 1 - \nu v_1(t)$.

The third set is just right for ν small. The factor ν^2 in the denominator produces a regular perturbation problem in which the leading order gravitational acceleration doesn't depend on height. This is just what will happen if the initial velocity is much less than the escape velocity. If ν is large, then this scaling is not as good as (16).

In summary, we do best to set Y_r/T_r as the initial velocity, which is equivalent to saying that the time scale represents the time required for the projectile to initially move a distance comparable to the length scale. If the initial velocity is not large relative to escape velocity, then we should require a dominant balance in the differential equation. This requirement should be waived, however, if the initial value is so large that the velocity is essentially unchanged while the projectile escapes from gravity.

3. A Damped Non-Linear Spring

Consider a solid object subject to a restoring force proportional to the cube of displacement and a damping force proportional to velocity. If the system starts with a fixed initial momentum I, we have

$$m\frac{d^2X}{dT^2}+a\frac{dX}{dT}+kX^3=0, \qquad X(0)=0, \quad m\frac{dX}{dT}(0)=I. \tag{18}$$

Direct scaling for displacement and velocity The initial condition suggests that I/m might be a good scale for velocity. But we should try to confirm that this value is a bound on velocity using an energy argument. Multiplying by velocity and integrating from time 0 to time T yields

$$m\int_0^T \frac{dX}{dT}\frac{d^2X}{dT^2}\,dt+a\int_0^T\left(\frac{dX}{dT}\right)^2 dt+k\int_0^T X^3\frac{dX}{dT}\,dt,$$

or

$$E(T)=\frac{1}{2}m\left(\frac{dX}{dT}\right)^2+\frac{1}{4}kX^4=\frac{I^2}{2m}-a\int_0^T\left(\frac{dX}{dT}\right)^2 dt. \tag{19}$$

The quantity E is the total energy, with the kinetic energy given in the first term and the potential energy given in the second. The calculation shows that energy can only decrease from its initial value. Since both energies are positive, the term $I^2/2m$ serves as an upper bound on each, giving us

$$\left|\frac{dX}{dT}\right|\le V_0=\frac{I}{m}, \qquad |X|<X_M=\left(\frac{2I^2}{mk}\right)^{1/4}; \tag{20}$$

hence, we have possible scale relations

$$\frac{X_r}{T_r}=\frac{I}{m}, \qquad X_r=\left(\frac{I^2}{mk}\right)^{1/4}. \tag{21}$$

Depending on the parameter values, the total energy might decrease quickly or slowly and the displacement and velocity could be either oscillating or monotone. The velocity V_0 is a tight upper bound, since it is the initial value. The length X_M could be a tight upper bound if the damping is relatively small, but it could also be considerably larger than the achieved maximum if there is a lot of damping.

Indirect scaling by dominant balance There are three possible dominant balance arguments, depending on which two of the three terms in the differential equation are assumed to be important.

1. **If the restoring force dominates the damping force:**
   If we neglect the damping force, then the integral in (19) is relegated to correction terms in a perturbation expansion. The system oscillates with the bounds on displacement and velocity given by X_M and V_0. With no other physically-motivated candidates, the scales

$$X_r=\left(\frac{I^2}{mk}\right)^{1/4}, \qquad T_r=\left(\frac{m^3}{I^2k}\right)^{1/4} \tag{22}$$

   follow from (21).[2] Careful accounting of factors yields a dimensionless model with a single parameter:

$$x''+\alpha x'+x^3=0, \qquad x(0)=0, \quad x'(0)=1, \qquad \alpha=\frac{a}{(mkI^2)^{1/4}}. \tag{23}$$

[2] This example demonstrates beyond any question that finding scales by looking for quantities with the right dimensions is a poor strategy.

This version of the model is relevant if α is small, or at least not large. The formula for α gives a clear meaning to "small damping"; the statement "a is small" doesn't have a clear meaning because the numerical values of a depend on an arbitrary choice of units.

2. **If the damping force dominates the restoring force:**
   Without the restoring force, the differential equation is given approximately by

$$m\frac{d^2X}{dT^2} + a\frac{dX}{dT} = 0.$$

   We can use this simple linear differential equation to identify a time scale

$$T_r = \frac{m}{a}, \tag{24}$$

   but not a length scale. Of course we can combine this result with either of the scales in (21) to get a complete set.

3. **If the system is close to equilibrium:**
   Without the acceleration term, the differential equation is

$$a\frac{dX}{dT} + kX^3 = 0,$$

   from which we obtain a scaling relation

$$X_r^2 T_r = \frac{a}{k}. \tag{25}$$

   As in case 2, this relation needs to be combined with one other relation to obtain a complete set of scales.

Choosing the scales when $\alpha \gg 1$: The case $\alpha \gg 1$ could correspond to either of cases 2 and 3. It is helpful to choose a different version of the dimensionless parameter. Thinking of it as a scaled version of k, we should use

$$\kappa = \alpha^{-4} = \frac{mkI^2}{a^4} \ll 1. \tag{26}$$

The large damping case could have a singular perturbation structure, which would mean that there is a single correct length scale and both an outer and an inner time scale.

The cases below have numbers matching the cases in the dominant balance listing. Subcase (a) for each will use the velocity scale from (21), while subcase (b) will use the length scale instead. Computational details are left to the reader.

2. (a) Using (21a) with (24) yields the scales

$$X_r = \frac{I}{a}, \qquad T_r = \frac{m}{a}, \tag{27}$$

   and the dimensionless problem

$$x'' + x' + \kappa x^3 = 0, \qquad x(0) = 0, \quad x'(0) = 1. \tag{28}$$

(b) Using (21b) with (24) yields the scales

$$X_r = \left(\frac{I^2}{mk}\right)^{1/4}, \qquad T_r = \frac{m}{a}, \tag{29}$$

and the dimensionless problem

$$x'' + x' + \sqrt{\kappa}\, x^3 = 0, \qquad x(0) = 0, \quad \kappa^{-1/4} x'(0) = 1. \tag{30}$$

3. (a) Using (21a) with (25) yields the scales

$$X_r = \left(\frac{aI}{mk}\right)^{1/3}, \qquad T_r = \left(\frac{am^2}{kI^2}\right)^{1/3}, \tag{31}$$

and the dimensionless problem

$$\kappa^{1/3} x'' + x' + x^3 = 0, \qquad x(0) = 0, \quad x'(0) = 1. \tag{32}$$

(b) Using (21b) with (25) yields the scales

$$X_r = \left(\frac{I^2}{mk}\right)^{1/4}, \qquad T_r = \left(\frac{a^2 m}{kI^2}\right)^{1/2}, \tag{33}$$

and the dimensionless problem

$$\sqrt{\kappa}\, x'' + x' + x^3 = 0, \qquad x(0) = 0, \quad \kappa^{1/4} x'(0) = 1. \tag{34}$$

The key to identifying scalings that work is being able to tell when any small parameters in the differential equation and initial condition are properly balanced for a singular perturbation structure. The outer scaling should have a small parameter in front of the first term of the differential equation and in front of the derivative in the initial condition. Case (3b) appears to meet this requirement, but the two small parameters are different.[3] Thus, none of the four cases represents an outer scaling for the large damping case!

The inner scaling should have a small parameter in front of the last term and none in the derivative initial condition. Thus, case (2a) is the correct inner scaling for the large damping case.

Once we have the correct inner scaling, we can immediately identify the corresponding outer scaling, which is

$$\kappa x'' + x' + x^3 = 0, \qquad x(0) = 0, \quad \kappa x'(0) = 1. \tag{35}$$

We get this from (28) by applying the rescaling relationship $t = \kappa\tau$, where τ is the inner time given in case (2a) by $T = T_r\tau = m\tau/a$. Combining these identifies the correct outer time scale as

$$T_r = \frac{a^3}{kI^2}, \tag{36}$$

a time scale that did not arise in any of our exhaustively delineated cases! The reason why it didn't arise is that case 2 uses the correct dominant balance for large damping, but subcase (2a) uses a velocity scale derived from the initial condition while subcase (2b) uses an incorrect length scale. Large damping means that the velocity decreases rapidly; hence the velocity scale of subcase (2a) only works for the inner region. The correct velocity scale for the outer region is a fraction κ of that for the inner region.

[3] It doesn't matter whether the small parameter is κ or some non-unit power of κ, but it has to be the same power in both locations.