USING THE SECOND TRANSLATION THEOREM

As we know from the formula

$$\mathcal{L}\left\{H(t-a)\right\} = \frac{e^{-as}}{s}, \tag{1}$$

Heaviside functions in t space are associated with exponential functions in s space. Note that only switches that turn on after time 0 function as switches, so the step function $H(t-a)$ implicitly assume $a > 0$, which means that the exponential function in any formula for $Y(s)$ will have a negative power.

To address problems that have a Heaviside function in t space or an exponential in s space, we need the second translation theorem:

$$\mathcal{L}\left\{f(t-a)H(t-a)\right\} = e^{-as}F(s). \tag{2}$$

Using the theorem to compute Laplace transforms

Example 1

Suppose a function $g(t)$ is 0 up until time $t = 3$ and then increases with a slope of 1 after that. This means that we need two different formulas for g. When $t < 3$, we have $g = 0$. When $t \geq 3$, we have a linear function with slope 1 and t intercept 3. Thus,

$$g(t) = \begin{cases} 0, & t < 3 \\ t-3, & t \geq 3 \end{cases} = (t-3)H(t-3).$$

To find the Laplace transform of g, we need to identify the components on the left side of the second translation theorem, then identify the components on the right side, and then apply the theorem.

1. The components on the left side are the constant $a = 3$ and the function $f(t) = t$. This is a bit tricky, but just note that if $f(t) = t$, then $f(t-3) = t-3$. We'll see a more surefire method to get this right later.
2. We already have the value of a. Given $f(t) = t$, we have $F(s) = 1/s^2$ from the formula for the transform of a power function.
3. Now that we have the pieces, we can put them together:

$$\mathcal{L}\left\{g\right\} = \mathcal{L}\left\{f(t-3)H(t-3)\right\} = e^{-3s}F(s) = \frac{e^{-3s}}{s^2}.$$

◇

The standard form of the second translation theorem is tricky to use to compute Laplace transforms because it takes some work to identify the function $f(t)$. Instead, there is a second form that is often more convenient for converting from t space to s space:

$$\mathcal{L}\left\{f(t)H(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t+a)\right\}. \tag{3}$$

Example 2
As in Example 1, let

$$g(t) = \begin{cases} 0, & t < 3 \\ t-3, & t \geq 3 \end{cases} = (t-3)H(t-3).$$

We follow the usual translation theorem procedure, but this time using form (3) instead of form (2).

1. The components on the left side are the constant $a = 3$ and the function $f(t) = t - 3$. With this version, there is no difficulty identifying the function f.
2. We already have the value of a. Given $f(t) = t - 3$, we first need $f(t+3) = (t-3) + 3 = t$. Then we need $\mathcal{L}\{f(t+3)\} = \mathcal{L}\{t\} = 1/s^2$.
3. Now we put the pieces together:

$$\mathcal{L}\{g\} = \mathcal{L}\{f(t)H(t-3)\} = e^{-3s}\mathcal{L}\{f(t+a)\} = \frac{e^{-3s}}{s^2}.$$

$\diamond$

Using the theorem to compute inverse Laplace transforms

Example 3
In a previous example, we applied the Laplace transform to the problem

$$y'' + 4y = \begin{cases} 0, & t < \pi \\ 12, & t \geq \pi \end{cases} = 12H(t-\pi), \qquad y(0) = 1, \quad y'(0) = 0.$$

and obtained the result

$$Y(s) = \frac{s}{s^2+4} + \frac{12}{s(s^2+4)}e^{-\pi s},$$

which allowed us to get the partial solution

$$y(t) = \cos 2t + \mathcal{L}^{-1}\left\{\frac{12}{s(s^2+4)}e^{-\pi s}\right\}.$$

The second term has the form $F(s)e^{-\pi s}$, with

$$F(s) = \frac{12}{s(s^2+4)}.$$

We can now write the solution for y as

$$y(t) = \cos 2t + f(t-\pi)H(t-\pi).$$

We still need to compute $f(t-\pi)$, but first we need $f(t)$. This requires a partial fraction decomposition:

$$F(s) = \frac{12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + 2C}{s^2+4}.$$

The use of $2C$ rather than C as the last coefficient is a matter of taste. The advantage of including the 2 is that we can immediately write

$$f(t) = A + B\cos 2t + C\sin 2t.$$

Multiplying the F equation by the common denominator gives

$$12 = A(s^2+4) + (Bs + 2C)s.$$

Special values of s don't help much here, so we just multiply out the polynomial to get

$$12 = (A + B)s^2 + 2Cs + 4A.$$

Setting the coefficients equal because this equation has to hold for all s, we get $4A = 12$, $2C = 0$, and $A + B = 0$, so $A = 3$ and $B = -3$. Thus,

$$f(t) = 3 - 3\cos 2t.$$

From here, we need

$$f(t - \pi) = 3 - 3\cos(2t - 2\pi) = 3 - 3\cos 2t.$$

Note that in this example we got $f(t - a) = f(t)$, but that won't happen very often. We finally have the last piece of the puzzle. The solution of the problem is

$$y(t) = \cos 2t + [3 - 3\cos 2t]H(t - \pi).$$

$\diamond$