Singular Perturbation

1 A Model First-Order Problem

$$\epsilon v' + v = g(t), \qquad y(0) = 0,$$

where g is smooth, $g(0) \neq 0$, and $g'(0) \neq 0$

Synopsis

We begin with an example of this class of problems, in which we will see that the regular perturbation method does not work, find the exact solution, obtain two expansions of the correct solution for different scalings of time, and describe a perturbation method for obtaining these expansions directly from the original problem. We'll then apply the new method to the more general model problem.

1.1 An Example

$$\epsilon v' + v = e^{-kt}, \qquad v(0) = 0.$$

Regular Perturbation Solution

Suppose we look for an approximation

$$v(t; \epsilon) \sim v_0(t) + \epsilon v_1(t) + O(\epsilon^2).$$

This assumption gives us the sequence of problems

$$v_0 = e^{-kt}, \qquad v_0(0) = 0;$$

$$v_1 = -v_0' = k e^{-kt}, \qquad v_1(0) = 0$$

and so on. Clearly these problems have no solution because the functions obtained from the differential equation do not satisfy the initial conditions.

Exact Solution

$$v = \frac{e^{-kt} - e^{-t/\epsilon}}{1 - \epsilon k}.$$

This solution can be found by the integrating factor method or as a combination of a complementary and particular solution, with the particular solution from either undetermined coefficients or variation of parameters.

Outer Expansion

If we expand the exact solution as $\epsilon \to 0$, the second term in the numerator is exponentially small provided $t = O(1)$, so the correct expansion is

$$v \sim e^{-kt} + \epsilon k e^{-kt} + \epsilon^2 k^2 e^{-kt} + \cdots + \text{EST}, \qquad \epsilon \to 0, \quad t = O(1). \tag{1}$$

This *outer* expansion is exactly what we would have got by naively applying the regular perturbation method to the original problem and ignoring the initial condition. Regular perturbation does not yield a correct approximation for the problem, but it does yield an approximation that is correct for t bounded away from 0.

Inner Expansion

The problem with the outer expansion is that it misses the rapid transient term $e^{-t/\epsilon}$, which is necessary to satisfy the initial condition. If we want to see the effect of that term, we need to zoom in on the point $t = 0$, which requires a rescaling of time so that $t = O(\epsilon)$. Rather than having two small parameters (ϵ and t) in the expansion, we rescale the time with

$$t = \epsilon\tau, \qquad \tau = O(1).$$

The exact solution is then

$$v = \frac{e^{-\epsilon k\tau} - e^{-\tau}}{1 - \epsilon k}.$$

Expanding as $\epsilon \to 0$ with $\tau = O(1)$ yields the *inner* expansion

$$v \sim \left(1 - e^{-\tau}\right) + \epsilon k \left(1 - \tau - e^{-\tau}\right) + \epsilon^2 k^2 \left(1 - \tau + \frac{1}{2}\tau^2 - e^{-\tau}\right), \qquad \epsilon \to 0, \quad \tau = O(1). \quad (2)$$

Note that this expansion satisfies the initial condition to all orders.

Summary

Having a small parameter ϵ in front of the highest order derivative means that the regular perturbation problem is of lower order than the original problem; hence it cannot generally satisfy the initial condition. It does yield an outer approximation that is valid for $t = O(1)$. Rescaling the time by $t = \epsilon\tau$ leads to an inner approximation that is only valid for $t = O(\epsilon)$ but does satisfy the initial condition. Singular perturbation problems are characterized by having regimes with different scalings of the independent variable rather than one uniform scaling.

The Singular Perturbation Method

The key idea of the singular perturbation method is that both the outer and inner approximations can be obtained directly from the original problem by applying the regular perturbation method, provided the independent variable is appropriately scaled.

In the example, the original problem is given in the scaling for the outer region. Using regular perturbation on the differential equation without considering the initial condition yields the outer approximation (1).

For the inner region, we define an inner variable τ by $t = \epsilon\tau$. Thinking of v as a function of τ rather than t changes $\epsilon\frac{d}{dt}$ to $\epsilon\frac{d\tau}{dt}\frac{d}{d\tau} = \frac{d}{d\tau}$. With τ in place of t, we get the inner problem

$$\frac{dv}{d\tau} + v = e^{-\epsilon k\tau}, \qquad v(0) = 0.$$

Transcendental functions of ϵ must always be expanded, so the inner differential equation needs to be rewritten as

$$\frac{dv}{d\tau} + v = 1 - \epsilon k\tau + \frac{1}{2}\epsilon^2 k^2 \tau^2 + O(\tau^3).$$

We can find the inner expansion by applying the regular perturbation method to the inner problem. Assume

$$v \sim V_0(\tau) + \epsilon V_1(\tau) + \epsilon^2 V_2(\tau) + O(\epsilon^3).$$

We then have

$$\left(\frac{dV_0}{d\tau} + \epsilon\frac{dV_1}{d\tau} + \epsilon^2\frac{dV_2}{d\tau}\right) + (V_0 + \epsilon V_1 + \epsilon^2 V_2) = 1 - \epsilon k\tau + \frac{1}{2}\epsilon^2 k^2 \tau^2 + O(\tau^3), \quad V_0(0) + \epsilon V_1(0) + \epsilon^2 V_2(0) = 0.$$

The individual problems are

$$V_0' + V_0 = 1, \qquad V_0(0) = 0,$$

$$V_1' + V_1 = -k\tau, \qquad V_1(0) = 0,$$

and

$$V_2' + V_2 = \frac{1}{2}\epsilon^2 k^2 \tau^2, \qquad V_2(0) = 0.$$

The solutions of these problems give the correct inner approximation (2).

1.2 The General Case

$$\epsilon v' + v = g(t), \qquad v(0) = 0, \tag{3}$$

where g is smooth, $g(0) \neq 0$, and $g'(0) \neq 0$

Outer Expansion from the Original Problem
Assume

$$v \sim v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + O(\epsilon^3).$$

Substituting this into the differential equation of (3) yields the results

$$v_0 = g, \qquad v_1 = -v_0' = -g', \qquad v_2 = -v_1' = g'', \qquad \cdots,$$

for a complete outer expansion of

$$v \sim g(t) - \epsilon g'(t) + \epsilon^2 g''(t) + \cdots, \qquad \epsilon \to 0, \quad t = O(1). \tag{4}$$

Inner Approximation from the Original Problem
The rescaling $t = \epsilon\tau$ changes the original problem to

$$\frac{dv}{d\tau} + v = g(\epsilon\tau) \sim g(0) + \epsilon g'(0)\tau + \frac{1}{2}\epsilon^2 g''(0)\tau^2 + O(\epsilon^3), \qquad v(0) = 0.$$

The form

$$v \sim V_0(\tau) + \epsilon V_1(\tau) + \epsilon^2 V_2(\tau) + O(\epsilon^3),$$

then yields the problems

$$V_0' + V_0 = g(0), \qquad V_0(0) = 0,$$

$$V_1' + V_1 = g'(0)\tau, \qquad V_1(0) = 0,$$

$$V_2' + V_2 = \frac{1}{2}g''(0)\tau^2, \qquad V_2(0) = 0,$$

and so on, with solutions

$$V_0 = g(0)(1 - e^{-\tau}), \qquad V_1 = -g'(0)(1 - \tau - e^{-\tau}), \qquad V_2 = g''(0)\left(1 - \tau + \frac{1}{2}\tau^2 - e^{-\tau}\right).$$

The inner approximation is

$$v \sim g(0)(1 - e^{-\tau}) - \epsilon g'(0)(1 - \tau - e^{-\tau}) + \epsilon^2 g''(0)\left(1 - \tau + \frac{1}{2}\tau^2 - e^{-\tau}\right) + O(\epsilon^3). \tag{5}$$

3

Exact Solution and Confirmation of Results

The exact solution is most easily found by the integrating factor method. Observe that the complementary solution is $v_c = c_1 e^{-t/\epsilon}$. Let μ be the reciprocal of v_c, omitting the integration constant, so $\mu = e^{t/\epsilon}$. Differentiating the product μv and applying the differential equation yields

$$\frac{d}{dt} e^{t/\epsilon} v(t) = e^{t/\epsilon} v' + \frac{1}{\epsilon} e^{t/\epsilon} v = \frac{1}{\epsilon} e^{t/\epsilon} (\epsilon v' + v) = \frac{1}{\epsilon} e^{t/\epsilon} g(t).$$

We can now integrate both sides from the initial condition to an arbitrary point, changing the symbol used for the integration variable:

$$e^{t/\epsilon} v(t) - e^0 v(0) = \int_0^t \frac{1}{\epsilon} e^{s/\epsilon} g(s) \, ds,$$

or

$$v(t) = \int_0^t \frac{1}{\epsilon} e^{-(t-s)/\epsilon} g(s) \, ds.$$

The substitution $u = (t - s)/\epsilon$ converts the integral into

$$v(t) = \int_0^{t/\epsilon} e^{-u} g(t - \epsilon u) \, du, \tag{6}$$

from which we can obtain the outer approximation (4) by expanding g about t. To obtain the inner approximation (5), we first rescale the exact solution as

$$v = \int_0^\tau e^{-u} g(\epsilon[\tau - u]) \, du = e^{-\tau} \int_0^\tau e^s g(\epsilon s) \, ds \tag{7}$$

and then expand g about 0.

2   A Nonlinear Example

Consider the problem
$$\epsilon y' + y = ty^3, \qquad y(0) = 1.$$

Outer Approximations

The leading order outer problem is
$$y_0 = ty_0^3,$$

which has three possible solutions:

$$y_0 = 0, \quad y_0 = \frac{1}{\sqrt{t}}, \quad y_0 = -\frac{1}{\sqrt{t}}.$$

Normally we do not accept 0 as an approximate solution, but in this case $y = 0$ is an exact solution of the differential equation, so it is a serious possibility. We'll need something more to determine which of these is correct.

Inner Approximation

With rescaling, the inner problem is

$$\frac{dy}{d\tau} + y = \epsilon\tau y^3, \qquad y(0) = 1.$$

The substitution

$$y \sim Y_0(\tau) + \epsilon Y_1(\tau)$$

yields the leading order problem

$$Y_0' + Y_0 = 0, \qquad Y_0(0) = 1,$$

with solution

$$Y_0 = e^{-\tau}.$$

Then the $O(\epsilon)$ problem is

$$Y_1' + Y_1 = \tau Y_0^3 = \tau e^{-3\tau}, \qquad Y_1(0) = 0,$$

with solution

$$Y_1 = \frac{1}{4}e^{-\tau} - \left(\frac{1}{4} + \frac{1}{2}\tau\right)e^{-3\tau}.$$

We have the inner approximation

$$y \sim e^{-\tau} + \epsilon\left[\frac{1}{4}e^{-\tau} - \left(\frac{1}{4} + \frac{1}{2}\tau\right)e^{-3\tau}\right], \qquad \epsilon \to 0, \quad \tau = O(1).$$

Matching

The key to selecting the correct outer approximation is to realize that the outer and inner approximations are for a single continuous function; hence, they cannot be completely unrelated. The large τ and small t behaviors of the approximations should match. In this case, the inner approximation vanishes as $\tau \to \infty$, so the outer approximation must go to 0 as $t \to 0$. Thus, $y = 0$ is the correct outer approximation. In this example, only the inner approximation is needed.

This example gives us a first look at what will become a necessary complement to the outer and inner approximations in most singular perturbation problems–asymptotic matching.

3 A Model Second-Order Initial Value Problem

$$\epsilon v'' + v' = g'(t), \qquad v(0) = 0, \quad \epsilon v'(0) = 1, \tag{8}$$

where g is smooth and $g'(0) \neq 0$. Without loss of generality we may assume $g(0) = 1$.

Outer and Inner Approximations from the Exact Solution

We can integrate the differential equation once and apply the initial conditions to obtain

$$\epsilon v' + v = g(t), \qquad v(0) = 0.$$

This is problem (3), so we already have asymptotic approximations in the outer (4) and inner (5) regions (but with $g(0) = 1$).

The Outer Approximation by Regular Perturbation

The assumption $v \sim v_0(t) + \epsilon v_1(t)$ gives us the problems

$$v_0' = g', \qquad v_1' = -v_0'',$$

with results

$$v_0(t) = g(t) + C_0, \qquad v_1 = -g'(t) + C_1. \tag{9}$$

These results are consistent with the correct approximation (4), but they are not complete. How do we get $C_0 = 0$ and $C_1 = 0$? This will require asymptotic matching after we have the inner approximation.

The Inner Approximation by Regular Perturbation

Replacing t by $t = \epsilon\tau$ introduces a factor of ϵ^{-1} for each derivative. Multiplying through by ϵ, we get the inner problem

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} = \epsilon g'(\epsilon\tau) \sim \epsilon g'(0) + O(\epsilon^2), \qquad v(0) = 0, \quad \frac{dv}{d\tau}(0) = 1. \tag{10}$$

With $v \sim V_0(\tau) + \epsilon V_1(\tau)$, we get the problems

$$V_0'' + V_0' = 0, \qquad V_0(0) = 0, \quad V_0'(0) = 1$$

and

$$V_1'' + V_1' = g'(0), \qquad V_1(0) = 0, \quad V_1'(0) = 0,$$

with solutions

$$V_0 = 1 - e^{-\tau}, \qquad V_1 = -g'(0)(1 - \tau - e^{-\tau}), \tag{11}$$

for the same asymptotic expansion (5) as before.

Leading Order Matching

We have leading order outer and inner approximations

$$v_0 = g(t) + C_0, \qquad V_0 = 1 - e^{-\tau}.$$

To match them, consider a simple thought experiment. Suppose an observer sitting in the outer region looks toward the inner region. This outer observer is looking at V_0 on the $t = O(1)$ time scale, but the inner approximation all takes place in the tiny $t = O(\epsilon)$ region, which is a mere dot to the outer observer. What the outer observer sees is $\lim_{\tau\to\infty} V_0 = 1$. A second observer sits in the inner region looking toward the outer region. This inner observer is looking at v_0 on the $t = O(\epsilon)$ time scale, but everything is impossibly far away. All that (s)he sees is $\lim_{t\to 0} v_0 = g(0) + C_0 = 1 + C_0$. This is our justification for the claim $C_0 = 0$. The two observers are looking at the same continuous function, so when they look toward each other they have to see the same thing. This is the *leading order matching principle*:

- Suppose $v \sim v_0(t)$ for $t = O(1)$ and $v \sim V_0(\tau)$ for $\tau = O(1)$, where $\tau \ll t$. Then

$$\lim_{t\to 0} v_0(t) = \lim_{\tau\to\infty} V_0(\tau).$$

Higher Order Matching Using Van Dyke's Principle

Notice that there is no limit as $\tau \to \infty$ of V_1. Leading order matching does not work except for leading order. Instead we need a generalization of the basic principle that when the two observers look at each other's solutions, they should see the same thing. Suppose we have outer and inner approximations up to some order ϵ^p, with $p \geq 0$:[1]

$$v \sim v^O(t;\epsilon) + o(\epsilon^p), \quad \epsilon \to 0, \quad t = O(1); \qquad v \sim v^I(\tau;\epsilon) + o(\epsilon^p), \quad \epsilon \to 0, \quad t = O(\epsilon).$$

In this notation, the ϵ after the semicolon indicates that the function is a power series in ϵ that includes terms that are present with the given truncation order.

The outer observer is looking at v^I on the $t = O(1)$ time scale, which means $v^I(t/\epsilon)$ rather than $v^I(\tau)$. But because the scaling has changed, some of the terms no longer meet the truncation order requirements and others are nonlinear in ϵ. The outer observer actually sees only those terms in $v^I(t/\epsilon)$ which are present after any terms that are nonlinear in ϵ under the new scaling have been expanded and less important terms removed. This process defines the *outer expansion of the inner approximation* by

$$v^I\left(\frac{t}{\epsilon};\epsilon\right) \sim v^{IO}(t;\epsilon) + o(\epsilon^p).$$

Similarly, the inner observer looks at $v^O(\epsilon\tau)$ and obtains the *inner expansion of the outer approximation* by expanding with the original truncation order to get

$$v^O(\epsilon\tau;\epsilon) \sim v^{OI}(\tau;\epsilon) + o(\epsilon^p).$$

The expansions v^{IO} and v^{OI} are orginally calculated using different time scalings, but they must agree, and this pins down the correct values of any previously undetermined integration constants. In this example, the truncation order is ϵ, so

$$v^I\left(\frac{t}{\epsilon}\right) \sim (1 - e^{-t/\epsilon}) - \epsilon g'(0)\left(1 - \frac{t}{\epsilon} - e^{-t/\epsilon}\right) + o(\epsilon) \sim 1 + g'(0)t - \epsilon g'(0) + o(\epsilon),$$

which means

$$v^{IO} = 1 + g'(0)t - \epsilon g'(0) = 1 - \epsilon g'(0)(1 - \tau).$$

Similarly,

$$v^O(\epsilon\tau) \sim g(\epsilon\tau) + C_0 + \epsilon[-g'(\epsilon\tau) + C_1] + o(\epsilon) \sim [1 + \epsilon\tau g'(0) + C_0] + \epsilon[-g'(0) + C_1] + o(\epsilon),$$

which means

$$v^{OI} = 1 + C_0 - \epsilon g'(0)(1 - \tau) + \epsilon C_1.$$

The asymptotic matching requirement forces $C_0 = C_1 = 0$.

The method we have developed here was first described in 1964 by Milton Van Dyke and is known as *Van Dyke's Principle*. The earlier *intermediate variable* method is more complicated. Some authors who use the intermediate variable method mistakenly claim that Van Dyke's Principle does not always work. This is an unfortunate consequence of a subtle error in Van Dyke's original statement, which counted the number of terms rather than focusing on the truncation order. In the form presented here, Van Dyke's Principle is always correct.

---

[1] Note that the number of terms needed for v^O and v^I do not matter; it is only the truncation order that matters.

4   A Model Boundary Value Problem

Consider the problem

$$\epsilon y'' + y' = f'(x), \qquad y'(0) = 0, \quad y(1) = 0, \tag{12}$$

where f is smooth, $f'(0) \neq 0$, and without loss of generality we may assume $f(1) = 0$. As with the model initial value problem, the regular perturbation solution cannot satisfy the boundary condition at $x = 0$. The outer region is where $x = O(1)$, so the appropriate problem is

$$\epsilon y'' + y' = f'(x), \qquad y(1) = 0.$$

The assumption $y \sim y_0(t) + \epsilon y_1(t)$ yields the problems

$$y_0' = f', \qquad y_0(1) = 0,$$

$$y_1' = -y_0'', \qquad y_1(0) = 0,$$

with result

$$y \sim f(x) + \epsilon[f'(1) - f'(x)] + o(\epsilon). \tag{13}$$

Rescaling with $x = \epsilon\xi$ yields the inner problem

$$\frac{d^2 y}{d\xi^2} + \frac{dy}{d\xi} = \epsilon f'(\epsilon\xi) \sim \epsilon f'(0) + O(\epsilon^2), \qquad \frac{dy}{d\xi}(0) = 0. \tag{14}$$

Note that we do not include the boundary condition at $x = 1$ because it is at $\xi \to \infty$, which means that information from that boundary condition must propagate through the outer region to the inner region via the matching requirement. We can find a leading order boundary condition at $\xi \to \infty$ using the leading order matching principle:

$$Y_0(\infty) = \lim_{\xi \to \infty} Y_0(\xi) = \lim_{x \to 0} y_0 = f(0);$$

hence, the inner problem reduces to

$$Y_0'' + Y_0' = 0, \qquad Y_0'(0) = 0, \quad Y_0(\infty) = f(0),$$

with solution

$$Y_0 = f(0),$$

$$Y_1'' + Y_1' = f'(0), \qquad Y_1'(0) = 0,$$

with solution

$$Y_1 = C_1 + f'(0)\left[\xi + e^{-\xi}\right].$$

The full inner approximation is

$$y \sim f(0) + \epsilon\left(C_1 + f'(0)\left[\xi + e^{-\xi}\right]\right) + o(\epsilon). \tag{15}$$

Expansions of (13) and (15) yield

$$y^O(\epsilon\xi) \sim f(\epsilon\xi) + \epsilon[f'(1) - f'(\epsilon\xi)] + o(\epsilon) \sim [f(0) + \epsilon f'(0)\xi] + \epsilon[f'(1) - f'(0)] + o(\epsilon)$$

and

$$y^I\left(\frac{x}{\epsilon}\right) \sim f(0) + f'(0)x + \epsilon C_1 + o(\epsilon) = f(0) + \epsilon f'(0)\xi + \epsilon C_1 + o(\epsilon).$$

Thus, $C_1 = f'(1) - f'(0)$.

The boundary value problem can be solved exactly, with the result

$$y(x) = f(x) - \epsilon \int_0^{\epsilon^{-1}x} e^{-s} f'(x - \epsilon s)\, ds + \epsilon \int_0^{\epsilon^{-1}} e^{-s} f'(1 - \epsilon s)\, ds.$$

Asymptotic approximation of this solution confirms the results obtained much more easily using the singular perturbation method with Van Dyke's matching principle.