Solving Linear Partial Differential Equations on Finite Spatial Domains[1]

This procedure assumes that the problem is "mostly homogeneous" (a homogeneous equation is one for which 0 is a solution);[2] that is,

- The differential equation is homogeneous.
- If time is a variable, all boundary conditions must be homogeneous and all but one initial condition homogeneous.
- If time is not a variable, there must be exactly one nonhomogeneous boundary condition.

1. Find a complete set of component solutions $u_n = h_n \phi_n$, where the complement function h_n depends only on the variable associated with the nonhomogeneous condition and the eigenfunction ϕ_n depends on all of the variables that are associated only with homogeneous conditions.
   (a) Substitute $u = h\phi$ into the differential equation and separate variables. Define the eigenvalues so that they are normally positive (so $\phi'' = -\lambda\phi$, not $\phi'' = +\lambda\phi$).
   (b) Determine the appropriate boundary/initial conditions for ϕ and h. Note that ϕ should have as many boundary conditions as the order of its differential equation.
   (c) Make appropriate assumptions about the sign of the eigenvalues (take $\lambda = \omega^2$, $\lambda = 0$ if needed, and rarely $\lambda = -\omega^2$). Solve the corresponding differential equations with boundary conditions to get the eigenvalues and a 1-parameter family of eigenfunctions for each eigenvalue. Arbitrarily take the parameter to be 1.
   (d) Solve the corresponding problem for h_n to obtain the component solutions $u_n = h_n \phi_n$
2. Construct a general solution $u = \sum C_n u_n$, where C_n are generalized Fourier coefficients to be determined.
3. Determine the generalized Fourier coefficients for the nonhomogeneous condition.
   (a) Write down the algebra problem that the coefficients must satisfy for the nonhomogeneous condition with general function.
   (b) Derive the coefficient formula by multiplying the equation by ϕ_m and the appropriate weight function σ, integrating over the appropriate interval, and using orthogonality.
   (c) Evaluate the coefficient formula for the specific nonhomogeneous function.

[1] Problems with infinite spatial domains require completely different methods.

[2] There are modifications that can be used if there is a nonhomogeneity in the PDE or in a BC for a time-dependent problem.