MATH 918: LOCAL COHOMOLOGY (SUMMER 1999) ①

6/7 Defn: A module E is <u>injective</u> if whenever $\exists$ a diagram of the form

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[r] \arrow[u] & N \arrow[ul, dashed, "\exists g"'] \end{tikzcd}$$

$\exists\ g: N \to E$ making the diagram commute.

Remark: ① E is injective $\Leftrightarrow \mathrm{Hom}_R(-, E)$ is exact.

~~② E is injective $\Leftrightarrow$ whenever $0 \to E \xrightarrow{i} M$ is exact, it splits (i.e., $M \cong E \oplus M/i(M)$)~~

Proposition: If E is an injective R-module and S is an R-algebra, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.

Proof: We'll use the Hom-$\otimes$ adjointness theorem, which says the following:

Let S be an R-algebra
M_1, M_2 S-modules
N R-module.

Then

$$\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, N)) \cong \mathrm{Hom}_R(M_1 \otimes_S M_2, N)$$

So to prove the Prop, ETS

$$\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, E)) \text{ is exact.}$$

But this functor is naturally equivalent to

$$\mathrm{Hom}_R(- \otimes_S S, E) = \mathrm{Hom}_R(-, E)$$

which is exact on R-modules (hence on S-modules).

<u>Example</u>: Suppose I is an ideal of R and E is an injective R-module. Then $\mathrm{Hom}_R(R/I, E)$ $\cong (0 :_E I)$ is an injective R/I-module.

<u>Theorem</u>: (Baer's criterion)
E is injective $\iff$ given any ideal I of R and a diagram

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & I \arrow[r] \arrow[u, "f"] & R \arrow[ul, dashed, "\exists g"']
\end{tikzcd}$$

$\exists\ g : R \to E$ making the diagram commute.

Proof: $\Rightarrow$: a fortiori

$\Leftarrow$:

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "i"'] & N
\end{tikzcd}$$

Let $\Lambda = \{(K, f_K) \mid M \subseteq K \subseteq N,\ f_K|_M = f\}$.

Partially order Λ in the obvious way.
As $(M, f) \in S$, S is nonempty.

By Zorn's lemma, $\exists$ a max'l element (K, f_K).

Suppose $K \neq N$. Let $x \in N \setminus K$.

Let $I = \{r \in R \mid rx \in K\}$

Define $\phi : I \to E$ by
$i \to f_K(ix)$

By assumption, $\exists\ \tilde{\phi} : R \to E$ s/t $\tilde{\phi}(i) = \phi(i) = f_K(ix)$.

Now define $g : K + Rx \to E$ by
$k + rx \to f_K(k) + \tilde{\phi}(r)$.

well-defined: suppose $k + rx = 0$. Then $r \in I$ so $f_K(rx) = \tilde{\phi}(r)$
$\therefore\ f_K(k) + f_K(rx) = f_K(k + rx) = f_K(0) = 0$. //

Defn: An R-module is divisible if $\forall\, x \in R$, x NZD and $\forall\, u \in M$, $\exists\, u' \in M$ s/t $xu' = u$.

Examples/Remarks:

1) Any vector space over a field is divisible.

2) An injective R-module is a divisible R-module.

3) If R is a domain then $Q(R)$ (quotient field) is a divisible R-module.

4) If M is divisible and $N \subseteq M$ then M/N is divisible.

5) Direct sums of ~~injectives~~ divisible modules are divisible.

Proof of 2):

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & R \arrow[u, shift right, "u"] \arrow[u, shift left] \arrow[r, "x"] & R \arrow[ul, dashed, "g"']
\end{tikzcd}$$

$x \cdot g(1) = u$.

Let R be a domain

Prop: 1) If M is torsion-free and divisible then M is injective.

2) If R is a PID then every divisible module is injective.

Proof:

1): Consider

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & I \arrow[u, "\phi"] \arrow[r] & R
\end{tikzcd}$$

Let $i \in I \setminus \{0\}$. Since M is divisible, $\exists\ x \in M$ s/t $\phi(i) = ix$.

Now let $i' \in I \setminus \{0\}$. Then

$$\phi(ii') = i\phi(i') = i'\phi(i) = i'ix$$

As M is torsion-free, $\phi(i') = i'x$.

Define $\tilde{\phi} : R \to M$ by $\tilde{\phi}(r) = rx$.

2): Consider

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & I \arrow[u, "\phi"] \arrow[r] & R
\end{tikzcd}$$

. Then $I = (a) \neq 0$.

Write $\phi(a) = ax$, some $x \in M$. Define $\tilde{\phi}(r) = rx$

Corollary: $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Corollary: If R is a domain then $Q(R)$ is an injective R-module.

Proposition: Let R be a ring, M an R-module. Then M can be embedded in an injective R-module.

Proof: Case 1: $R = \mathbb{Z}$.

~~Map a fre~~ Map a free $\mathbb{Z}$-module $F = \oplus\mathbb{Z}$ onto M. Then $M \cong \oplus\mathbb{Z}/K$.

Now

$$M = \oplus\mathbb{Z}/K \subseteq \oplus\mathbb{Q}/K$$

$\oplus\mathbb{Q}/K$ is a divisible $\mathbb{Z}$-module $\therefore$ $\oplus\mathbb{Q}/K$ is injective as $\mathbb{Z}$ is a PID.

Case 2: Let M be an R-module. Then $\exists$ an injective $\mathbb{Z}$-module E s/t $M \subseteq E$.

Then $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an injective R-module

Define

$$g : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, E) \text{ by}$$

$$m \longrightarrow f_m \quad \text{where } f_m(r) = rm \in E.$$

This is an R-module map (check).

ETS g is 1-1.

But this is clear, since if $m \neq 0$ then $f_m(1) = m \neq 0$.

<u>Remark</u>: Let $\{E_i\}_{i \in \Lambda}$ be a family of R-modules. Then $\prod E_i$ is injective $\iff$ each E_i is injective.

<u>Pf</u>:

$\Leftarrow$:

$$\begin{tikzcd}
 & & E_i \\
 & \prod E_i \arrow[ur, "\pi_i"] & \\
0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[uu, "\exists f_i"'] \arrow[ul, dashed, "F = \prod f_i"']
\end{tikzcd}$$

$\Rightarrow$:

$$\begin{tikzcd}
 & & \prod E_i \arrow[dl, bend right, "\pi_i"'] \\
 & E_i \arrow[ur] & \\
0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[uu, "g"'] \arrow[ul, dashed, "\pi_i g"']
\end{tikzcd}$$

<u>Proposition</u>: Let R be Noetherian. Then direct sums of injectives are injective.

<u>Proof</u>: Let $\{E_i\}_{i \in \Lambda}$ be a family of injectives.

Consider

$$\begin{tikzcd}
 & \oplus E_i & \\
0 \arrow[r] & I \arrow[u, "\phi"] \arrow[r] & R
\end{tikzcd}$$

⑧

Write $I = (x_1, \dots, x_n)$.

$\{\phi(x_i)\}$ involves only finitely many non-zero components, say $E_1, \dots, E_n$ (after rearrangement).

$\therefore \operatorname{im}\phi \subseteq \bigoplus_{i=1}^{n} E_i = \prod_{i=1}^{n} E_i$

Hence, ϕ extends to $\tilde{\phi} : R \longrightarrow \prod_{i=1}^{n} E_i \hookrightarrow \bigoplus_{i \in \Lambda} E_i$.

Note: Bass has proved that if the direct sum of injectives is injective, then R is Noeth.

Prop: An R-module E is injective $\Longleftrightarrow$ whenever $0 \to E \to M \to C \to 0$ is exact, the sequence splits.

Proof:

$\Longrightarrow$):

$$\begin{tikzcd} & E & & & \\ 0 \arrow[r] & E \arrow[r] \arrow[u, "="'] & M \arrow[r] \arrow[ul, dotted, "\pi, i"'] & C \arrow[r] & 0 \end{tikzcd}$$

$\Longleftarrow$: Embed E in an injective module M.

$0 \to E \to M \to M/E \to 0$. This splits, so

$M \cong E \oplus M/E$. As M is injective, E is also.

<u>Defn</u>: Let $M \subseteq N$ be R-modules. N is an <u>essential extension</u> of M if ~~every submodule~~ $\forall\, x \in N - \{0\}$ $\quad Rx \cap M \neq 0$.

<u>Examples/Remarks</u>:

1. R domain, then $R \subseteq Q(R)$ is an essential extension.

2. If $M \subseteq K \subseteq N$. N is essential over M $\Leftrightarrow$ $M \subseteq K$, $K \subseteq N$ are essential.

3. Consider

$$\begin{tikzcd} & E & \\ & & \\ 0 \arrow[r] & M \arrow[r] \arrow[uu, "f"] & N \arrow[uul, dashed, "\exists g"'] \end{tikzcd}$$

Assume E is injective and f is 1-1, and $M \subseteq N$ is essential $\exists\, g: N \to E$ and $\ker g = 0$. For if $\ker g \neq 0$ then $\ker f = \ker g \cap M \neq 0$.

<u>Prop</u>: An R-module E is injective $\Leftrightarrow$ $\nexists$ a proper essential extension of E.

<u>Pf</u>. $\Rightarrow$: Suppose $E \subseteq N$ is essential.
So $0 \to E \to N$ splits $\Rightarrow$ $N \cong E \oplus N/E$.
But $N/E \cap E = 0 \Rightarrow N/E = 0 \Rightarrow E = N$.

$\Leftarrow$: ~~Suppose E is not injective~~

There exists an injective R-module I s/t

$E \subseteq I$.

Suppose $E \neq I$.

By assumption, $E \subseteq I$ is not essential.

Let $\Lambda = \{ K \subseteq I \mid K \cap E = 0 \}$.

By Zorn's lemma, $\exists$ a max'l $K \in \Lambda$.

Then $0 \to E \xrightarrow{f} I/K$ is exact.

By maximality, this is essential.

So $\therefore$ f is onto.

Hence, $I = E + K \Rightarrow I = E \oplus K$

$\Rightarrow$ E is injective. //

<u>Theorem</u>: Let $M \subseteq E$ be R-modules. TFAE:

a) E is a maximal essential extension of M.

b) E is a minimal injective module over M.

c) E is injective and essential over M.

Also, given M, such an E exists and is called an <u>injective hull</u> (<u>envelope</u>) of M.

6/8 <u>Theorem</u>: Let $M \subseteq E$ be an extension of R-modules.

TFAE:

a) E is a maximal essential extension of M

b) E is a minimal injective containing M.

c) E is injective and essential over M.

Moreover, such an E exists ~~is unique~~ and is unique up to isomorphism. E is called the <u>injective</u> <u>hull</u> of M, denoted $E_R(M)$.

<u>Pf</u>: a) ⇒ c): If L is essential over E, L is essential over M. $\therefore L = E$. Hence, E has no proper essential ext ⇒ E is injective.

c) ⇒ b): Suppose $M \subseteq E' \subseteq E$, where E' is injective. As E is essential over M, E is essential over E'. $\therefore E = E'$.

b) ⇒ a): Let N be a maximal essential ext of M contained in E. (Zorn).

<u>Claim</u>: N is injective.

Suppose $N \subseteq K$ is an essential ext.

Then
$$\begin{tikzcd} & E & \\ 0 \arrow[r] & N \arrow[u, "\cup|"] \arrow[r, "\subseteq"'] & K \arrow[ul, dashed, "\varphi"'] \end{tikzcd}$$

φ is 1-1 as $N \subseteq K$ is essential. Hence, we may assume $N \subseteq K \subseteq E$.

But N is max'l ess. ext of M in E, so $N = K$.
$\therefore$ N has no proper essential extension, so N is injective.
Hence, $N = E$.

For existence, embed M in any injective R-module I and let E be a maximal essential ext of M in I. The proof of b) $\Rightarrow$ a) shows E is injective.

For uniqueness, suppose E and E' are injective hulls of M.
Then

$$\begin{tikzcd}
 & E' & \\
0 \arrow[r] & M \arrow[u, "f"] \arrow[r] & E \arrow[ul, dashed, "g"']
\end{tikzcd}$$

g is 1-1 as f is 1-1 and $M \subseteq E$ is essential

$\therefore$ we may assume $E \subseteq E'$. Then $E = E'$ by b).

Examples:

① Let R be a domain. Then $E_R(R) = Q(R)$.

Pf. $R \subseteq Q(R)$ is essential and $Q(R)$ is injective.

② Let $R = k[x]$, k a field.

What is $E_R(k)$?

Define $F = \bigoplus_{i \geq 0} kx^{-i} \subseteq k[x^{-1}]$.

~~Let $F = \{x^{i} \mid i \geq 0\} \subseteq k[x^{-1}]$.~~

Define an R-module structure on F by

$$x \cdot x^{-i} = \begin{cases} x^{-i+1} & , \text{ if } -i+1 < 0 \\ 0 & , \text{ if } -i+1 = 0 \end{cases}$$

Claim: $E_R(k) = F$

Identify k with $kx^{-1} \subseteq F$.

Clearly, F is essential over kx^{-1}.

ETS F is an injective R-module.

As R is a PID, ETS F is divisible.

Let $u = \sum_{i=1}^{N} \alpha_i x^{-i}$, $\alpha_i \in k$.

Let $p(x) \in k[x] - \{0\}$.

Write $p(x) = x^n \cdot q(x)$, $n \geq 0$, $q(0) \neq 0$.

$\frac{1}{q(x)}$ is a unit in $K[[x]]$, so write

$$\frac{1}{q(x)} = \sum_{j=0}^{\infty} \beta_j x^j .$$

Let $\ell(x) = \sum_{j=0}^{N+n-1} \beta_j x^j \in K[x]$.

Then $q(x) \cdot \ell(x) \equiv 1 \pmod{x^{N+n}}$

~~Now, [illegible]~~ $q(x)\ell(x) - 1 = r(x) x^{N+n}$.

But $x^{N+n} \cdot (x^{-n} u) = 0$, so

$$q(x)\ell(x)(x^{-n} \cdot u) = x^{-n} \cdot u$$

$$x^n q(x) \underbrace{\left[\ell(x) x^{-n} \cdot u\right]}_{u'} = u$$

$$p(x)\, u' = u . //$$

Remark: If $R = k[x_1, \dots, x_n]$ then

$$E_R(k) = \bigoplus_{\substack{(i_1,\dots,i_n) \in \mathbb{Z}^n \\ i_j < 0\ \forall j}} k\, x_1^{i_1} \cdots x_n^{i_n}$$

with obvious R-module structure.

<u>Theorem:</u> Let R be a Noetherian ring, S a mc. set of R.

① If E is an injective R-module, E_S is an injective R_S-module.

② For any R-module M, $E_R(M)_S = E_{R_S}(M_S)$.

<u>Proof:</u>

(i): Let I be an ideal of R. We need to show we can complete the diagram

$$\begin{tikzcd}
 & E_S & \\
0 \arrow[r] & I_S \arrow[r] \arrow[u, "\phi"] & R_S \arrow[ul, dashed, "\text{show } \exists\, g"']
\end{tikzcd}$$

As I is finitely presented,

$\mathrm{Hom}_{R_S}(I_S, E_S) \cong \mathrm{Hom}_R(I, E)_S$.

$\therefore\ \phi = \frac{f}{s}$ where $f \in \mathrm{Hom}_R(I, E)$.

We can complete the diagram

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & I \arrow[r] \arrow[u, "f"] & R \arrow[ul, dashed, "h"']
\end{tikzcd}$$

Let $g = \frac{h}{s}$.

②: As $E_R(M)_S$ is injective, ETS $M_S \subseteq E_R(M)_S$ is essential.

Let $\frac{x}{s} \in E_R(M)_S - \{0\}$. ETS $R_S \cdot \frac{x}{s} \cap M_S \neq 0$,

or $R_S \cdot x \cap M_S \neq 0$.

Let $\Lambda = \{ (0 :_R tx) \mid t \in S \}$.

As R is Noetherian, $\exists$ a max'l ideal in Λ.

As $R_S x = R_S (tx)$, we may assume $(0 :_R x)$ is maximal in Λ.

Now, $Rx \cap M \neq 0$ as $M \subseteq E_R(M)$ is essential.

Now $Rx \cap M = (M :_R x) x$.

Let $(M :_R x) = (a_1, \dots, a_n)$.

Suppose $a_i x = 0$ in $\overset{E_R(M)_S}{\cancel{R_S}}$ $\forall i$.

Then $\exists\, t \in S$ s/t $t a_i x = 0$ in $E_R(M)$ $\forall i$.

As $(0 : tx) = (0 : x) \Rightarrow a_i x = 0 \ \forall i$

$\Rightarrow Ix = 0, *$.

$\therefore$ Some $a_i x \neq 0$ in $E_R(M)_S \Rightarrow R_S x \cap M_S \neq 0$.

<u>Theorem</u>: Let R be a Noetherian ring.
Then

① An non-zero injective R-module E is indecomposable $\Leftrightarrow E \cong E_R(R/p)$ for some $p \in \operatorname{Spec} R$.

② Every non-zero injective R-module is a direct sum of some $E(R/p)$'s, $p \in \operatorname{Spec} R$

<u>Proof</u>:

① $\Leftarrow$: Suppose $E(R/p) = E_1 \oplus E_2$, $E_1, E_2 \neq 0$.
Then $E_1 \cap R/p \neq 0$ and $E_2 \cap R/p \neq 0$.
Since R/p is a domain,

$$(E_1 \cap R/p) \cap (E_2 \cap R/p) \neq 0 \;, \; *.$$

$\Rightarrow$: Let $u \in E \setminus \{0\}$ and $N = Ru$.
Let $p \in \operatorname{Ass} N$. Then $\exists$ an injective map

$$\begin{tikzcd} R/p \arrow[r, hook] \arrow[rr, bend right, "f"'] & N \arrow[r, hook] & E. \end{tikzcd}$$

Consider

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & R/p \arrow[u, "f"] \arrow[r] & E_R(R/p) \arrow[ul, dashed, "\phi"'] \end{tikzcd}$$

As f is 1-1, ϕ is 1-1.

Then $0 \to E_R(R/p) \to E$ splits $\Rightarrow E = E_R(R/p)$ as E is indec.

Note: This also shows any injective R-module has $E(R/p)$ as a direct summand, for some $p \in \operatorname{Spec} R$.

②: Let E be an injective R-module and

$$\Lambda = \left\{ \{E_\lambda\}_{\lambda \in J} \;\middle|\; \begin{array}{l} E_\lambda \text{ is indec. injective R-submodule of } E \\ \text{and } \sum E_\lambda \cong \bigoplus E_\lambda . \end{array} \right\}.$$

$\Lambda \neq \phi$ by Note above.

By Zorn, $\exists$ a max'l set $S = \{E_\lambda\}$ in Λ.
Let $N = \sum E_\lambda = \bigoplus E_\lambda$
As R is Noeth, N is injective.

$$0 \to N \to E \to N' \to 0 \quad \text{splits}$$

$\therefore$ $E = N \oplus N'$. N' is also injective.
If $N' \neq 0$, it has an indecomp. submodule, say E'.
Then $S \cup \{E'\} \in \Lambda$, $\ast$.
$\therefore$ $N' = 0$.

Prop: Let R be a Noetherian ring. Then

$$\operatorname{Ass}_R M = \operatorname{Ass} E_R(M).$$

Pf.:

As $M \subseteq E_R(M)$, $\operatorname{Ass}_R M \subseteq \operatorname{Ass} E_R(M)$.

Suppose $q \in \operatorname{Ass} E_R(M)$.

Then ~~$R/q \hookrightarrow E_R(M)$~~.

$q = (0 :_R x)$ for some $x \in E_R(M)$.

$Rx \cap M \neq 0$. Say $rx \in M - \{0\}$. (so $r \notin q$).

$\therefore$ ~~Since~~ $(0 : rx) = (0 : x) = q$, so $q \in \operatorname{Ass} M$.

Corollary: Let R be Noetherian. Then

$$E_R(R/p) \cong E_R(R/q) \iff p = q.$$

Prop: Let R be Noeth, $p \in \operatorname{spec} R$. Then

$$E_R(R/p) \cong E_R(R/p)_p \cong E_{R_p}(R_p/pR_p).$$

Proof: ETS $E = E_R(R/p)$ is an R_p-module.

Let $s \in R-p$ ~~and s is a NZD on $E_R(R/p)$~~.

and $x \in E$. ~~As $\operatorname{Ass} E = \{p\}$, s is~~

Then $Rx \cong R/I$ where $I = (0 :_R x)$.

As $\operatorname{Ass} R/I \subseteq \operatorname{Ass} E = \{p\}$, s is a NZD on R/I.

$$\begin{tikzcd} & x & E & \\ 0 \arrow[r] & \bar{1} \arrow[u, mapsto] & R/I \arrow[u] \arrow[r, "s"] & R/I \arrow[ul, dashed, "f"'] \end{tikzcd}$$

$\therefore \quad f(\bar{s}) = s \cdot f(\bar{1}) = x$.

~~Call~~ Let $x' = f(\bar{1})$. x' is unique s/t $sx' = x$ as s is a NZD on E.

For any $r \in R$, define $\frac{r}{s} \cdot x = rx'$.

This makes E into an R_p-module (check).

~~Prop: Let R be Noetherian.~~

__Lemma:__ Let R be Noetherian, $p \in \operatorname{Spec} R$. Every element in $E_R(R/p)$ is annihilated by a power of p.

__Pf:__ Let $x \in E_R(R/p)$, $x \neq 0$. ~~$Rx \subseteq R_p$~~ ~~Ass R~~ ~~Ass $Rx = \{p\}$~~. $Rx \cong R/(0:x)$,

so $\operatorname{Ass} R/(0:x) = \{p\}$. $\therefore \sqrt{(0:x)} = p$

$\Rightarrow p^n \subseteq (0:x)$.

(G. Leuschke)

6/9 **Lemma:** Let R be a Noeth. ring, $I \subseteq R$ and M an ~~R-module. Then~~ R/I-module.

Then $\mathrm{Hom}_R(R/I, E_R(M)) = E_{R/I}(M)$.

Pf. We know

$\mathrm{Hom}_R(R/I, E_R(M))$ is an injective R/I-module. ETS ~~it is enough~~ it is an essential ext of M.

Define $\phi: M \longrightarrow \mathrm{Hom}_R(R/I, E_R(M))$
$$x \longrightarrow f_x : \bar{1} \longrightarrow x$$

This is 1-1.
Let $f: R/I \longrightarrow E_R(M)$ be non-zero.
Need $R/I f \cap M \neq 0$.
Let $x = f(\bar{1}) \in E_R(M)$. Then $\exists\, r \in R - \{0\}$ s/t $rx \in M - \{0\}$. Also, $r \notin I$.

~~$(\bar{r}f)(y)$~~ $(\bar{r} f)(R/I) \subseteq M$.
$\neq 0$.

From now on, work with a local ring (R, m, k). Let $E = E_R(k)$.

~~Define~~ [illegible]

Define a (contravariant) (exact) functor on R-modules

$$- ^{\vee} = \mathrm{Hom}_R(-, E).$$

Corollary: $k^{\vee} = k$

Pf: $k^{\vee} = \mathrm{Hom}_R(R/m, E_R(R/m)) = E_{R/m}(R/m) = k$.

Prop: Let M be an R-module.

① The natural map

$$\begin{aligned} \theta_M : M &\longrightarrow M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) \\ x &\longrightarrow ev_x \end{aligned}$$

is 1-1.

② If $\lambda(M) < \infty$ then ⓐ $\lambda(M^{\vee}) = \lambda(M)$

ⓑ θ_M is an iso.

Pf: ①: ETS that for any $0 \neq x \in M$, $\exists$ $\phi: M \to E$ s/t $\phi(x) \neq 0$.
Let $x \in M - \{0\}$

$$\begin{tikzcd}
Rx \arrow[r, "\cong"] \arrow[rrr, bend right, "f"'] & R/(0:x) \arrow[r] & R/m \arrow[r, hook] & E
\end{tikzcd}$$

Note $f(x) \neq 0$.

Consider
$$\begin{tikzcd}
0 \arrow[r] & Rx \arrow[r] \arrow[d, "f"'] & M \arrow[dl, dashed, "\phi"] \\
 & E &
\end{tikzcd}$$

So f extends $\phi: M \to E$, $\phi(x) \neq 0$.

②: ⓐ Suppose $\lambda(M) = n < \infty$
Induce on n.

If $n=1$, see Cor.

If $n>1$, $\exists$ $N \subseteq M$ s/t $\lambda(N) = n-1$, and $\exists$ an ex seq

$$0 \to N \to M \to k \to 0$$

Apply $^\vee$: $0 \to k^\vee \to M^\vee \to N^\vee \to 0$.

But $k^\vee = k$, $\lambda(N^\vee) = n-1$ by induction.
So $\lambda(M^\vee) = n$.

ⓑ $\lambda(M^{\vee\vee}) = \lambda(M^\vee) = \lambda(M)$.

By ①, the nat map

$\theta_M : M \longrightarrow M^{\vee\vee}$ is injective, so must be onto as they have the same length.

connected

Prop: Let R be a Noeth ring, $p \in \operatorname{Spec} R$, $E = E_R(R/p)$. Then $\operatorname{Ann}_R E = 0$.
(in fact, this holds for any injective).

False

If $\frac{r}{1} = 0$ on R_p
Then $r \in \operatorname{Ann}_R E_R(R_p)$

Holds for $E_R(R/m)$

Pf: Since $E_R(R/p) = E_{R_p}(R_p/pR_p)$, we may assume (R, m, k) is local and $p = m$.

?!
Not true.

Let $r \in \operatorname{Ann}_R E \setminus \{0\}$. Then $r \notin m^n$ for some $n > 0$. Since $rE = 0$,

$$\bar{r} \operatorname{Hom}_R(R/m^n, E) = 0. \quad (\bar{r} \in R/m^n \setminus \{0\}).$$

$$\shortparallel$$

$$\bar{r} \cdot E_{R/m^n}(k) = 0.$$

So WMA that R is 0-dim'l, $E = E(R/m)$.
Now, E is an $R/(\bar{r})$-module.

$E = \mathrm{Hom}_R(R/(r), E) = E_{R/(r)}(k)$.

Now $\lambda(R) = \lambda(R^\vee) = \lambda(\mathrm{Hom}_R(R,E))$

$\qquad$ ~~$= \lambda(E)$~~.

$\qquad = \lambda(E_R(k))$

$\qquad = \lambda(E_{R/(r)}(k))$

$\qquad = \lambda(R/(r))$

$\therefore$ $r = 0$, $*$.

<u>Prop:</u> Let (R, m, k) be a 0-dim'l local ring. Then $R^\vee = E_R(k)$ and $E^\vee = R$.

<u>Pf.</u> ✓

<u>Prop:</u> $E_R(k) \cong E_{\hat{R}}(k)$ (so $E_R(k)$ is an $\hat{R}$-module).

<u>Pf.</u> First, let's show it's an $\hat{R}$-module. Let $x \in E_{\hat{R}}(k)$, $\hat{r} \in \hat{R}$. Then $m^n x = 0$, some n. Then $\exists\ r \in R$ s/t $\hat{r} - r \in m^n \hat{R}$ ~~Define~~ $\hat{r} - r \in m^n$. Define $\hat{r}x = rx$. ✓

Now, $k \subseteq E_R(k)$ is an essential ext of R-modules, and so is an essential ext of $\hat{R}$-modules.

~~We have~~

<u>Claim</u>: $k \subseteq E_R(k) \subseteq E_{\hat{R}}(k)$.

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "ess"] \arrow[d] & E_R(k) \arrow[ld, dashed, "1-1"] \\
 & E_{\hat{R}}(k) &
\end{tikzcd}$$

as $\hat{R}$-modules.

✓ claim.

We know $K \subseteq E_{\hat{R}}(k)$ is essential as $\hat{R}$-modules.
Let's show it's essential as R-mods.
Let $y \in E_{\hat{R}}(k) - \{0\}$.
Then $\exists\ \hat{r} \in \hat{R}$ s/t $\hat{r}y \in k - \{0\}$.
Also, $\exists\ n$ s/t $\hat{m}^n y = 0$.
So choose $r \in R$ s/t $\hat{r} - r \in m^n \hat{R}$.
Then $ry = \hat{r}y \in k - \{0\}$.

$\therefore$ $E_R(k) = E_{\hat{R}}(k)$.

<u>Theorem:</u> Let (R, m, k) be a local ring. $E = E_R(k)$. Then $E^\vee = \mathrm{Hom}_R(E,E) \cong \hat{R}$.

<u>Pf.</u> $\phi: \hat{R} \longrightarrow \mathrm{Hom}_R(E,E)$ by

$$\hat{r} \longrightarrow \mu_{\hat{r}}: E \longrightarrow E$$

This is an $\hat{R}$-module homomorphism and 1-1 as $\mathrm{Ann}_{\hat{R}} E = 0$.

We'll show ϕ is surjective. Let $f: E \to E$ be an R-hom.

$$\begin{aligned} \text{Let } E_n = \{x \in E \mid m^n x = 0\} &\subseteq E \\ &\cong \mathrm{Hom}_R(R/m^n, E) \\ &= E_{R/m^n}(k). \end{aligned}$$

Let $f_n = f|_{E_n}$. Now $f_n(E_n) \subseteq E_n$.

We have

$$\begin{tikzcd}
f_{n+1}: E_{n+1} \arrow[r] & E_{n+1} \\
f_n: E_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[ur, phantom, "\hookrightarrow"] & E_n \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Since $E_n = E_{R/m^n}(k)$

$\mathrm{Hom}_{R/m^n}(E_n, E_n) \cong R/m^n$. by the 0-dim'l case.

$\therefore \exists\ \bar{r}_n \in R/m^n$ s/t $f_n = \mu_{\bar{r}_n}$

Lift these to $r_n \in R$.

<u>Claim:</u> (r_n) is a Cauchy seq.

$\mu_{r_{n+1} - r_n} : E_n \longrightarrow E_n$.

is the zero map.

But $\mathrm{Ann}_R(E_n) = \mathrm{Ann}_R(E_{R/m^n}(k)) = m^n$.

So $r_{n+1} - r_n \in m^n$.

Let $\hat{r} = \lim r_n$

<u>Claim:</u> $f = \mu_{\hat{r}}$

Let $x \in E$. Then $m^n \cdot x = 0$ for some n, so $x \in E_n$.

$$f(x) = f_n(x) = r_n x = \hat{r} x$$

So ϕ is surjective. ▪

$$\begin{tikzcd}[column sep=small] \underline{\text{Cor:}}\ \mathrm{Hom}_R(E,E) \arrow[rd, phantom, "\parallel" sloped] & = & \mathrm{Hom}_{\hat{R}}(E,E) \arrow[ld, phantom, "\parallel" sloped] \\ & \hat{R} & \end{tikzcd}$$

<u>Theorem:</u> (Matlis Duality) Let (R, m, k) be Noeth.

① If M is a f.g. $\hat{R}$-module, then

$M^\vee$ is an Artinian $\hat{R}$-module. (or R-mod)

~~(the functor must be $\mathrm{Hom}_{\hat{R}}(-,E)$~~

Here we must have $^\vee = \mathrm{Hom}_{\hat{R}}(-,E)$ $\longrightarrow$

$\downarrow$

see remarks on 7/6

② If M is an Artinian $\hat{R}$-module, then

$M^\vee$ is a f.g. $\hat{R}$-module.

③ If M is as in either ① or ②, then the natural map

$$\theta_M : M \longrightarrow M^{\vee\vee} \text{ is an iso.}$$

I.e., The functor $^\vee$ defines an anti-equivalence from $\langle\langle$Noeth $\hat{R}$-mods$\rangle\rangle \longleftrightarrow \langle\langle$Artinian $\hat{R}$-modules$\rangle\rangle$

Pf: ① Let M be a f.g. $\hat{R}$-mod

$$\hat{R}^a \to M \to 0 \quad \text{is exact.}$$

Apply $-^\vee$:

$$0 \to M^\vee \to (\hat{R}^a)^\vee$$

Now

$$\hat{R}^\vee = \mathrm{Hom}_R(\hat{R}, E) = E$$

not true! See remarks on 7/6.

So $M^\vee$ embed in E^a, an Artinian R-module.

②: Let M be Artinian. Let

$$V = \mathrm{Soc}(M) = \{x \in M \mid m \cdot x = 0\}.$$

Then $\dim_k V < \infty$, so $V \cong k^n$, some n.

Consider

$$\begin{tikzcd} 0 \arrow[r] & k^n \arrow[r, "ess"] \arrow[d] & M \arrow[dl, dashed, "g"] \\ & E^n & \end{tikzcd}$$

Now $V \subseteq M$ is essential.

So g is 1-1.

So we have $0 \to M \to E^n$

Apply $-^\vee$: $\underset{\overset{\shortparallel}{\hat{R}}}{(E^n)^\vee} \to M^\vee \to 0$

$\therefore$ M is a f.g. $\hat{R}$-module.

③ If M is as in ①, we know

$\hat{R} \to \hat{R}^{\vee\vee}$ is an iso.

So
$$\begin{tikzcd}
\hat{R}^a \arrow[r] \arrow[d] & \hat{R}^b \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\text{by 5-lemma}"] & 0 \\
(\hat{R}^a)^{\vee\vee} \arrow[r] & (\hat{R}^b)^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

If M is as in ②, first note

$E \to E^{\vee\vee}$ is an iso.

By the proof of ②:

we have $0 \to M \xrightarrow{\phi} E^{a}$. Let $C = \operatorname{coker} \phi$.

As C is Artinian, $\exists$ an ex seq $0 \to C \to E^b$.

Then
$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & E^a \arrow[r] \arrow[d, "\cong"] & E^b \arrow[d, "\cong"] \\
0 \arrow[r] & M^{\vee\vee} \arrow[r] & (E^a)^{\vee\vee} \arrow[r] & (E^b)^{\vee\vee}
\end{tikzcd}$$

iso by 5-lemma.

6/10

Lemma: Let (R, m) be a local ring and M an R-module. Then $M = 0 \iff M^\vee = 0$.

Pf: $\Rightarrow$: clear.

$\Leftarrow$: Suppose $M \neq 0$. Let $x \in M \setminus \{0\}$.

As $0 \to Rx \to M$ exact,

$M^\vee \to (Rx)^\vee \to 0$ is exact.

Hence, if we show $(Rx)^\vee \neq 0$ then $M^\vee \neq 0$.

Now,

$$\begin{tikzcd} Rx \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "f"'] & R/(0:x) \arrow[r] & R/m \arrow[r, hook] & E \end{tikzcd}$$

$f \neq 0$, $f \in (Rx)^\vee$.

Lemma: Suppose $f: M \to N$ is a R-map. If $f^\vee: N^\vee \to M^\vee$ is an isomorphism, then f is.

Pf. $0 \to K \to M \xrightarrow{f} N \to C \to 0$ exact.

Then $0 \to C^\vee \to N^\vee \xrightarrow{f^\vee} M^\vee \to K^\vee \to 0$ exact.

$f^\vee$ is an $\cong$ $\Rightarrow C^\vee = K^\vee = 0$. $\therefore K = C = 0$.

Theorem: (R, m) local. Then $E = E_R(R/m)$ is Artinian.

Pf: Suppose

$E \supseteq M_0 \supseteq M_1 \supseteq \cdots$ is a descending chain of R-modules.

Let $i_n : M_n \hookrightarrow M_{n-1}$ be the inclusion map.

So

$$E \hookleftarrow M_0 \hookleftarrow M_1 \hookleftarrow M_2 \hookleftarrow \cdots$$

Apply $^\vee$:

$$E^\vee \xrightarrow{i_0^\vee} \!\!\!\!\rightarrow M_0 \xrightarrow{i_1^\vee} \!\!\!\!\rightarrow M_1 \xrightarrow{i_2^\vee} \!\!\!\!\rightarrow M_2 \twoheadrightarrow$$

$E^\vee \cong \hat{R}$.

Let $I_n = \text{kernel } \hat{R} \twoheadrightarrow M_n$.

Then I_n is an ascending chain of ideals. As $\hat{R}$ is Noetherian, $I_n = I_{n+1} = \cdots$ for $n \gg 0$.

Hence, $i_n^\vee$ are iso for $n \gg 0$.

$\therefore$ i_n are iso for $n \gg 0$.

$\Longrightarrow$ $M_n = M_{n+1} = \cdots$ //

Lemma: Let M be any R-module.
Then ~~Hom~~ $p \in \text{Ass } M \iff \text{Hom}_{R_p}(k(p), M_p) \neq 0$.

Pf. Suppose $p \in \text{Ass} M$.
Localize at p:

$$0 \to R/p \to M$$

$$0 \to k(p) \to M_p$$

$\therefore \text{Hom}_{R_p}(k(p), M_p) \neq 0$.

Conversely, suppose $\text{Hom}_{R_p}(k(p), M_p) \neq 0$.

Then $\exists$ a ~~map~~ non-zero map

$k(p) \xrightarrow{\phi} M_p$, which must be 1-1.

As $\text{Hom}_{R_p}(k(p), M_p) \cong \text{Hom}_R(R/p, M)_p$, $\phi = \frac{f}{s}$

some $f \in \text{Hom}_R(R/p, M)$.

$\therefore 0 \to R/p \xrightarrow{f} M$ is 1-1

$\Rightarrow p \in \text{Ass} M$.

False (see 7/6)

~~<u>Lemma</u>: Let (R, m, k) be a local ring, $E = E_R(R/m)$. Then $\operatorname{Hom}_R(\hat{R}, E) = \operatorname{Hom}_{\hat{R}}(\hat{R}, E) = E$.~~

<u>Proof</u>: Let $E' = \operatorname{Hom}_R(\hat{R}, E) \supseteq \operatorname{Hom}_{\hat{R}}(\hat{R}, E) = E$.

<u>Claim</u>: $\operatorname{Ass}_{\hat{R}} E' = \{\hat{m}\}$.

<u>Pf</u>: Let $p \in \operatorname{Ass}_{\hat{R}} E'$.

Then $\operatorname{Hom}_{\hat{R}}(\hat{R}/p, E')_p \neq 0$.

But ~~$\operatorname{Hom}_{\hat{R}}(\hat{R}/p, E') = \operatorname{Hom}_{\hat{R}}(\hat{R}/p, \operatorname{Hom}_R(\hat{R}, E))$~~

~~$= \operatorname{Hom}_{\hat{R}}(R/p \otimes_R \hat{R}, E)$~~

~~$= \operatorname{Hom}_{\hat{R}}(\hat{R}/p, E)$~~

~~$= E_{\hat{R}/\hat{p}}(R/m) = E_{R/p}(R/m)$.~~

And $E_{R/p}(R/m)_p = 0$ ~~as every element~~ if $p \neq m$, as every element of $E_{R/p}(R/m)$ is annihilated by a power of m.

~~Thus~~ Now, E' is an injective $\hat{R}$-module. Hence,

$$E' = \oplus E_R(R/m) \qquad (\text{since } \operatorname{Ass}_{\hat{R}} E' = \{\hat{m}\}).$$

Now,

$$\mathrm{Hom}_R(R/m, E') = \mathrm{Hom}_R(R/m, \bigoplus_i E(R/m))$$
$$= \bigoplus_i R/m$$

On the other hand,

$$\mathrm{Hom}_R(R/m, \mathrm{Hom}_R(\hat{R}, E)) = \mathrm{Hom}_{\hat{R}}(R/m \otimes \hat{R}, E)$$
$$= \mathrm{Hom}_{\hat{R}}(\hat{R}/\hat{m}, E)$$
$$= R/m.$$

$\therefore \quad E' = E_{\hat{R}}(R/m)$ .//

<u>Defn</u>: An <u>injective</u> <u>resolution</u> of R-module M is an exact sequence

$$0 \to M \to E^0 \xrightarrow{\phi_0} E^1 \xrightarrow{\phi_1} E^2 \xrightarrow{\phi_2} \cdots$$

where E^i is an injective R-module.

The resolution is <u>minimal</u> if

E^i = injective hull of ~~coker ϕ~~ im ϕ_{i-1}

such resolutions always exist.

(Let $E^0 = E_R(M)$ and $E^1 = E_R(\operatorname{coker}\phi_0)$, etc.)

~~Exercise~~:

Example: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to$ is a minimal injective resolution of $\mathbb{Z}$.

Recall $\operatorname{Ext}^i_R(M,N)$ can be computed in two ways:

1) Take a projective resolution $P_\circ$ of M.

Then $\operatorname{Ext}^i_R(M,N) = H^i(\operatorname{Hom}_R(P^\circ, N))$

2) Take an injective resolution I° of N and

$$\operatorname{Ext}^i_R(M,N) = H^i(\operatorname{Hom}_R(M, I^\circ)).$$

<u>Proposition:</u> Let R be a Noetherian ring. Suppose $0 \to M \xrightarrow{i} E$ where E is injective. Then $E \cong E_R(M) \iff \forall\ p \in \operatorname{Spec} R$

$$\operatorname{Hom}_{R_p}(k(p), M_p) \xrightarrow{i_p} \operatorname{Hom}_{R_p}(k(p), E_p) \text{ is an}$$

isomorphism.

<u>Proof:</u>

$\Longrightarrow$): i_p is 1-1, as localization and $\operatorname{Hom}_{R_p}(k(p), -)$ are left exact.

Let $\phi \in \operatorname{Hom}_{R_p}(k(p), E_p)$.

As R is Noeth, $\phi = \frac{\psi}{s}$ where

$\psi \in \operatorname{Hom}_R(R/p, E)$. Let $x = \psi(\bar{1})$.
As $M \subseteq E$ is essential, $\exists\ t \notin p$ s/t $tx \in M$.

$\therefore\ \phi(\bar{1}) = \frac{\psi(\bar{1})}{s} = \frac{x}{ts} \in M_p$.

$\therefore\ \phi : k(p) \longrightarrow M_p$.

$\Longleftarrow$: Let $x \in E \setminus \{0\}$. Choose $p \in \operatorname{Ass}(Rx)$.

Then $\begin{array}{ccc} R/p & \hookrightarrow & Rx \\ \bar{1} & \longrightarrow & rx = y. \end{array}$ ETS $Ry \cap M \neq 0$.

As $Ry \cong R/p$, the ~~embedding~~ containment $Ry \subseteq E$

gives an embedding $R/p \hookrightarrow E$
$\bar{1} \longrightarrow y$

By assumption, $\frac{y}{1} \in M_p$.

$\therefore \exists\ s \notin p$ s/t $sy \in M$ and $sy \neq 0$ as $\operatorname{ann}_R y = p$.

Corollary: Let R be Noetherian, M an R-module and I° an injective resolution of M. Then I° is minimal $\iff$ $\forall\ p \in \operatorname{Spec} R$, the maps

$$\operatorname{Hom}_{R_p}(k(p), E_p^i) \longrightarrow \operatorname{Hom}_{R_p}(k(p), E_p^{i+1})$$ are zero $\forall i$.

Pf: Consider the exact seq

$$0 \longrightarrow Z_i \xrightarrow{\alpha_i} I^i \xrightarrow{\phi_i} I^{i+1}, \quad Z_i = \ker\phi_i = \operatorname{im}\phi_{i-1}.$$

I° is minimal $\iff \forall i$, $I^i = E_R(Z_i)$

~~$\iff$ I^i is essential over Z_i~~

~~$\Leftarrow$ $\forall p \in \operatorname{Spec} R$~~

~~$0 \to \operatorname{Hom}_{R_p}(k(p),(Z_i)_p) \to \operatorname{Hom}_{R_p}(k(p), I^i_p)$~~

~~is an~~ [illegible]

Now,

$$0 \to \operatorname{Hom}_{R_p}(k(p), (Z_i)_p) \xrightarrow{\alpha_{ip}} \operatorname{Hom}_{R_p}(k(p), I^i_p) \xrightarrow{\phi_{ip}} \operatorname{Hom}_{R_p}(k(p), I^{i+1}_p)$$

is exact

$I^\cdot$ is minimal $\iff$ α_{ip} is an $\cong$ $\forall p$

$\iff (\phi_i)_p = 0 \quad \forall p$.

<u>Corollary 2</u>: Let R be Noeth, M an R-module, $I^\cdot$ a minimal inj. resolution of M. If $p \in \operatorname{Spec} R$, then $I^\cdot_p$ is a min'l inj. resolution of M_p.

<u>Pf</u>:
- localization preserves exactness
- I^i_p is an injective R_p-module $\forall i$
- still minimal by Cor 1.

6/11

Defn: Let M be an R-module, R Noeth.
The Bass numbers of $\mu_i(p, M)$ (where $p \in \operatorname{Spec} R$) is defined to be the number of copies of $E_R(R/p)$ in I^i, where $I^\cdot$ is any minimal injective resolution of M.

Prop: $\mu_i(p, M) = \dim_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), M_p)$.

Pf: Let $I^\cdot$ be any min'l injective resolution of M.

$$0 \to M \to I^0 \to \cdots \to I^i \to \cdots$$

Write $I^i = E(R/p)^{\alpha} \oplus A$ where $p \notin \operatorname{Ass} A$.
Need to show $\alpha =$ RHS.
Then $I^i_p = E_{R_p}(k(p))^{\alpha_i} \oplus A_p$, $p_p \notin \operatorname{Ass} A_p$.

As I_p is a min'l inj. resol. of M_p, ETS the claim when (R, m) is local and $p = m$.

Suppose $I^i = E_R(R/m)^{\alpha_i} \oplus I'$, $m \notin \operatorname{Ass} I'$.

Then $\operatorname{Hom}_R(R/m, E_R(R/m)^{\alpha_i}) = (R/m)^{\alpha_i}$

$\operatorname{Hom}_R(R/m, I') = 0$ as $m \notin \operatorname{Ass} I'$.

$\therefore$

$\mathrm{Hom}_R(R/\mathfrak{m}, I^\circ)$:

$$0 \to (R/\mathfrak{m})^{\alpha_0} \to (R/\mathfrak{m})^{\alpha_1} \to (R/\mathfrak{m})^{\alpha_2} \to \cdots$$

Furthermore, all the maps are zero by the Cor.

Hence, $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) = H^i(\mathrm{Hom}_R(R/\mathfrak{m}, I^\circ))$

$$= \alpha_i .$$

<u>Corollary</u>: If R is Noeth and M is f.g., then $\mu_i(p, M) < \infty$.

<u>Pf</u>: $\mathrm{Ext}^i_{R_p}(k(p), M_p)$ is a f.g. $k(p)$-module.

<u>Notation</u>: use $\mu_i(M)$ for $\mu_i(\mathfrak{m}, M)$.

<u>Defn</u>: The <u>injective dimension</u> of M is R defined by

$\mathrm{id}_R M :=$ length of the ~~smallest~~ shortest injective resolution of M

Defn: A local ring (R, m) is **Gorenstein** if $id_R R < \infty$.

Theorem: Let (R, m) be a local ring.
TFAE:

① R is Gorenstein

② R is CM and $\mu_d(R) = 1$

③ $id_R R = \dim R$.

Corollary: ① If R is Gor and $p \in \operatorname{Spec} R$, R_p is Gor.

② If R is Gor, x is a NZD then $R/(x)$ is Gor.

③ $\mu_i(p, R) = \begin{cases} 1, & \text{if } i = \operatorname{ht}(p) \\ 0, & \text{otherwise} \end{cases}$

Pf: ① From defn

② $R/(x)$ is CM. Use that $\operatorname{Ext}^{d-1}_{R/(x)}(R/m, R/(x))$
$$\cong \operatorname{Ext}^{d}_{R}(R/m, R) \cong R/m.$$

③ $\mu_i(p, R) = \mu_i(pR_p, R_p)$

As R_p is Gor, ETP the case $p = m$.

$\mu_d(R) = 1$ by thm.

$\operatorname{Ext}^i_R(R/m, R) = 0$ for $i < d$ as R is CM.

$\operatorname{Ext}^i_R(R/m, R) = 0$ for $i > d$ as $id_R R = d$.

<u>Proposition</u>: If R is a RLR then R is Gorenstein.

<u>Pf</u>: We know RLR are CM.

Let $m=(x_1,\dots,x_d)$, $d=\dim R$.

$x_1,\dots,x_d$ is a regular sequence.

∴ The Koszul complex

$$\begin{tikzcd}
K_\bullet : 0 \arrow[r] & R \arrow[r, "1 \mapsto \pm(x_1,\dots,x_d)"'] & R^d \arrow[r] & \cdots \arrow[r] & R \arrow[r] & R/m \arrow[r] & 0
\end{tikzcd}$$

is a proj. resolution of R/m.

Apply $\mathrm{Hom}_R(-,R)$

$$\begin{tikzcd}
 & d-1 & d & \\
\cdots \arrow[r] & \mathrm{Hom}_R(R^d,R) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Hom}_R(R,R) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
\arrow[r] & R^d \arrow[r] & R \arrow[r] & 0 \\
 & e_i \arrow[r] & \pm x_i &
\end{tikzcd}$$

So $\mathrm{Ext}^d_R(R/m,R) = R/m$.

∴ $\mu_d(R)=1$.

Defn: If (R, m) is CM, $\mu_d(R)$ is called the C-M type of R.

Defn: Let R be a ring, I an ideal, M an R-module.

$$\text{Define } \Gamma_I(M) = \bigcup_{n \geq 1}^{\infty} (0 \underset{M}{:} I^n)$$

$$= \{ m \in M \mid I^n m = 0 \text{ some } n \}.$$

Given a map $f : M \to N$, define

$$\Gamma_I(f) = f|_{\Gamma_I(M)} : \Gamma_I(M) \longrightarrow \Gamma_I(N).$$

(Note: $f(\Gamma_I(M)) \subseteq \Gamma_I(N)$).

Clearly, $\Gamma_I(f+g) = \Gamma_I(f) + \Gamma_I(g)$.

∴ Γ_I is an additive covariant functor on the category of R-modules.

Claim: Γ_I is left exact

Suppose $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L$ is exact.

$$0 \longrightarrow \Gamma_I(M) \xrightarrow{\Gamma_I(f)} \Gamma_I(N) \xrightarrow{\Gamma_I(g)} \Gamma_I(L)$$

Clearly, $\Gamma_I(f)$ is 1-1 as $\Gamma_I(f) = f|_{\Gamma_I(M)}$.

Also, $gf = 0$ so $\Gamma_I(g)\Gamma_I(f) = 0$.

Let $x \in \ker \Gamma_I(g) \subseteq \ker g$.

$\therefore \exists\ m \in M$ s/t $f(m) = x$.

But $I^n x = 0$ for some n. $\therefore$ $I^n f(m) = f(I^n m) = 0$.
$\Rightarrow I^n m = 0$. $\therefore$ $m \in \Gamma_I(M)$.

Defn: The i^{th} local cohomology of of M with support in I (actually $V(I)$) is

$$H^i_I(M) := R^i\Gamma_I(M).$$

Recall R^iF, the right derived functor of a covariant, left exact functor:

Let $0 \to M \to I^\bullet$ be an injective resolution of M. Define $R^iF(M) := H^i(F(I^\bullet))$

Facts: ① This does not depend on the choice of inj. resolution of M. (comparison theorem)

② If $f: M \to N$, there is a natural map

$R^iF(f): R^iF(M) \to R^iF(N)$.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "f_0"] \arrow[rd, phantom, "\circlearrowleft"] & I^1 \arrow[r] \arrow[d, "f_1"] \arrow[rd, phantom, "\circlearrowleft"] & I^2 \arrow[r] \arrow[d, "f_2"] & \cdots \\
0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

Apply F:

$$\begin{tikzcd}
F(I^i) \arrow[r] \arrow[d, "F(f_i)"] \arrow[rd, phantom, "\circlearrowleft"] & F(I^{i+1}) \arrow[r] \arrow[d, "F(f_{i+1})"] \arrow[rd, phantom, "\circlearrowleft"] & F(I^{i+2}) \arrow[d, "F(f_{i+2})"] \\
F(E^i) \arrow[r] & F(E^{i+1}) \arrow[r] & F(E^{i+2})
\end{tikzcd}$$

$F(f_{i+1})^*$, the induced map on homology, is $R^iF(f)$.

③ $R^0F = F$ ~~(as F is left ex~~

$0 \to M \to I^0 \to I^1$ ex.

$0 \to F(M) \to F(I^0) \to F(I^1)$ ex.

$\therefore H^0(F(I^\bullet)) = F(M)$.

④ If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a s.e.s., then ∃ a l.e.s.

$$\cdots \longrightarrow R^iF(M_1) \longrightarrow R^iF(M_2) \longrightarrow R^iF(M_3) \longrightarrow R^{i+1}F(M_1) \longrightarrow \cdots$$

⑤ If E is injective,

$$R^iF(E) = \begin{cases} F(E), & \text{if } i=0 \\ 0, & \text{if } i>0. \end{cases}$$

Remarks about local cohomology:

1. $H^i_I(E) = 0$ if E is injective and $i>0$.

$$H^0_I(E_R(R/p)) = \begin{cases} 0, & \text{if } I \not\subseteq p \\ E_R(R/p), & I \subseteq p \end{cases}$$

2. Every element of $H^i_I(M)$ is killed by a power of I.

Pf: $H^i_I(M) = H^0_I(E^\circ)$ where E° is an inj resolution

But every elt in $H^0_I(E^i)$ is killed by a power of I.

3. Suppose every element of M is killed by a power of I. Then

$$H^0_I(M) = M$$
$$H^i_I(M) = 0 \quad \text{for } i > 0.$$

<u>Pf</u>: Clearly, $H^0_I(M) = \Gamma_I(M) = M$.

<u>Claim</u>: If $\mu_i(p, M) > 0$ then $p \geq I$.

Suppose not. Let

$0 \to M \to I^{\circ}$ be a minimal inj. res'l of M.

Then $0 \to M_p \to I_p^{\circ}$ is min'l.

$M_p = 0$.

$\therefore 0 \to I_p^{\circ}$ is min'l.

As each I^i is injective, $0 \to I_p^{\circ}$ is split exact.

$$\operatorname{Hom}_{R_p}(k(p), I_p^{i-1}) \xrightarrow{0} \operatorname{Hom}_{R_p}(k(p), I_p^{i}) \xrightarrow{0} \operatorname{Hom}_{R_p}(k(p), I_p^{i+1})$$

exact.

$\therefore \operatorname{Hom}_{R_p}(k(p), I_p^i) = 0$, $*$.

$$\begin{tikzcd} \therefore \; 0 \arrow[r] & \Gamma_I(M) \arrow[r] \arrow[d, equal] & \Gamma_I(I^0) \\ 0 \arrow[r] & M \arrow[r] & I^0 \end{tikzcd}$$ exact

$\Rightarrow H^i_I(M) = 0$ for $i > 0$.

4. Let R be Noeth, M a f.g. R-module.

$\text{depth}_I M = \min\{ i \mid H^i_I(M) \neq 0 \}$.

Pf: Induct on $\text{depth}_I M$.

$\text{depth}_I M = 0 \Rightarrow I \subseteq Z(M)$

$\Rightarrow I \subset p = (0:x),\ x \neq 0.$

$\Rightarrow Ix = 0$

$\Rightarrow H^0_I(M) \neq 0.$

Suppose $t = \text{depth}_I M > 0$. Then I contains a NZD on M, so $H^0_I(M) = 0$.

~~For if $H^0_I(M) \neq 0$ then $\exists\, u \in M,\ n \in \mathbb{N}$ s/t $uI^n = 0$.~~

Let $x \in I$ be a NZD on M.

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

$\operatorname{depth}_I M/xM = \operatorname{depth}_I M - 1. = t-1$

By induction, $H^i_I(M/xM) = 0$ for $i < t-1$

$H^{t-1}_I(M/xM) \neq 0$.

$$\underset{\overset{\|}{0}}{H^{i-1}_I(M/xM)} \longrightarrow H^i_I(M) \xrightarrow{x} H^i_I(M)$$

for $i-1 < t-1$.

$H^i_I(M)$ is killed by some power of x.

$\therefore\ H^i_I(M) = 0$ for $i < t$.

If $i = t$,

$$\underset{\overset{\|}{0}}{H^{t-1}_I(M)} \longrightarrow \underset{\neq 0}{H^{t-1}_I(M/xM)} \longrightarrow \underset{\neq 0.}{H^t_I(M)}$$

Corollary: $(R, \mathfrak{m})$ local.
R is CM $\Longleftrightarrow$ $H^i_{\mathfrak{m}}(R) = 0 \quad \forall\, i < \dim R$.

6/14

Corollary: $(R, \mathfrak{m})$ local

$$R \text{ is Gorenstein} \Longleftrightarrow H^i_{\mathfrak{m}}(R) = \begin{cases} 0, & i \neq \dim R \\ E_R(R/\mathfrak{m}), & i = \dim R. \end{cases}$$

Proof: Let $I^\bullet$ be a minimal injective resolution of R. By previous remarks, we have

$$H^0_{\mathfrak{m}}(I^i) = E^{\mu_i(R)} \quad \text{where } E = E_R(R/\mathfrak{m}).$$

$\Longrightarrow$:
As R is Gorenstein, $\mu_i(R) = 0$ if $i \neq d = \dim R$
$\mu_d(R) = 1$.

$$\therefore H^0_{\mathfrak{m}}(I^\bullet) = \overset{0}{0} \to 0 \to \cdots \to \overset{d-1}{0} \to \overset{d}{E} \to 0 \to 0$$

$$\therefore H^d_{\mathfrak{m}}(R) = E, \quad H^i_{\mathfrak{m}}(R) = 0 \quad \forall\, i \neq d.$$

$\Longleftarrow$: By the previous corollary, R is CM.
ETS $\mu_d(R) = 1$. Consider $H^0_{\mathfrak{m}}(I^\bullet)$:

$$0 \to E^{\mu_d(R)} \to E^{\mu_{d+1}(R)} \to \cdots$$

(Note: as R is CM, $\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \ \forall\, i < d$. $\therefore \mu_i(R) = 0 \ \forall\, i < d$)

By assumptions,

$$\begin{tikzcd}
0 \arrow[r] & H^d_m(R) \arrow[r] & E^{\mu_d(R)} \arrow[rr] \arrow[dr] & & E^{\mu_{d+1}(R)} \arrow[r] & \cdots \\
& & & C \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

is exact.

As $H^d_m(R) \cong E$, $E^{\mu_d(R)} \cong H^d_m(R) \oplus C$. Thus, $C \cong E^{\mu_d(R)-1}$. Hence

$\mu_d(R)=1 \iff C=0$.

Apply $\operatorname{Hom}_R(R/m, -)$:

(#)

$$\begin{tikzcd}
\operatorname{Hom}_R(R/m, E^{\mu_d(R)}) \arrow[rr] \arrow[dr] & & \operatorname{Hom}_R(R/m, E^{\mu_{d+1}(R)}) \\
& \operatorname{Hom}_R(R/m, C) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0
\end{tikzcd}$$

This map is still surjective, as the map $E^{\mu_d(R)} \to C$ splits.

In general, note that

$$\underset{\underset{(0 :_N m)}{\parallel}}{\operatorname{Hom}_R(R/m, N)} \cong \underset{\underset{(0 :_{H^0_m(N)} m)}{\parallel}}{\operatorname{Hom}_R(R/m, H^0_m(N))} \text{ naturally}.$$

Hence, the map

$$\begin{tikzcd}
\operatorname{Hom}_R(R/m, E^{\mu_d(R)}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Hom}_R(R/m, E^{\mu_{d+1}(R)}) \arrow[d, phantom, sloped, "\cong"] \\
\operatorname{Hom}_R(R/m, H^0_m(I^d)) \arrow[r] \arrow[d, phantom, sloped, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Hom}_R(R/m, H^0_m(I^{d+1})) \arrow[d, phantom, sloped, "\cong"] \\
\operatorname{Hom}_R(R/m, I^d) \arrow[r] & \operatorname{Hom}_R(R/m, I^{d+1})
\end{tikzcd}$$

This last map is zero as $I^\cdot$ is minimal.

∴ From the diagram #, $\operatorname{Hom}_R(R/m, C) = 0$.

But $\operatorname{Hom}_R(R/m, C) = \operatorname{Hom}_R(R/m, E^{\mu_d(R)-1})$

$= k^{\mu_d(R)-1}$.

∴ $\mu_d(R) = 1$ and R is Gorenstein. //

Prop: Let R be Noeth. Then for any ideal I of R and any R-module M,

$$H^i_I(M) \cong H^i_{\sqrt{I}}(M) \quad \forall\, i \geq 0.$$

Pf: ETS $H^0_I(M) = H^0_{\sqrt{I}}(M)$.
But $\exists\, n$ s/t $(\sqrt{I})^n \subseteq I \subseteq I$ as $\sqrt{I}$ is f.g. //

R Noeth.

<u>Prop:</u> Let S be a m.c. set, M an R-module, I an ideal. Then

$$H^i_{I}(M)_S \cong H^i_{I_S}(M_S) \quad \forall i.$$

<u>Proof:</u>

$H^i_I(M)_S$ is computed by taking an inj. resolution of M, applying $H^0_I(-)$, take homology, the localize.

As localization is flat, it commutes with taking homology. ∴ ETS

localization commutes with the functor $H^0_I(-)$:

i.e., is $H^0_I(M)_S = H^0_{I_S}(M_S)$

Clearly, $H^0_I(M)_S \subseteq H^0_{I_S}(M_S)$.

Suppose $(I_S)^n \cdot \left(\frac{m}{s}\right) = 0$,

As I is f.g., $\exists\ s' \in S$ s/t $s' I^n m = 0$.
$\Rightarrow s'm \in H^0_I(M) \Rightarrow \frac{m}{s} \in H^0_I(M)_S$.

Prop: Let $(R, \mathfrak{m})$ be a local ring, M a f.g. R-module. Then

$$H^i_{\mathfrak{m}}(M) \text{ are Artinian } \forall i.$$

Proof:

$$0 \to M \to I^{\circ} \quad \text{min'l inj. resol of } M.$$

But $H^0_{\mathfrak{m}}(I^i) \cong E_R(R/\mathfrak{m})^{\mu_i(M)}$

(as $H^0_{\mathfrak{m}}(E(R/p)) = 0$ if $p \neq \mathfrak{m}$).

$\mu_i(M) < \infty$ and $E_R(R/\mathfrak{m})$ is Artinian.

$\therefore$ $H^0_{\mathfrak{m}}(I^i)$ is Artinian, and $H^i_{\mathfrak{m}}(M)$ is a subquotient of $H^0_{\mathfrak{m}}(I^i)$.

Prop: Let I be an ideal, M an R-module. Then

$$H^i_I(M) \cong \varinjlim \operatorname{Ext}^i_R(R/I^n, M).$$

Pf: $\underline{\underline{i=0}}$: $\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n)$.

①

K. Kattche: "A Note on Factorial Rings" - Murthy (1964).

6/15 <u>Theorem:</u> Let A be a UFD which is a quotient of a RLR. TFAE

① A is CM

② A is Gorenstein

<u>From now on:</u> B is a RLR, $n = \dim B$.
$A = B/p$, $p \in \operatorname{Spec} B$, $r = \operatorname{ht} p$.

<u>Facts:</u> ① B is Gorenstein ($\therefore \omega_B \cong B$)

② A is Gorenstein

$$A \cong \operatorname{Ext}^r_B(A, B)$$

<u>reason:</u> see theorem 3.3.7 in Bruns - Herzog.

Let $B \to A$ local homomorphism of CM rings s/t A is a finite B-module.
If B has a canonical module the A has one,

$$\omega_A = \operatorname{Ext}^t_B(A, \omega_B) \quad , \ t = \dim B - \dim A.$$

Lemma 1: Let M be a CM B-module, $h = \mathrm{pd}_B M$.

Then $\mathrm{Ext}^i_B(M, B) = 0 \quad \forall\, i < h$.

and $M' = \mathrm{Ext}^h_B(M, B)$ is CM with $\mathrm{pd}_B M = h$.

Proof: See Prop 3.3.3 B-H.

Sketch: Induct on $t = \operatorname{depth} M = \dim M$.

$t=0$: $\lambda(M) < \infty$. Ok. (??!)

$t>0$:

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

Apply $\mathrm{Hom}_B(-, B)$ to the LES

$$\cdots \to \mathrm{Ext}^{i-1}_B(M, B) \to \mathrm{Ext}^i_B(M/xM, B) \to \mathrm{Ext}^i_B(M, B)$$

$$\xrightarrow{x} \mathrm{Ext}^i_B(M, B) \to \mathrm{Ext}^{i+1}_B(M/xM, B) \to \cdots$$

$\operatorname{depth} M/xM = t-1$. $\therefore\ \mathrm{pd}_B M/xM = h-1$.

Follows by induction.

Lemma 2: Let M be a f.g. B-module.
Then $p \in \operatorname{Ass} M \Rightarrow \operatorname{pd}_B M \geq \operatorname{ht} p$.

Proof: Since B is a RLR,

$$\operatorname{pd}_B M = \dim B - \operatorname{depth}_B M$$

$$\operatorname{ht} p = \dim B - \dim B/p$$

Thus,

$$\operatorname{pd}_B M \geq \operatorname{ht} p$$

$$\Leftrightarrow \operatorname{depth}_B M \leq \dim B/p$$

But if $p \in \operatorname{Ass} M$, this inequality holds. //

Lemma 3: Suppose $A = B/p$ is a CM ring.
Then

$M := \operatorname{Ext}^r_B(A,B) \cong A$ or an "unmixed" ht one ideal

recall "unmixed": I is unmixed if

every member of $\operatorname{Ass}_B B/I$ has the same height.

proof: Induction on $\ell = \dim A = \dim B/p = n-r$

$\ell = 0$: In this case $p = m_B$.

So $M = \mathrm{Ext}^n_B(B/m, B) \cong B/m = A$

$\ell > 0$: In this case $p \neq m_B$.
Let $\bar{q} = q/p \in \operatorname{Spec} A$, $p \subsetneq q \subsetneq m_B$.

We have

$$M_{\bar{q}} = \mathrm{Ext}^r_{B_q}(A_{\bar{q}}, B_{\bar{q}})$$

By induction,

$M_{\bar{q}}$ is a torsion-free $A_{\bar{q}}$-module of rank 1.

$\therefore$ $\bar{q} \notin \mathrm{Ass}_A M$. So $\mathrm{Ass}_A M \subseteq \{(0), \bar{m}\}$.

Since A is C-M,

depth $A = \dim A = \ell$.

$$\begin{aligned} \mathrm{pd}_B A &= \dim B - \operatorname{depth} A \\ &= \dim B - \dim A = n - \ell < \dim B \end{aligned}$$

By the lemma above,

$M = \mathrm{Ext}^r_B(A, B)$ is CM and $\mathrm{pd}_B M = r$

Hence, $\mathrm{depth}_A M = \mathrm{depth}_B M = \dim B - \mathrm{pd}_B M$

$= \dim B - r$

$> 0.$

$\therefore$ $\bar{m} \notin \mathrm{Ass}\, M$.

Hence, $\mathrm{Ass}\, M = \{(0)\}$.

Hence, M is torsion-free.

Now, $M_{(0)} = M_p$

$= \mathrm{Ext}^r_B(A, B)_p = \mathrm{Ext}^r_{B_p}(k(p), B_p) = k(p)$

So $\mathrm{rank}_A M = 1$. Thus $M \cong \bar{I}$, where $I \subseteq B$ is an ideal.

If $I = B$, then $M \cong A$. Done.

Assume I is proper.

We have the following s.e.s.:

ⓐ $0 \to p \to B \to A \to 0$

ⓑ $0 \to p \to I \to \underset{\bar{I} \cong M}{\underset{\shortparallel}{I/p}} \to 0$

ⓒ $0 \to I \to B \to B/I \to 0$

From ⓐ, $pd_B p = pd_A p - 1 = r-1$.

we already have $pd_B M = r$, so from ⓑ we get ~~$pd_B M \leq r$~~ $pd_B I \leq r$. (Hershoe lemma).

Then by ⓒ, we have $pd_B B/I \leq r+1$.

By lemma 2, if $q \in \text{Ass } B/I$ then $\text{ht } q \leq pd\, B/I \leq r+1$.

$\therefore$ I is unmixed of ht $r+1$.

Hence, $M \cong \bar{I} = I/p$ is unmixed of ht 1. //

Proof of Theorem: ② ⇒ ①: ✓

~~Theorem~~

① ⇒ ②: Write $A = B/p$ as always.

By Lemma 3,

$$\mathrm{Ext}^r_B(A,B) \cong A \text{ or } \bar{I}, \ \bar{I} \text{ unmixed of ht } 1.$$

If $\cong A$, then done.

If $\cong \bar{I}$, then note that in a UFD, ht 1 primes are principal.

Now, $\bar{I}$ has a primary decomp, but each ideal in the decomposition is principal.

(Fact: If q is primary to a prime ideal $p = (x)$, then q is principal.)

$\therefore$ $\bar{I}$ is principal.

$\bar{I} \cong A$ //

Tensor product of co-complexes:

6/16 Let C°, D° be two (co-) complexes.

Define

$(C\otimes_R D^\circ)^\circ$ by

$$(C\otimes_R D)^n := \bigoplus_{i+j=n} C^i\otimes_R D^j$$

and define ∂ on $C\otimes_R D$ as follows:

for $c\otimes d\in C^i\otimes D^j$, $\quad \partial(c\otimes d) = \partial c\otimes d + (-1)^i c\otimes\partial d$

Note: $\partial^2(c\otimes d) = \partial\left(\partial c\otimes d + (-1)^i c\otimes\partial d\right)$

$$= \underset{\overset{\shortparallel}{0}}{\partial^2 c\otimes d} + (-1)^{i+1}\partial c\otimes\partial d + (-1)^i\partial c\otimes\partial d + \underset{\overset{\shortparallel}{0}}{(-1)^{i+1} c\otimes\partial^2 d}$$

$$= 0.$$

Facts: ① $(C\otimes_R D)^\circ \cong (D\otimes_R C)^\circ$ (as complexes)

② $C\otimes(D\otimes E) \cong (C\otimes D)\otimes E$.

Pf: Exercise.

<u>Defn</u>: Let $\underline{x} = x_1, \ldots, x_n \in R$. Define the <u>Čech complex</u> on R wrt $x_1, \ldots, x_n$ by

$$C^\bullet(x_1; R) := \quad 0 \longrightarrow R \longrightarrow R_{x_1} \longrightarrow 0$$
$$r \longrightarrow \frac{r}{1}$$

$$C^\bullet(x_1, \ldots, x_n; R) := \quad C^\bullet(x_1, \ldots, x_{n-1}; R) \otimes_R C^\bullet(x_n; R)$$
$$= \bigotimes_{i=1}^{n} C^\bullet(x_i; R)$$

Let's compute $C^\bullet(x, y; R)$:

$$\left(0 \longrightarrow \overset{0}{R} \longrightarrow \overset{1}{R_x} \longrightarrow 0\right) \otimes \left(0 \longrightarrow \overset{0}{R} \longrightarrow \overset{1}{R_y} \longrightarrow 0\right)$$
$$r \longrightarrow \frac{r}{1} \qquad\qquad r \longrightarrow \frac{r}{1}$$

we get

$$0 \longrightarrow R \otimes R \longrightarrow R_x \otimes R \oplus R \otimes R_y \longrightarrow R_x \otimes R_y \longrightarrow 0$$
$$1 \otimes 1 \longrightarrow \frac{1}{1} \otimes 1 \oplus 1 \otimes \frac{1}{1}$$
$$\left(\frac{1}{1} \otimes 1, 0\right) \longrightarrow (-1)\, \frac{1}{1} \otimes \frac{1}{1}$$
$$\left(0, 1 \otimes \frac{1}{1}\right) \longrightarrow \frac{1}{1} \otimes \frac{1}{1}$$

Simplified:

$$0 \longrightarrow R \longrightarrow R_x \oplus R_y \longrightarrow R_{xy} \longrightarrow 0$$
$$1 \longrightarrow (1, 1)$$
$$(1, 0) \longrightarrow -1$$
$$(0, 1) \longrightarrow 1$$

In general, $C^\circ(\underline{x};R)$ looks like

$$0 \longrightarrow \overset{0}{R} \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_ix_j} \longrightarrow \cdots \longrightarrow R_{x_1\cdots x_n} \longrightarrow 0$$

$$1 \longmapsto (1,1,\dots,1)$$

where the differentials are the same as the ~~koszul~~ maps in the koszul co-complex, with 1's in place of x_i's.

If M is an R-module, define

$$C^\circ(\underline{x};M) = C^\circ(\underline{x};R) \otimes_R M$$

The i^{th} Čech cohomology of M is

$$H^i_{\underline{x}}(M) := H^i(C^\circ(\underline{x};M)).$$

Want to show $H^i_{\underline{x}}(M) = H^i_{(\underline{x})}(M)$.

<u>Lemma</u>: Let M be an R-module, $\underline{x}=x_1,\dots,x_n \in R$, $I=(\underline{x})$. Then $H^0_{\underline{x}}(M) \cong H^0_I(M)$.

<u>Pf</u>: $C^\circ(\underline{x};M)$ starts out

$$0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{i=1}^{n} M_{x_i}.$$

$m \in H^0_{\underline{x}}(M) \iff m \in \ker d_0$

$\iff \frac{m}{1} = 0 \text{ in } M_{x_i} \quad \forall i$

$\iff \exists\, t \geq 0 \text{ s/t } x_i^t m = 0 \quad \forall i$

$\iff \exists\, t \geq 0 \text{ s/t } I^t m = 0$

$\iff m \in H^0_I(M)$. //

<u>Proposition</u>: Suppose $0 \to L \to M \to N \to 0$ is a s.e.s. of R-modules, and $\underline{x} = x_1, \ldots, x_n \in R$. Then $\exists$ a natural l.e.s.

$$\cdots \to H^n_{\underline{x}}(L) \to H^n_{\underline{x}}(M) \to H^n_{\underline{x}}(N) \to H^{n+1}_{\underline{x}}(L) \to \cdots$$

<u>Proof</u>:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
C^\bullet(\underline{x};L): & 0 \arrow[r] & L \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & \oplus L_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & L_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
C^\bullet(\underline{x};M): & 0 \arrow[r] & M \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & \oplus M_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & M_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
C^\bullet(\underline{x};N): & 0 \arrow[r] & N \arrow[r] \arrow[d] & \oplus N_{x_i} \arrow[r] \arrow[d] & \cdots \arrow[r] & N_{x_1\cdots x_n} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & & 0 &
\end{tikzcd}$$

columns are exact as localization is exact.

∴ We have a s.e.s. of co-complexes

$$0 \to C^\cdot(\underline{x}; L) \to C^\cdot(x;M) \to C^\cdot(x;N) \to 0.$$

The l.e.s. now follows //

Prop: ~~Lemma~~ Let M be an R-module and $\underline{x} = x_1, \dots, x_n \in R$. Let $y \in R$. Then $\exists$ a l.e.s.

$$\cdots \to H^i_{\underline{x},y}(M) \longrightarrow H^i_{\underline{x}}(M) \xrightarrow{(-1)^i} H^i_{\underline{x}}(M)_y \longrightarrow H^{i+1}_{\underline{x},y}(M) \to \cdots$$

Proof: Let $C^\cdot = C^\cdot(\underline{x};M)$

$$C^\cdot(y) = C^\cdot(\underline{x},y;M) = C^\cdot(\underline{x};M) \otimes C^\cdot(y;R)$$

Now, $C^\cdot(y) = C^\cdot \otimes (0 \to \overset{0}{R} \to \overset{1}{R_y} \to 0)$

Hence, $C^\cdot(y)^n = C^{n-1} \otimes_R R_y \oplus C^n \otimes_R R \cong C^{n-1}_y \oplus_R C^n.$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^{n-1}_y \arrow[r, "a \mapsto (a,0)"] \arrow[d, "\partial_y"'] & C^{n-1}_y \oplus C^n \arrow[r, "(a,b) \mapsto b"] \arrow[d, shift right=4, "d_y"'] \arrow[d, "(-1)^n"] \arrow[d, shift left=4, "\partial"] & C^n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^n_y \arrow[r] & C^n_y \oplus C^{n+1} \arrow[r] & C^{n+1} \arrow[r] & 0
\end{tikzcd}$$

this commutes.

Hence, we have the s.e.s. of co-complexes:

$$0 \to C^\cdot_y[-1] \to C^\cdot(y) \to C^\cdot \to 0,$$

which gives the l.e.s.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{\underline{x}}(M)_y \arrow[r] & H^{i}_{\underline{x},y}(M) \arrow[r] & H^{i}_{\underline{x}}(M) \arrow[r, "\delta"] & H^{i}_{\underline{x}}(M)_y \arrow[r] & \cdots \\
& H^{i-1}(C^\cdot_y) \cong H^{i-1}(C^\cdot)_y \arrow[u] & & & &
\end{tikzcd}$$

where δ is the connecting homomorphism given by the snake lemma ~~from the~~ applied to the previous diagram. It is clear that $\delta = (-1)^n$. //

~~Lemma~~

<u>Corollary</u>: Let M be an R-module and $x_1, \ldots, x_n \in R$. Suppose some x_i acts as a unit on M. (i.e., M is an R_{x_i}-module). Then

$$H^i_{\underline{x}}(M) = 0 \text{ for all } i.$$

<u>Proof</u>: $i=0$: $H^i_{\underline{x}}(M) = H^0_{(\underline{x})}(M) = 0$ clear.

<u>$i>0$</u>: As $C^\cdot(\underline{x}; M) = \left[\bigotimes_{i=1}^{n} C_\cdot(x_i; R)\right] \otimes_R M$,

wlog, we can assume x_n acts as a unit on M.

Let $\underline{x}' = x_1, \dots, x_{n-1}$. By the Prop, ∃ a l.e.s.,

$$\cdots \longrightarrow H^i_{\underline{x}}(M) \longrightarrow H^i_{\underline{x}'}(M) \xrightarrow{(-1)^i} H^i_{\underline{x}'}(M)_{x_n} \longrightarrow \cdots$$

As M is an R_{x_n}-module, certainly each module in $C^{\cdot}(\underline{x}';M)$ is an R_{x_n}-module. Hence, the map

$$\begin{tikzcd} H^i_{\underline{x}'}(M) \arrow[r, "(-1)^i"] & H^i_{\underline{x}'}(M)_{x_n} \\ m \arrow[r] & (-1)^i\frac{m}{1} \end{tikzcd}$$

is an iso ∀ i.

∴ $H^i_{\underline{x}}(M) = 0$ ∀ i.//

<u>Prop:</u> Let R be a Noetherian ring, $\underline{x} = x_1, \dots, x_n \in R$. For any injective R-module I,

$$H^i_{\underline{x}}(I) = 0 \quad \forall\, i \geq 1.$$

<u>Proof:</u> As $I = \bigoplus E_R(R/p)$, ETS the Prop in the case $E = E_R(R/p)$ for some $p \in \operatorname{Spec} R$.

<u>Case 1:</u> $x_1, \dots, x_n \in p$. As every element in E is annihilated by a power of p, $E_{x_i} = 0$ ∀ i.

∴

$$C^{\cdot}(\underline{x};E) = \quad 0 \to E \to 0 \to 0 \to \cdots$$

So $H^0_{\underline{x}}(E) = E$ and $H^i_{\underline{x}}(E) = 0$ ∀ $i \geq 1$.

Case 2: Some $x_i \notin p$. Then x_i acts as a unit on E. Hence $H^i_{\underline{x}}(E)=0$ $\forall\, i \geq 1$ by the Corollary.//

Theorem: Let R be Noeth, $I=(x_1,\dots,x_n)$, M any R-module. Then $\exists$ natural isomorphisms

$$H^i_{\underline{x}}(M) \cong H^i_I(M) \quad \forall\, i \geq 0.$$

Proof: Induct on i. We've already shown this for $i=0$.

Suppose $i>0$:

Let $E=E_R(M)$ and consider the s.e.s.,

$$0 \to M \to E \to C \to 0.$$

Then $\exists$ l.e.s.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}_{\underline{x}}(E) \arrow[r] \arrow[d, "\cong"] \arrow[rd, phantom, "\circlearrowright"] & H^{i-1}_{\underline{x}}(C) \arrow[r] \arrow[d, "\cong"] & H^i_{\underline{x}}(M) \arrow[r] \arrow[d, dotted] & H^i_{\underline{x}}(E) \overset{\text{as } i>0}{=0} \\
\cdots \arrow[r] & H^{i-1}_I(E) \arrow[r] & H^{i-1}_I(C) \arrow[r] & H^i_I(M) \arrow[r] & H^i_I(E) =0
\end{tikzcd}$$

By the 5-lemma, $H^i_{\underline{x}}(M) \cong H^i_I(M)$.//

6/17

Defn: If I is an ideal of R, the *arithmetic rank* of I, $\operatorname{ara}(I)$, is defined by

$$\operatorname{ara}(I) = \min\{n \geq 0 \mid \exists\, a_1,.,a_n \text{ s/t } \sqrt{I} = \sqrt{(a_1,.,a_n)}\}$$

Corollary: Let I be an ideal of a Noeth ring R and M an R-module.
Then $H^i_I(M) = 0 \quad \forall\, i > \operatorname{ara}(I)$.

Pf: Let $t = \operatorname{ara}(I)$. Then $\exists\, a_1,.,a_t \in R$ s/t

$$\sqrt{(a_1,.,a_t)} = \sqrt{I}.$$

$$\begin{aligned} \text{Then } H^i_I(M) &\cong H^i_{\sqrt{I}}(M) \\ &\cong H^i_{\sqrt{(\underline{a})}}(M) \\ &\cong H^i_{(\underline{a})}(M) \\ &= H^i_{\underline{a}}(M) = 0 \text{ for } i > t. \end{aligned}$$

Defn: Let R be a CM local ring and P a prime of ht h. P is called a *set-theoretic complete intersection* if $\operatorname{ara}(P) = h$.

Corollary: Let R be CM, ht P = h and $H_P^{h+1}(R) \neq 0$.
Then P is not a s.t.c.i.

Example: Let $R = k[X_{ij}]_{\substack{1 \le i \le 2 \\ 1 \le j \le 3}}$, char k = 0.

Let $I = I_2((X_{ij}))$ = ideal of 2×2 minors of the matrix (X_{ij}).

I is prime of ht 2.
Hochster proved that $H_I^3(R) \neq 0$.
∴ I is not a s.t.c.i.

Lemma: Let R be a Noetherian ring, I an ideal.
For any integer $r \geq 1$, $\exists\, f_1, \ldots, f_r \in I$ s/t for any prime P s/t ht $P \le r-1$, $P \supseteq I \iff P \supseteq (f_1, \ldots, f_r)$.

Proof: Induct on r.
r=1: Choose $f_1 \in I \setminus \bigcup_{\substack{\text{ht} P_i = 0 \\ I \not\subseteq P_i}} P_i$ f_1 works.

r>1: By induction, we have $f_1, \ldots, f_{r-1} \in I$
s/t if ht $P \le r-2$, $P \supseteq (f_1, \ldots, f_{r-1}) \iff P \supseteq I$.

Choose $f_r \in I \setminus \bigcup_{\text{ht} P_i = r-1,\ P_i \text{ min over } (f_1, \ldots, f_{r-1}),\ I \not\subseteq P_i} P_i$

Claim: $(f_1, \dots, f_r)$ works.

Pf: Let $P \supseteq (f_1, \dots, f_r)$, ht $P \le r-1$.
If ht $P \le r-2$, done by inductive assumption.
Suppose ht $P = r-1$. If P is not minimal over $(f_1, \dots, f_{r-1})$, then $\exists$ prime Q, $P \supsetneq Q \supseteq (f_1, \dots, f_{r-1})$, so ht $Q \le r-2$. $\therefore$ $Q \supseteq I$.
If P is minimal over $(f_1, \dots, f_{r-1})$ then $I \subseteq P$ by choice of f_r.

Theorem: Let R be a Noetherian ring of dimension d, I an ideal of R.

Then $\operatorname{ara}(I) \le d+1$
If R is local, then $\operatorname{ara}(I) \le d$.

Proof: By the lemma, $\exists$ $f_1, \dots, f_{d+1} \in I$ s/t $\forall$ $P \in \operatorname{Spec} R$, $P \supseteq I \iff P \supseteq (f_1, \dots, f_{d+1})$.
Hence, $\sqrt{I} = \sqrt{(f_1, \dots, f_{d+1})}$.

If (R, m) is local, we know $\exists$ $f_1, \dots, f_d \in I$ s/t $\forall$ $p \neq m$, $p \supset (f_1, \dots, f_d) \iff p \supset I$.
Since m contains both ideals,

$$\sqrt{I} = \sqrt{(f_1, \dots, f_d)}.$$

Theorem: Let R be a Noetherian ring of dim d, I an ideal, M an R-module.
Then
$$H^i_I(M) = 0 \quad \text{for } i > d.$$

Proof: If R is local, then $\operatorname{ara}(I) \leq d$. ~~[illegible]~~. Otherwise, let $p \in \operatorname{spec} R$.
Then for $i > d$

$$H^i_I(M)_p \cong H^i_{IR_p}(M_p) = 0 \quad \text{as } \dim R_p \leq d.$$

$$\text{Hence, } H^i_I(M) = 0 \quad \forall\, i > d.$$

Theorem: (Change of Rings ~~principal~~ Principle)

~~Let S be an~~
Let S be an R-algebra, where R, S are Noetherian. Let I be an ideal of R and M an S-module.

$$\text{Then } \underset{\substack{\nearrow \\ \text{considered as} \\ \text{an } R\text{-module}}}{H^i_I(M)} \cong \underset{\substack{\uparrow \\ \text{considered} \\ \text{as an } S\text{-module.}}}{H^i_{IS}(M)} \quad \forall\, i$$

Proof: Let $I = (x_1, \dots, x_n)R$
Then

$$C^\circ_R(\underline{x}; M) = C^\circ(\underline{x}; R) \otimes_R M$$

↗ Čech complex of R-modules

$$= C^\circ(\underline{x}; R) \otimes_R (S \otimes_S M)$$

$$= C^\circ(\underline{x}; S) \otimes_S M$$

$$= C^\circ_S(\underline{x}; M)$$

← Čech complex of S-modules.

$$\therefore\ H^i_I(M) = H^i_{\underline{x}}(M)$$

$$= H^i_{\underline{x}S}(M)$$

$$= H^i_{IS}(M). //$$

Corollary: Let R be a Noetherian ring, I an ideal of R, M an finite R-module.
Then

$$H^i_I(M) = 0 \quad \forall\ i > \dim M.$$

Proof: $\dim M = \dim R/\operatorname{Ann}_R M$. M is an $R/\operatorname{Ann}_R M$-mod.

$$\therefore\ H^i_I(M) \cong H^i_{IS}(M) \quad \text{where } S = R/\operatorname{Ann}_R M.$$

Hence, $H^i_{IS}(M) = 0$ for $i > \dim S$.

Proposition: Let S be a flat R-algebra (R, S Noeth.), I an ideal of R, M an R-module. Then

$$H^i_I(M)\otimes_R S \cong H^i_{IS}(M\otimes_R S) \quad \forall\, i\geq 0.$$

Proof: $H^i_I(M)\otimes_R S = H^i(C^\circ(\underline{x};M))\otimes_R S$ where $I=(\underline{x})R$.

$$\cong H^i(C^\circ(\underline{x};M)\otimes_R S)$$

S is flat, so $-\otimes_R S$ is exact

$$\cong H^i(C^\circ(\underline{x}S;M\otimes_R S))$$

$$= H^i_{\underline{x}S}(M\otimes_R S)$$

$$= H^i_{IS}(M\otimes_R S).$$

Corollary: (R,m) local, I an ideal, M a finite R-module. Let $\hat{R}$ be the m-adic completion of R. Then

$$H^i_I(M)\otimes_R \hat{R} \cong H^i_{I\hat{R}}(M\otimes_R \hat{R})$$

$$\cong H^i_{I\hat{R}}(\hat{M}) \quad \forall\, i.$$

⑦

Proposition: Let M be an R-module (R Noeth), $I = (x_1, \dots, x_n)$ an ideal.

Then $H^n_I(M) \cong M_{x_1 \cdots x_n} \Big/ \sum_{i=1}^{n} M_{x_1 \cdots \hat{x}_i \cdots x_n}$.

Pf. $H^n_I(M)$ is the homology of

$$\bigoplus_i M_{x_1 \cdots \hat{x}_i \cdots x_n} \xrightarrow{\phi} M_{x_1 \cdots x_n} \longrightarrow 0$$

$$(0, \dots, \underset{\overset{\uparrow}{i^{\text{th}}\text{ spot}}}{w}, \dots, 0) \longrightarrow (-1)^i w$$

$\therefore$ im $\phi = \sum_i M_{x_1 \cdots \hat{x}_i \cdots x_n} \underset{\uparrow}{\subseteq} M_{x \cdots x_n}$

(by this containment, I mean the image of the natural map $M_{x_1 \cdots \hat{x}_i \cdots x_n} \to M_{x_1 \cdots x_n}$).

Hence, $H^n_I(M) = M_{x_1 \cdots x_n} \Big/ \sum M_{x_1 \cdots \hat{x}_i \cdots x_n}$.

Corollary: Let (R,m) be a Gorenstein local ring. Let $x_1,\ldots,x_d$ be an s.o.p. for R. Then.

$$E_R(R/m) \cong R_{x_1\cdots x_d} \Big/ \sum_i R_{x_1\cdots\hat{x}_i\cdots x_d}$$

Pf: $H^d_{(\underline{x})}(R) = H^d_m(R) \cong E_R(R/m)$.

Example: Let $R = k[x_1,\ldots,x_d]$, k a field.
$m = (x_1,\ldots,x_d)$.

Then $E_R(R/m) = R_{x_1\cdots x_d} \Big/ \sum R_{x_1\cdots\hat{x}_i\cdots x_d}$

$$\cong \bigoplus_{(i_1,\ldots,i_d)\in\mathbb{N}^d} k\, x_1^{-i_1}\cdots x_d^{-i_d}$$

(**Note:** we don't need to localize.

$E_R(R/m) \cong E_{R_m}(R_m/m)$ (already shown).

$H^i_m(R) \cong H^i_{mR_m}(R_m)$ (exercise).)

①

R. Karr : "Direct Limits".

6/18 Defn: Let D be a set, partially ordered by $\leq$. Then D is directed if $\alpha, \beta \in D$ $\exists$ $\gamma \in D$ s/t $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Examples: (1) $\mathbb{N}$

(2) The class of all subsets of a given set S, ordered by $\subseteq$.

Defn: Let $\mathcal{C}$ be any category, let D be a directed set. Then a directed system in $\mathcal{C}$ over D is a collection of objects M_α with $\alpha \in D$. ~~Suppose~~ such that whenever $\alpha \leq \beta$ $\exists$ $M_\beta^\alpha : M_\alpha \longrightarrow M_\beta$. satisfying

(1) $M_\alpha^\alpha = 1_{M_\alpha}$ $\forall$ α.

(2) whenever $\alpha \leq \beta \leq \gamma$

$$\begin{tikzcd}
M_\alpha \arrow[r] \arrow[dr] & M_\beta \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
 & M_\gamma
\end{tikzcd}$$

Examples: (1) Let $A \in \mathcal{C}$ and D any directed set.
Then A is a directed system as follows

$$A_\alpha = A, \quad A^\alpha_\beta = id_A, \quad \forall \alpha \le \beta.$$

(2) Let $D = \mathbb{N}$ ordered by $\le$.
Suppose M_n are R-modules
s/t $M_n \subseteq M_k$ if $n \le k$.
Then this is a directed system
where M^n_k is inclusion.

(3) Let M be an R-module and D ~~the class~~ any set
of submodules of M, ordered by inclusion.
Then D is directed system over "itself".

Def: Let M, N be directed systems over D.
(M consists of objects ($\{M_\alpha\}$) and maps ($\{M^\alpha_\beta\}$)).

Suppose $\forall \alpha \in D$ $\exists$ $F_\alpha : M_\alpha \to N_\alpha$ satisfying.

$$\begin{tikzcd} M_\alpha \arrow[r, "F_\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & N_\alpha \arrow[d] \\ M_\beta \arrow[r, "F_\beta"] & N_\beta \end{tikzcd} \qquad \forall \beta \geq \alpha.$$

We say $F : M \to N$ is a directed map over D.

Def: Let $\mathcal{M}$ be a direct system (in $\mathcal{C}$ over $\mathcal{D}$)

Suppose $M_\infty \in \mathcal{C}$ and consider M_∞ as a constant directed system (as in Example 1).

Also, suppose $M_\infty^\alpha : M_\alpha \longrightarrow M_\infty$ ~~such that~~ gives a map of directed systems.

i.e.,

$$\begin{tikzcd} M_\alpha \arrow[rr] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & & M_\infty \\ M_\beta \arrow[rru] & & \end{tikzcd} \quad \forall\ \alpha \le \beta.$$

We say M_∞ is the direct limit of $\mathcal{M}$, written $\varinjlim \mathcal{M}$, if for all other direct maps $\mathcal{M} \longrightarrow M_*$ with $M_* \in \mathcal{C}$ (a constant direct system) $\exists$ unique

$F : M_\infty \longrightarrow M_*$ s/t

$$\begin{tikzcd} M_\alpha \arrow[r] \arrow[d] \arrow[rd, phantom, "\circlearrowright"] & M_* \\ M_\infty \arrow[ru] & \end{tikzcd}$$

Examples:

(1) $D = \mathbb{N}$, $M_n \subseteq M_k$ $\forall$ $n \leq k$

(R-modules, submodules of a module N)

Then $\varinjlim M = \bigcup_{n \in \mathbb{N}} M_n$

(2) Suppose D is <u>any</u> set trivially ordered.
Suppose M_α is an R-module $\forall \alpha \in D$.

Then $\varinjlim M = \bigoplus_{\alpha \in D} M_\alpha$

Now we will fix $\mathcal{C}$ as the category of R-modules.

<u>Prop</u>: If M is a direct system then $\varinjlim M$ exists.

<u>Pf:</u> Let $X = \dot{\bigcup}_{\alpha \in D} M_\alpha$ be the disjoint union

as sets.

Define an equivalence relation on X by

$x \sim y \iff x \in M_\alpha, y \in M_\beta$ and

$\exists \gamma$ with $\alpha \leq \gamma, \beta \leq \gamma$ s/t

$M_\gamma^\alpha(x) = M_\gamma^\beta(y)$

We can put an R-module structure on X which respects this equivalence relation

$$x+y = M^{\alpha}_{\gamma}(x) + M^{\beta}_{\gamma}(y)$$

Let $M_\infty := X/\sim$

Define $M^{\alpha}_{\infty} : M_\alpha \longrightarrow M_\infty$

$x \longrightarrow [x]$ (equiv class of x).

Facts:

(1) If $x \in M_\infty$ then $\exists\, \alpha$ and $x_\alpha \in M_\alpha$ s/t

$M^{\alpha}_{\infty}(x_\alpha) = x$.

(2) If $M^{\alpha}_{\infty}(x) = 0$ then $\exists\, \beta \geq \alpha$ s/t

$M^{\alpha}_{\beta}(x) = 0$.

Prop: Suppose $F : M \to N$ is a direct map. Then $\exists$ unique $F_\infty : \varinjlim M \to \varinjlim N$

s/t $\forall \alpha$

$$\begin{tikzcd} M_\alpha \arrow[r, "F_\alpha"] \arrow[d] & N_\alpha \arrow[d] \\ M_\infty \arrow[r, "F_\infty"] & N_\alpha \end{tikzcd}$$

$N^{\alpha}_{\infty} F_\alpha = F_\infty M^{\alpha}_{\infty}$

Pf: omit.

THM: If $L \to M \to N$ is an exact sequence of directed systems then

$$\varinjlim L \to \varinjlim M \to \varinjlim N \text{ is exact.}$$

Pf:

Let $g : L \to M$, $f : M \to N$.

(1) $f_\infty g_\infty(x) = f_\infty g_\infty L^\alpha_\infty(\hat{x})$ some $\hat{x} \in L_\alpha$, some $\alpha \in D$.

$$= f_\infty M^\alpha_\infty g_\alpha(\hat{x})$$

$$= N^\alpha_\infty f_\alpha g_\alpha(\hat{x})$$

$$= N^\alpha_\infty(0) = 0.$$

(2) If $x \in \ker F_\infty$ then

$\exists\, \alpha \in D$ s/t $F_\infty M^\alpha_\infty(\hat{x}) = 0$.

So $0 = F_\infty M^\alpha_\infty(\hat{x}) = N^\alpha_\infty F_\alpha(\hat{x})$

By Fact 2 $\exists\, \beta \geq \alpha$ s/t

$N^\alpha_\beta F_\alpha(\hat{x}) = 0$.

Thus $F_\beta M^\alpha_\beta(\hat{x}) = 0$

so by exactness, $\exists\, y$ s/t

$$g_\beta(y) = M^\alpha_\beta(\hat{x}).$$

So

$$x = M^\beta_\infty M^\alpha_\beta(\hat{x}) = M^\beta_\infty g_\beta(y)$$

$$= g_\infty L^\beta_\infty(y) \in \operatorname{im} g_\infty.$$

<u>Theorem:</u> Let N be any R-module.
Let M be a direct system.
Then

$$\varinjlim (M_\alpha \otimes N) \cong (\varinjlim M_\alpha) \otimes N$$

<u>Pf:</u> (Sketch).

Note $L_\alpha = M_\alpha \otimes N$ is a direct system.

~~Call the~~ Let $L_\infty = \varinjlim L$.

<u>Claim:</u> $L_\infty \cong M_\infty \otimes N$

The maps $F_\alpha = M^\alpha_\infty \otimes \mathrm{id}_N$ gives a direct map

$$f : L \longrightarrow M_\infty \otimes N.$$

By universality, we get $f_\infty : L_\infty \longrightarrow M_\infty \otimes N$

Conversely,

let $g_\alpha : M_\alpha \times N \longrightarrow M_\alpha \otimes N$

By universality, we get $g_\infty : M_\infty \times N \longrightarrow L_\infty$

check g_∞ is R-bilinear and so gives a linear map

$$h : M_\infty \otimes N \longrightarrow L_\infty$$

Then $fh = 1 = hf$.

6/21 <u>Theorem</u>: Let $I \subset R$, M an R-module.

Then $H^i_I(M) \cong \varinjlim_n \operatorname{Ext}^i_R(R/I^n, M)$ $\forall i$.

<u>Pf:</u> First note that ~~the maps~~ $\operatorname{Ext}^i_R(-, M)$ applied

to $R/I^{n+2} \longrightarrow R/I^{n+1} \longrightarrow R/I^n \longrightarrow \cdots$

gives a directed system

$$\operatorname{Ext}^i_R(R/I^n, M) \longrightarrow \operatorname{Ext}^i_R(R/I^{n+1}, M) \longrightarrow \operatorname{Ext}^i_R(R/I^{n+2}, M) \longrightarrow \cdots$$

In the case $i=0$,

$$\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n) \quad \text{(naturally)}$$

So

$$\varinjlim \operatorname{Hom}_R(R/I^n, M) \cong \varinjlim (0 :_M I^n)$$

$$\cong \bigcup_n (0 :_M I^n) = H^0_I(M).$$

In general, let $E^\bullet$ be an inj. resol of M. Then

$$\varinjlim \mathrm{Ext}^i_R(R/I^n, M) = \varinjlim H^i(\mathrm{Hom}_R(R/I^n, E^\bullet))$$

$$\cong H^i(\varinjlim \mathrm{Hom}_R(R/I^n, E^\bullet))$$

$\varinjlim$ is exact

$$\cong H^i(H^0_I(E^\bullet))$$

$$= H^i_I(M).$$

Koszul Cohomology

Defn: Let $\underline{x} = x_1, \dots, x_n \in R$. Define the Koszul co-complex on R wrt $\underline{x}$ as follows:

$n=1$: $$K^\bullet(x_1; R) := \quad 0 \to \overset{0}{R} \xrightarrow{x_1} \overset{1}{R} \to 0$$

$n>1$: $$K^\bullet(\underline{x}; R) := K^\bullet(x_1, \dots, x_{n-1}; R) \otimes K^\bullet(x_n; R)$$

$$= \bigotimes_{i=1}^{n} K^\bullet(x_i; R)$$

looks like

$$0 \to \underset{1 \to (x_1, \dots, x_n)}{\overset{0}{R}} \to \overset{1}{R^n} \to R^{\binom{n}{2}} \to \cdots \to \underset{e_i \to \pm x_i}{R^n \to \overset{n}{R}} \to 0$$

This is essentially the same as $K_\circ(\underline{x};R)$, the koszul complex, except it is written as a co-complex and the signs in the maps differ.

If M is an R-module, define the koszul co-complex on M wrt to $\underline{x}$ by

$$K^\circ(\underline{x};M) = K^\circ(\underline{x};R)\otimes_R M.$$

The i^{th} *koszul cohomology* on M wrt $\underline{x}$ is

$$H^i(\underline{x};M) = H^i(K^\cdot(\underline{x};M)).$$

In the same way as for the koszul complex, one can prove the following

Prop: Let $\underline{x} = x_1,\dots,x_n \in R$, M an R-module. Then

a) $H^0(\underline{x};M) \cong (0:_M(\underline{x}))$

b) $H^n(\underline{x};M) \cong M/(\underline{x})M$

c) If $x_1,\dots,x_n$ is an M-regular seq, then

$$H^i(\underline{x};M) = 0 \quad \forall \ i<n.$$

<u>Defn:</u> Let $M = \{M_\alpha\}$, $N = \{N_\alpha\}$ be directed systems of R-modules. Define a directed system $M \otimes_R N$ by $(M \otimes_R N)_\alpha = M_\alpha \otimes N_\alpha$

and $M_\alpha \otimes N_\alpha \xrightarrow{M^\alpha_\beta \otimes N^\alpha_\beta} M_\beta \otimes N_\beta$ for $\alpha \leq \beta$.

<u>Lemma:</u> $\varinjlim (M_\alpha \otimes N_\alpha) \cong \varinjlim M_\alpha \otimes \varinjlim N_\alpha$.

<u>Pf:</u> Similar to the proof in the case N is an R-module, which Ryan sketched in class.

<u>Defn:</u> Let $\{C^\cdot_\alpha\}$ be a directed system of cocomplexes of R-modules.

i.e.
$$\begin{tikzcd}
\cdots \arrow[r] & C^n_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C^{n+1}_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C^{n+2}_\alpha \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^n_\beta \arrow[r] & C^{n+1}_\beta \arrow[r] & C^{n+2}_\beta \arrow[r] & \cdots
\end{tikzcd}$$

for $\alpha \leq \beta$.

Then $\varinjlim C^\cdot_\alpha$ is a co-complex:

$$\cdots \longrightarrow \varinjlim C^n_\alpha \longrightarrow \varinjlim C^{n+1}_\alpha \longrightarrow \varinjlim C^{n+2}_\alpha \longrightarrow \cdots$$

<u>Defn:</u> Let $C^\cdot, D^\cdot$ be directed systems of co-complexes of R-modules. Define a directed system $C^\cdot \otimes_R D^\cdot$ by

$$\begin{tikzcd}
(C^\cdot \otimes_R D^\cdot)^n_\alpha \arrow[d] & = & \sum_{i+j=n} C^i_\alpha \otimes D^j_\alpha \arrow[d, "(C^i)^\alpha_\beta \otimes (D^j)^\alpha_\beta"] \\
(C \otimes_R D)^{n+1}_\beta & = & \sum_{i+j=n} C^i_\beta \otimes D^j_\beta .
\end{tikzcd}$$

<u>Fact:</u> $\varinjlim (C^\cdot \otimes D^\cdot)_\alpha \cong (\varinjlim C^\cdot_\alpha) \otimes (\varinjlim D^\cdot_\alpha)$.

<u>Pf:</u> Exercise.

<u>Lemma:</u> Let $x \in R$. Then

$$\varinjlim (R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots) \cong R_x .$$

<u>Pf:</u> For $n \geq 0$, let $R(n) = R$. Map $R(n) \longrightarrow R_x$ by

$$r \longrightarrow \frac{r}{x^n}$$

Then
$$\begin{tikzcd}
R(n) \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowright"] & R_x . \\
R(n+1) \arrow[ur] &
\end{tikzcd}$$

By the universal property, $\exists$ an R-map $\phi : \varinjlim R(n) \longrightarrow R_x$

ϕ is clearly onto, as if $r \in R(n)$ then $\phi(\tilde{r}) = \frac{r}{x^n}$ (where $\tilde{r}$ is the equiv. class of r in the direct limit.) Suppose $\phi(\tilde{r}) = 0$ where $r \in R(n)$. Then $\frac{r}{x^n} = 0 \Rightarrow \exists\ l$ s/t $x^l \cdot r = 0$. But $\tilde{r} = \underset{\in R(n+l)}{\widetilde{x^l r}} = \tilde{0}$. $\therefore$ ϕ is an $\cong$.

Corollary: $\varinjlim (M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots) \cong M_x$.

Pf: $\varinjlim (M \xrightarrow{x} M \xrightarrow{x} \cdots) = \varinjlim (R \xrightarrow{x} R \xrightarrow{x} \cdots) \otimes_R M$

$$\cong R_x \otimes M$$
$$\cong M_x.$$

Defn: Let $\underline{x} = x_1, \ldots, x_n \in R$, M an R-module. Define a directed system $K^\bullet(\underline{x}^t; M)$ as follows:

$\underline{n=1}$: $K^\bullet(x^t; M)$:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "x"] \arrow[d, "="] & M \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & M \arrow[r, "x^2"] \arrow[d, "="] & M \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & M \arrow[r, "x^3"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots &
\end{tikzcd}$$

<u>n>1:</u> $K^\bullet(\underline{x}^t; M) := K^\bullet(x_1^t, \dots, x_{n-1}^t; M) \otimes K^\bullet(x_n^t; R)$.

<u>Theorem</u>. $\varinjlim K^\bullet(\underline{x}^t; M) \cong C^\bullet(\underline{x}; M)$ ← Čech complex.

<u>Proof:</u>

<u>n=1:</u> clearly, $\varinjlim (M \xrightarrow{=} M \xrightarrow{=} M \xrightarrow{=} \;) \cong M$

By the corollary, $\varinjlim (M \xrightarrow{x} M \xrightarrow{x} M \to \cdots) \cong M_x$. One easily checks that the induced map on direct limits is

$$\begin{aligned} M &\longrightarrow M_x \\ m &\longrightarrow \tfrac{m}{1} \end{aligned}$$

<u>n>1:</u>

$$\begin{aligned} \varinjlim K^\bullet(\underline{x}^t; M) &= \varinjlim \left(K^\bullet(x_1^t, \dots, x_{n-1}^t; M) \otimes_R K^\bullet(x_n^t; R) \right) \\ &= \left(\varinjlim K^\bullet(x_1^t, \dots, x_{n-1}^t; M) \right) \otimes_R \left(\varinjlim K^\bullet(x_n^t; R) \right) \\ &\cong C^\bullet(x_1, \dots, x_{n-1}; M) \otimes C^\bullet(x_n; R) \\ &= C^\bullet(\underline{x}; M). \end{aligned}$$

<u>Theorem</u>: Let R be Noeth, $I = (\underline{x})R$, M an R-module. Then

$$H^i_I(M) \cong \varinjlim H^i(\underline{x}^t; M)$$

<u>Proof</u>:

$$H^i_I(M) \cong H^i_{\underline{x}}(M)$$

$$\cong H^i(C^\cdot(\underline{x}; M))$$

$$\cong H^i(\varinjlim K^\cdot(\underline{x}^t; M))$$

$$\cong \varinjlim H^i(K^\cdot(\underline{x}^t; M)) \quad (\varinjlim \text{ is exact})$$

$$= \varinjlim H^i(\underline{x}^t; M).$$

<u>Corollary</u>: Let R be Noeth, $I = (x_1, \ldots, x_n)R$, M an R-module. Then

$$H^n_I(M) \cong \varinjlim M/(x_1^t, \ldots, x_n^t)M.$$

$$\left(\text{where } M/(x_1^t, \ldots, x_n^t)M \xrightarrow{x_1 \cdots x_n} M/(x_1^{t+1}, \ldots, x_n^{t+1})M\right).$$

<u>Remark</u>: Let $\{I_n\}, \{J_n\}$ be two decreasing chains of ideals. We say the chains are <u>cofinal</u> if $\forall\, n\ \exists\, k$ s/t $J_k \subseteq I_n$, and $\forall\, m\ \exists\, l$ s/t $I_l \subseteq J_m$.

If $\{I_n\}$ is a descending chain of ideals cofinal with $\{I^n\}$ then

$$H^0_I(M) = \bigcup_n (0 :_M I_n) = \varinjlim \operatorname{Hom}_R(R/I_n, M)$$

The same proof given before will ~~go~~ yield

$$H^i_I(M) = \varinjlim \operatorname{Ext}^i_R(R/I_n, M).$$

<u>Theorem</u>: (Mayer-Vietoris sequence) Let R be a Noeth ring, $I, J \subseteq R$, M an R-module. Then $\exists$ a natural l.e.s.

$$0 \to H^0_{I+J}(M) \to H^0_I(M) \oplus H^0_J(M) \to H^0_{I\cap J}(M) \to \cdots$$

$$\cdots \to H^i_{I+J}(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I\cap J}(M) \to \cdots$$

<u>Proof</u>: $\forall\, n\ \exists$ a s.e.s.

$$0 \to R/I^n \cap J^n \to R/I^n \oplus R/J^n \to R/I^n + J^n \to 0$$

Apply $\mathrm{Hom}_R(-, M)$ to get a l.e.s.

$$\cdots \to \mathrm{Ext}^i_R(R/I^n+J^n, M) \to \mathrm{Ext}^i_R(R/I^n \oplus R/J^n, M) \to \mathrm{Ext}^i_R(R/I^n \cap J^n, M) \to \cdots$$

This forms a directed system of l.e.s's.

Take direct limits. ETS $\{I^n+J^n\}$ is cofinal with $\{(I+J)^n\}$ and $\{I^n \cap J^n\}$ is cofinal with $\{(I \cap J)^n\}$.

$$I^n+J^n \subseteq (I+J)^n \text{ and } (I+J)^{2n} \subseteq I^n+J^n.$$

Now, $(I \cap J)^n \subseteq I^n \cap J^n$.

By the Artin-Rees lemma, $\exists$ $k = k(n)$ s/t $\forall$ $m \geq k$,

$$I^m \cap J^n = I^{m-k}(J^k \cap J^n) \subseteq I^{m-k}J^n.$$

$\therefore$ For $m \geq n+k$, $I^m \cap J^m \subseteq I^m \cap J^n \subseteq I^{m-k}J^n \subseteq I^nJ^n$
$\subseteq (I \cap J)^n$. //

Prop: (Hartshorne). Let (R,m) be a local ring s/t depth $R \geq 2$. Then $U = \operatorname{Spec}R - \{m\}$ is ~~open~~ connected.

Proof: Assume U is disconnected. Then $\exists$ clopen sets $V(I)\cap U \neq \emptyset$, $V(J)\cap U \neq \emptyset$ s/t (1) $(V(I)\cap U) \cup (V(J)\cap U) = U$ and (2) $V(I)\cap V(J)\cap U = \emptyset$.

(1) $\Longleftrightarrow$ $\sqrt{I\cap J} \subseteq \bigcap_{\substack{p\in\operatorname{Spec}R\\ p\neq m}} p = \sqrt{0}$ $\Longleftrightarrow$ $I\cap J$ nilpotent

(2) $\Longleftrightarrow$ $\sqrt{I+J} = m$. (as I and J must be proper).

Together with $V(I)\cap U \neq 0$ and $V(J)\cap U \neq \emptyset$, we have neither I nor J is m-primary or nilpotent.

By Mayer-Vietoris,

$$0 \to H^0_{I+J}(R) \to H^0_I(R)\oplus H^0_J(R) \to H^0_{I\cap J}(R) \to H^1_{I+J}(R)$$

Now, $\sqrt{I+J} = m$ and depth $R \geq 2$, so

$$H^0_{I+J}(R) = H^1_{I+J}(R) = 0.$$

Also, $H^0_{I\cap J}(R) = R$ as $I\cap J$ is nilpotent.

$\therefore$ $R \cong H^0_I(R)\oplus H^0_J(R)$. As R is local, R is indec.

$\therefore$ Say, $H^0_I(R) \cong R \Rightarrow H^0_I(R)$ is gen by a NZD $\Rightarrow$ I is nilp. $\ast$.

Let's recall some facts about canonical modules:

6/23 For reference, see Bruns-Herzog.

<u>Defn:</u> Let $(R, \mathfrak{m})$ be a CM local ring.
A finite R-module C is a <u>canonical module</u> of R if

(1) C is maximal CM (i.e., $\operatorname{depth} C = \dim R$).
(2) C has type 1 (i.e., $\mu_d(C) = 1$)
(3) $\operatorname{id}_R C < \infty$.

<u>Facts:</u> ① C is unique up to isomorphism.
We write ω_R for the canonical module of R.

② ω_R exists $\Longleftrightarrow$ R is the homomorphic image of a Gorenstein ring.

$\therefore$ Complete CM local rings always have canonical modules.

③ If x is a NZD on R then $\omega_{R/xR} \cong \omega_R / x\omega_R$

④ If $p \in \operatorname{Spec} R$, $(\omega_R)_p \cong \omega_{R_p}$

⑤ If $\hat{R}$ is the completion of R then
$\omega_{\hat{R}} \cong \widehat{\omega_R}$

⑥ R is Gorenstein $\Longleftrightarrow$ $\omega_R \cong R$.

Lemma: (Flat resolution lemma)

Let R be a ring, M, N R-modules and $F_\bullet$ a flat resolution of M.

That is, each F_i is a flat R-module and

$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is exact.

Then $\mathrm{Tor}_i^R(M,N) \cong H_i(F_\bullet \otimes_R N)$ $\forall\, i \geq 0$.

Proof:

Induct on i.

$i=0$: as $- \otimes_R N$ is right exact,

$$F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0 \text{ is exact,}$$

so $H_0(F_\bullet \otimes_R N) = M \otimes_R N = \mathrm{Tor}_0^R(M,N)$.

$i>0$: Let $K_0 = \ker(F_0 \to M)$.

Then $0 \to K_0 \to F_0 \to M \to 0$ is exact.

As F_0 is flat, $\mathrm{Tor}_i^R(F_0,N) = 0$ $\forall\, i \geq 1$. $\therefore$

$0 \to \mathrm{Tor}_1^R(M,N) \to K_0 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$ exact and $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i-1}^R(K_0,N) \quad \forall\, i \geq 2.$

$\underline{i=1}$: We have $\mathrm{Tor}_1^R(M,N) = \ker(K_0 \otimes N \to F_0 \otimes N)$

But from the diagram

$$\begin{tikzcd}
F_2 \otimes N \arrow[r] \arrow[rrrdd, dash, bend right, "\text{exact}"'] & F_1 \otimes N \arrow[rr] \arrow[rd] & & F_0 \otimes N \\
& & K_0 \otimes N \arrow[ru] \arrow[rd, "\text{onto}"] & \\
& & & 0
\end{tikzcd}$$

$\ker(K_0 \otimes N \to F_0 \otimes N)$

$\cong \ \ker\left({}^{F_1 \otimes N}/_{\mathrm{im}(F_2 \otimes N)} \longrightarrow F_0 \otimes N \right)$

$= \ H_1(F_\bullet \otimes_R N).$

$\underline{i>1}$: Use the isos $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i-1}^R(K_0,N)$ $\forall\, i \geq 2$ and that

$\to F_2 \to F_1 \to K_0 \to 0$ is a flat resolution

of K_0. //

Theorem (Local Duality)
Let (R, m) be a complete CM local ring of dimension d. Then $\forall$ f.g. R-modules M,

$$\mathrm{Ext}^{d-i}_R(M, \omega_R) \cong H^i_m(M)^\vee$$

$$\text{and} \quad \mathrm{Ext}^{d-i}_R(M, \omega_R)^\vee \cong H^i_m(M) \quad \text{for all } i,$$

where $^\vee = \mathrm{Hom}_R(-, E_R(R/m))$.

Proof: We'll prove the first iso. The 2^{nd} iso follows from the first by Matlis Duality (as $\mathrm{Ext}^{d-i}_R(M, \omega_R)$ is f.g. and $H^i_m(M)$ is Artinian.)

~~Let $K = H^d_m(R)$~~
Let $x_1, \ldots, x_d$ be an s.o.p. for R.

Then $C^\cdot(\underline{x}; R)$:

$$0 \to R \to \oplus R_{x_i} \to \cdots \to R_{x_1 \cdots x_d} \to 0$$

The homology at the i^{th} place is $H^i_{(\underline{x})}(R) = H^i_m(R)$.

As R is CM, $H^i_m(R) = 0 \ \forall\, i < d$.

$$\therefore \ 0 \to R \to \oplus R_{x_i} \to \cdots \to R_{x_1 \cdots x_d} \to H^d_m(R) \to 0$$

is exact.

⑤

Hence, $F_\circ = C^\cdot(\underline{x};R)$ is a flat resolution of $H^d_m(R)$.
(Let $F_i = C^{d-i}$, to make $F_\circ$ a complex.)

Now $H^i_m(M) = H^i(C^\cdot(\underline{x};R) \otimes_R M)$

$$= H_{d-i}(F_\circ \otimes_R M)$$

$$\cong \mathrm{Tor}^R_{d-i}(H^d_m(R), M)$$

~~[illegible]~~

Compute this Tor using a free resolution $G_\circ$ of M:

Then $H^i_m(M) = H_{d-i}(G_\circ \otimes_R H^d_m(R))$.

For all i,

$\therefore \quad H^i_m(M)^\vee = H_{d-i}(G_\circ \otimes_R H^d_m(R))^\vee$

as $^\vee$ is an exact functor $- - - - \to$ $\cong H^{d-i}((G_\circ \otimes_R H^d_m(R))^\vee)$

$$= H^{d-i}(\mathrm{Hom}_R(G_\circ \otimes H^d_m(R), E))$$

Hom, ⊗ adjointness $- - - \to$ $\cong H^{d-i}(\mathrm{Hom}_R(G_\circ, H^d_m(R)^\vee))$

$$= \mathrm{Ext}^{d-i}_R(M, H^d_m(R)^\vee).$$

for all i.

ETS $\omega_R \cong H^d_m(R)^\vee$. $H^d_m(R)^\vee$ is f.g. by Matlis Duality

Note: this isomorphism is true for all i, including $i<0$ and $i>d$.

$\therefore$ $\operatorname{Ext}^i_R(M, H^d_{\mathfrak m}(R)^\vee) = 0$ for $i>d$ and all ~~R-modules~~ finite R-modules M.

Hence, $\operatorname{Ext}^i_R(R/\mathfrak p, H^d_{\mathfrak m}(R)^\vee) = 0 \quad \forall \underset{i>d}{\mathfrak p \in \operatorname{spec} R}$

$$\Longrightarrow \mu_i(\mathfrak p, H^d_{\mathfrak m}(R)^\vee) = 0 \quad \forall\, i>d,\ \mathfrak p \in \operatorname{spec} R.$$

$$\Longrightarrow \operatorname{id}_R H^d_{\mathfrak m}(R)^\vee < \infty.$$

Also,
$$\operatorname{Ext}^i_R(R/\mathfrak m, H^d_{\mathfrak m}(R)^\vee) = H^{d-i}_{\mathfrak m}(R/\mathfrak m)^\vee = \begin{cases} 0, & \text{if } 0 \le i < d \\ R/\mathfrak m, & \text{if } i = d. \end{cases}$$

$\therefore$ $\operatorname{depth} H^d_{\mathfrak m}(R)^\vee = d$

and $\mu_d(H^d_{\mathfrak m}(R)^\vee) = 1.$

Hence, $\omega_R \cong H^d_{\mathfrak m}(R)^\vee$.

Remarks: Let (R, m) be a local ring and M an R-module. Let $\hat{R}$ denote the m-adic completion of R, $E = E_R(R/m) = E_{\hat{R}}(\hat{R}/\hat{m})$.

① $\mathrm{Hom}_{\hat{R}}(M \otimes_R \hat{R}, E) \cong \mathrm{Hom}_R(M, E)$

Pf: By Hom-⊗ adjointness,

$$\mathrm{Hom}_{\hat{R}}(M \otimes_R \hat{R}, E) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\hat{R}}(\hat{R}, E))$$

$$\cong \mathrm{Hom}_R(M, E)$$

② If M is Artinian then M is naturally an $\hat{R}$-module and $M \otimes_R \hat{R} \cong M$.

Pf: Exercise.

③ If M is a f.g. R-module,

$$H^i_m(M) \cong H^i_{m\hat{R}}(\hat{M}). \quad \forall i$$

Pf: We've seen that $H^i_{m\hat{R}}(\hat{M}) = H^i_m(M) \otimes_R \hat{R}$ and that $H^i_m(M)$ is Artinian.

<u>Theorem</u>: (Version of local duality for non-complete rings) Let (R, m) be a d-dimensional CM local ring which is the homomorphic image of a Gorenstein ring. Let ω_R be the canonical module of R. Then for all f.g. R-modules M and all i,

$$\mathrm{Ext}^{d-i}_R(M, \omega_R)^\vee \cong H^i_m(M)$$

<u>Proof</u>:

$$\mathrm{Ext}^{d-i}_R(M, \omega_R)^\vee = \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(M, \omega_R), E)$$

Remark ① $\longrightarrow$ $\cong \mathrm{Hom}_{\hat{R}}(\mathrm{Ext}^{d-i}_R(M, \omega_R) \otimes_R \hat{R}, E)$

$\widehat{\omega_R} = \omega_{\hat{R}}$ $\longrightarrow$ $\cong \mathrm{Hom}_{\hat{R}}(\mathrm{Ext}^{d-i}_{\hat{R}}(\hat{M}, \omega_{\hat{R}}), E)$

by the complete case of local duality $\longrightarrow$ $\cong H^i_{m\hat{R}}(\hat{M})$

$\cong H^i_m(M)$

Remark ③.

<u>Remark</u>: Let $(R, \mathfrak{m})$ be local CM ring which has a canonical module. Let K be a f.g. R-module. If $\widehat{K} \cong \widehat{\omega_R} (= \omega_{\widehat{R}})$ then $K \cong \omega_R$.

<u>Pf</u>: Exercise (see Bruns-Herzog Prop 3.3.14, for instance).

<u>Proposition</u>: Let $(R, \mathfrak{m})$ be a CM local ring which has a canonical module. Write $R \cong S/I$, where $(S, \mathfrak{n})$ is a Gorenstein local ring and let $\operatorname{ht} I = g$. Then $\omega_R \cong \operatorname{Ext}^g_S(R, S)$

<u>Proof</u>: By the Remark, ETS

$$\operatorname{Ext}^g_S(R,S) \otimes_R \widehat{R} \cong \omega_{\widehat{R}} = H^d_{\mathfrak{m}\widehat{R}}(\widehat{R})^\vee .$$

∴ we may assume R and S are complete.

$$\begin{aligned}
\text{Now, } \operatorname{Ext}^g_S(R,S)^\vee &= \operatorname{Hom}_R(\operatorname{Ext}^g_S(R,S), E_R(k)) \\
&= \operatorname{Hom}_R(\operatorname{Ext}^g_S(R,S), \operatorname{Hom}_S(R, E_S(k))) \\
&= \operatorname{Hom}_S(\operatorname{Ext}^g_S(R,S) \otimes_R R, E_S(k)) \\
&= \operatorname{Hom}_S(\operatorname{Ext}^g_S(R,S), E_S(k))
\end{aligned}$$

$= H_{\mathfrak{n}}^{\dim S - g}(R)$ (by local duality and as $\omega_S \cong S$)

$= H_{\mathfrak{m}}^{\dim R}(R).$ ([illegible] by the change of rings principal.)

By Matlis Duality, $\mathrm{Ext}_S^g(R,S) \cong H_{\mathfrak{m}}^{\dim R}(R)^{\vee} \cong \omega_R$ //

M. Arnavut

6/24 <u>Theorem</u>: (Chevalley's Theorem)

Let (R, m) be a complete local ring.
If I_n $(n=1,2,\ldots)$ are ideals of R s/t
$I_n \supseteq I_{n+1}$ $\forall n$ and $\bigcap_n I_n = 0$.

Then for any $n \in \mathbb{N}$ $\exists$ $s = s(n) \in \mathbb{N}$ s/t
$I_s \subseteq m^n$.

<u>Proof</u>: By contradiction: Assume $\exists$ $r \in \mathbb{N}$ s/t
$I_s \nsubseteq m^r$ for any $s \in \mathbb{N}$.

Then for any $n \geq r$, $I_s \nsubseteq m^n$, all s.
Now
$\dim R/m^n = 0$, so R/m^n is Artinian.

Thus, $\exists$ $t(n) \in \mathbb{N}$ s/t $I_{t(n)} + m^n = I_s + m^n$
$\forall$ $s > t(n)$.

Now, we may assume $t(n) < t(n+1)$ for any $n \geq r$.

Then, $I_{t(n)} \subseteq I_{t(n)} + m^n = \cancel{I_{t(n)} + m^n} = I_{t(n+1)} + m^n$

$\therefore$ For any $x_n \in I_{t(n)}$, $\exists$ $x_{n+1} \in I_{t(n+1)}$

s/t $x_n - x_{n+1} \in m^n$.

Start with $x_r \in I_{t(r)} \setminus \mathfrak{m}^r$

Then we have a seq $(x_n)_{n \geq r}$ s/t $x_n - x_{n+1} \in \mathfrak{m}^n$.

Clearly, (x_n) is a Cauchy sequence.
As R is complete, let $x^* = \lim_{n \to \infty} x_n$.

Now, $x_n, x_{n+1}, \ldots \in I_{t(n)}$

As ideals are closed in the $\mathfrak{m}$-adic topology,

$x^* \in I_{t(n)}$.

$\therefore \; x^* \in \bigcap_{n \geq r} I_{t(n)} = 0$.

On the other hand, $x_n - x_r \in \mathfrak{m}^r$ for all $n \geq r$.

$\therefore \; x^* - x_r \in \mathfrak{m}^r$

(As $\exists \, n \geq r$ s/t $x^* - x_n \in \mathfrak{m}^r$

Then $(x^* - x_n) + (x_n - x_r) \in \mathfrak{m}^r$.

$\therefore \; x_r \in \mathfrak{m}^r$, $\divideontimes$. //

<u>Recall</u>: If R is Noetherian then direct sums of injectives are injective.

<u>Theorem</u>: (Bass) Let R be a ring such that ~~every~~ direct sums of injectives is injective. Then R is Noetherian.

<u>Proof</u>: By contradiction. Then we will show there is an ideal I and a map from I to a sum of injective modules that cannot be extended.

There exists a strictly increasing sequence of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

Let $I = \bigcup_{n=1}^{\infty} I_n$.

Note that $I/I_n \neq 0 \quad \forall n$.

Embed I/I_n in an injective R-module E_n.

We claim: $\bigoplus_{n\geq 1} E_n$ is not injective.

Let $\pi_n : I \longrightarrow I/I_n$ be the nat. map

For each $a \in I$, $\pi_n(a) = 0$ for large n.

So the map

$$f : I \longrightarrow \bigoplus E_n \qquad \text{is well-defined.}$$
$$a \longrightarrow (\pi_n(a))$$

(consider $\pi_n(a) \in E_n$).

Suppose $\exists$ a map g making the diagram

$$\begin{tikzcd}
 & \bigoplus E_n & \\
0 \arrow[r] & I \arrow[u, "f"] \arrow[r, "\subseteq"'] & R \arrow[ul, dashed, "g"']
\end{tikzcd}$$

Say $g(1) = (x_n) \in \bigoplus E_n$.

~~Choose $m \in \mathbb{N}$~~

For each m, choose $a \in I$, $a \notin I_m$.

Now $\pi_m(a) \neq 0$.

Also, $g(a) = f(a) = (\pi_n(a))$ has a non-zero m^{th}-coordinate.

But $g(a) = ag(1) = a\cdot(x_n)$
$= (ax_n)$

So (ax_n) has a non-zero m^{th} coord.

$\therefore\ x_m \neq 0 \ \forall m$, ✱. (as $(x_n) \in \oplus E_n$)).

(End Meral's talk).

<u>Remark:</u> Let (R, m) be a 0-dim'l local ring
Then R is Gorenstein $\Leftrightarrow$ $R \cong E_R(R/m)$.

<u>Proof:</u> $\Leftarrow$: $R \cong E \Rightarrow R$ is injective
$\Rightarrow \mathrm{id}_R R < \infty$.

$\Rightarrow$: ~~$\lambda(E) = \lambda(R)$ so E is a finite R-module.~~

$$R \cong \omega_R \underset{\substack{\uparrow \\ \text{local duality}}}{\cong} H^0_m(R)^\vee \cong R^\vee = E. \ //$$

Theorem: Let (R, m) be a local ring and M a finite R-module of dimension s. Then

$$H^s_m(M) \neq 0.$$

(Hence, $\dim M = \sup\{ i \mid H^i_m(M) \neq 0 \}$.

Proof: Since $\dim \hat{M} = \dim M$ and $H^i_{\hat{m}}(\hat{M}) \cong H^i_m(M)$, we may assume R is complete.

Let $R = S/I$, where (S, n) is a complete RLR. By the change of rings principle, ETS

$H^s_n(M) \neq 0$ where M is considered as an S-module.

Let $g = \operatorname{ht} \operatorname{Ann}_S M$. As S is CM,

$\exists\ x_1, \dots, x_g \in \operatorname{Ann}_S M$ which form an S-sequence.

Let $T = S/(x_1, \dots, x_g)$. Then (T, n_1) is a complete Gorenstein local ring, M is a finite T-module, and $\dim M = \dim T = s$. By the change of rings principle, ETS

$H^s_{n_1}(M) \neq 0$, where M is considered as a T-module

6/2[illegible]5

Defn: Let (R, m) be a local ring and M a f.g. R-module. M is said to be a ~~generalized~~ Buchsbaum module iff for all $\underline{x} = x_1, \ldots, x_r \in R$ which are s.o.p. for M (i.e., $r = \dim M$ and $\lambda(M/(\underline{x})M) < \infty$),

$$\lambda(M/(\underline{x})M) - e_{(\underline{x})}(M) = C, \text{ a constant.}$$

Recall $e_{(\underline{x})}(M)$ = multiplicity of M wrt to $(\underline{x})$

$$= \lim_{n \to \infty} \frac{\lambda(M/(\underline{x})^n M)}{n^r} \cdot r!$$

Note: Since $e_{(\underline{x})}(M) = \lambda(M/(\underline{x})M)$ if $\underline{x}$ is an M-sequence, CM-modules are Buchsbaum.

Theorem: (Stückrad-Vogel) If M is a Buchsbaum module of dimension d, then

$$m \cdot H^i_m(M) = 0 \text{ for all } i < d.$$

The converse, however, does not hold.
(There is no known cohomological characterization of Buchsbaum modules.)

Note that as $H^i_m(M)$ are Artinian R/m-modules, this means $\dim_{R/m} H^i_m(M) < \infty \quad \forall \ i<d$.

This led to the following:

<u>Defn</u>: Let (R,m) be a local ring and M a f.g. R-module. M is said to be a <u>generalized CM module</u> if

$$\lambda\left(H^i_m(M)\right) < \infty \quad \forall \ i < \dim M.$$

<u>Remark</u>: Buchsbaum modules are generalized CM-modules

Let $$I(M) := \sup_{\substack{\underline{x} \in R \\ \text{s.o.p. for } M}} \left\{ \lambda(M/\underline{x}M) - e_{(\underline{x})}(M) \right\}$$

<u>Theorem</u>: (Cuong-Schenzel-Trung, 1978). Let (R,m) be a local ring and M ~~an R-m~~ a finite R-module. TFAE:

(1) M is generalized CM

(2) $I(M) < \infty$

Moreover, if either holds then

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda\left(H^i_m(M)\right). \qquad (d = \dim M)$$

Defn: A finite R-module M is equidimensional if $\dim R/p = \dim M$ for all $p \in \mathrm{Min}_R M = \mathrm{Min}_R(R/\mathrm{Ann}_R M)$. (i.e., $R/\mathrm{Ann}_R M$ is equidimensional.)

Remark: We always have $\dim R/p + \dim M_p \leq \dim M$ for all $p \supseteq \mathrm{Ann}_R M$. If R is local and catenary, then M is equidimensional $\Leftrightarrow$ $\dim R/p + \dim M_p = \dim M$ for all $p \supseteq \mathrm{Ann}_R M$.

Lemma: Let (R, m) be a local ring and N an R-module. Then $\mathrm{Ann}_R N \overset{?}{=} \mathrm{Ann}_R N^\vee$.

Pf: Certainly $\mathrm{Ann}_R N \subseteq \mathrm{Ann}_R \mathrm{Hom}_R(N, E) = \mathrm{Ann}_R N^\vee$. Thus, $\mathrm{Ann}_R N^\vee \subseteq \mathrm{Ann}_R N^{\vee\vee}$. But the natural map $N \longrightarrow N^{\vee\vee}$ is always 1-1, so $\mathrm{Ann}_R N^{\vee\vee} \subseteq \mathrm{Ann}_R N \Rightarrow \mathrm{Ann}_R N^\vee \subseteq \mathrm{Ann}_R N$. //

Theorem: Let (R, m) be a local ring which is the homomorphic image of a Gorenstein ring. ~~Then TFAE:~~ Let M be a finite R-module. TFAE:

(1) M is generalized CM

(2) M is equidimensional and M_p is CM for all $p \in \mathrm{Spec}\, R \setminus \{m\}$.

Proof: Let $R = S/I$ where (S, n) is a

local Gorenstein ring.

Then M is an S-module in the natural way. By the change of rings principle,

$$H^i_n(M) \cong H^i_m(M) \qquad \forall\, i.$$

↑ considered as an S-module

↑ as an R-module

$\therefore$ M is generalized CM as an R-module $\Leftrightarrow$ M is generalized CM as an S-module.

Likewise, M is equidimensional as an R-module $\Leftrightarrow$ M is equid as an S-module (since $S/\mathrm{Ann}_S M = R/\mathrm{Ann}_R M$) and M_q is CM $\forall\, q \in \operatorname{Spec} S - \{m\}$ $\Leftrightarrow$ M_p is CM $\forall\, p \in \operatorname{Spec} R - \{m\}$.

Thus, we may assume (R, m) is Gorenstein. Let $d = \dim R$.

Note that as $H^i_m(M)$ is Artinian,

$$\lambda(H^i_m(M)) < \infty \Leftrightarrow m^n H^i_m(M) = 0 \text{ some } n$$

~~$\Leftrightarrow m^n \subseteq \mathrm{Ann}_R H^i_m(M)$ some n~~

$$\Leftrightarrow m^n \subseteq \mathrm{Ann}_R H^i_m(M) \text{ for some } n.$$

By local duality, $H^i_m(M) = \mathrm{Ext}^{d-i}_R(M, R)^\vee$.

By the lemma,

$$\mathrm{Ann}_R H^i_m(M) = \mathrm{Ann}_R \mathrm{Ext}^{d-i}_R(M,R)$$

Thus,

$$\lambda(H^i_m(M)) < \infty \iff m^n \subseteq \mathrm{Ann}_R \mathrm{Ext}^{d-i}_R(M,R)$$

$$\iff \mathrm{Ext}^{d-i}_R(M,R)_p = 0 \quad \forall\, p \neq m$$

(as $\mathrm{Ext}^{d-i}_R(M,R)$ is f.g.) $$\iff \mathrm{Ext}^{d-i}_{R_p}(M_p, R_p) = 0$$

$$\forall\, p \neq m,\ p \supseteq \mathrm{Ann}_R M.$$

As R_p is Gorenstein, we can use local duality again!

$$\mathrm{Ext}^{d-i}_{R_p}(M_p, R_p)^\vee \cong H^{ht(p)-(d-i)}_{pR_p}(M_p)$$

Thus, (as in general $N=0 \iff N^\vee = 0$),

$$\mathrm{Ext}^{d-i}_{R_p}(M_p, R_p) = 0 \iff H^{ht(p)-d+i}_{pR_p}(M_p) = 0.$$

Thus, we arrive at the following:

$$(\#)\quad \lambda(H^i_m(M)) < \infty \iff H^{\overset{i-\dim R/p}{\cancel{ht(p)-d+i}}}_{pR_p}(M_p) = 0$$

$$\forall\, p \neq m,\ p \supseteq \mathrm{Ann}_R M.$$

⑥

(2) ⇒ (1): As M_p is CM $\forall\, p \neq m$,

$$H^{i-\dim R/p}_{pR_p}(M_p) = 0 \quad \forall\ i - \dim R/p < \dim M_p$$

$$\overset{\shortparallel}{\quad} = 0 \quad \forall\ i < \dim M$$

(by Remark).

$\therefore\ \lambda(H^i_m(M)) < \infty \quad \forall\ i < \dim M$.

(1) ⇒ (2): $H^{i-\dim R/p}_{pR_p}(M_p) = 0 \quad \forall\ i < \dim M$

$\forall\, p \neq m,\ p \supseteq \operatorname{ann}_R M$

or, $H^j_{pR_p}(M_p) = 0 \quad \forall\ j < \dim M - \dim R/p$.

$\forall\, p \neq m,\ p \supseteq \operatorname{ann}_R M$.

Since $H^{\dim M_p}_{pR_p}(M_p) \neq 0$, this

says that $\dim M_p \geq \dim M - \dim R/p$

$\forall\, p \neq m,\ p \supseteq \operatorname{ann}_R M$.

Since we always have $\dim M_p \leq \dim M - \dim R/p$ $\forall\, p \supseteq \operatorname{Ann}_R M$, we have

$$\dim M_p = \dim M - \dim R/p \quad \forall\, p \neq m,\ p \supseteq \operatorname{ann}_R M.$$

Thus, M is equidimensional and $H^j_{pR_p}(M_p) = 0$ $\forall\ j < \dim M_p$ so M_p is CM $\forall\, p \neq m$.

①

6/28

<u>Defn</u>: Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The <u>socle of M</u> is defined as

$$\mathrm{soc}(M) := (0 :_M \mathfrak{m}) = \{x \in M \mid \mathfrak{m}x = 0\}.$$

<u>Notes</u>:

- $\mathrm{soc}(M)$ is an $R/\mathfrak{m}$ - vector space
- $\mathrm{soc}(M) \cong \mathrm{Hom}_R(R/\mathfrak{m}, M)$
- If $E = E_R(R/\mathfrak{m})$, $\mathrm{soc}(E) \cong R/\mathfrak{m}$.

<u>Lemma</u>: Let $(R, \mathfrak{m})$ be a local ring and M a f.g. R-module. Then

$$\mu(M) = \dim_{R/\mathfrak{m}} \mathrm{soc}(M^\vee).$$

Since $\mu(M) = \mu(\hat{M})$ and $M^\vee \cong (\hat{M})^\vee$, we may assume R is complete!

<u>Pf</u>: $0 \to \mathfrak{m}M \to M \to L \to 0$.

$\mu(M) = \dim_k L \quad (k = R/\mathfrak{m})$.

Since $0 \to L^\vee \to M^\vee$ is exact and $\mathfrak{m} \cdot L^\vee = 0$, $\dim \mathrm{soc}(M^\vee) \geq \dim L^\vee = \mu(M)$.

On the other hand, let $V = \mathrm{soc}(M^\vee)$. From $0 \to V \to M^\vee \to B \to 0$, we get

$M^{\vee\vee} \to V^\vee \to 0$ exact. As R is complete,

$\mu(M) = \mu(M^{\vee\vee}) \geq \mu(V^\vee) = \dim V^\vee = \dim V.$ //

<u>Theorem</u>: Let (R, m) be a d-dim'l CM local ring s/t ~~where~~ $R = S/I$, where (S, n) is a RLR. Let $g = \text{ht } I$. Then $\text{pd}_S R = g$ (by Auslander-Buchsbaum). Let

$$0 \to F_g \xrightarrow{\phi_g} F_{g-1} \xrightarrow{\phi_{g-1}} \cdots \to F_0 \to R \to 0$$

be a minimal free resolution of R as an S-module. Then

$$\underset{\substack{\uparrow \\ (= \text{CM type})}}{\mu_d(R)} = \mu(\omega_R) = \text{rk } F_g$$

<u>Proof</u>: By a previous theorem,

$$\omega_R \cong \text{Ext}^g_S(R, S)$$

Calculate this Ext by applying $\text{Hom}_S(-, S)$ to the free resolution above. Then

$$\begin{tikzcd} F^*_{g-1} \arrow[r, "\phi^*_g"] \arrow[d, equal] & F^*_g \arrow[r] & \omega_R \arrow[r] & 0 \\ \text{Hom}_S(F_{g-1}, S) & & & \end{tikzcd}$$

is exact.

As the entries of a matrix (a_{ij}) representing ϕ are in m, then ϕ^*_g is rep. by $(a_{ij})^t$.

hence, $\operatorname{im} \phi_g^* \subseteq m F_g^*$.

$\therefore \ \mu(\omega_R) = \mu(F_g^*) = \operatorname{rk} F_g$.

Now, we know ~~$\omega_R \cong H^d_m$~~ by local duality that $\omega_R^\vee = \operatorname{Hom}_R(R, \omega_R)^\vee$

$$\cong H^d_m(R).$$

~~Since,~~ By the lemma,

$$\mu(\omega_R) = \dim \operatorname{soc}(\omega_R^\vee)$$

$$= \dim \operatorname{soc}(H^d_m(R)).$$

$\therefore$ ETS $\dim \operatorname{soc}(H^d_m(R)) = \mu_d(R)$.

Let $0 \to R \to I^0 \to I^1 \to \cdots$ by a minimal injective resolution of R.

Apply $H^0_m(-)$ to this resolution.
As $H^0_m(E(R/p)) = 0$ if $p \neq m$ and $= E_R(R/m)$ if $p = m$, we get the complex

$$0 \to E^{\mu_0(R)} \to E^{\mu_1(R)} \to \cdots \qquad E = E_R(R/m).$$

As R is CM, $\mu_i(R) = 0 \;\; \forall\, i < d$.

$\therefore$ we have $0 \to H^d_m(R) \to E^{\mu_d(R)} \to E^{\mu_{d+1}(R)}$ is exact.

Apply $\mathrm{Hom}_R(R/m, -)$ to get

$$0 \to \mathrm{Hom}_R(R/m, H^d_m(R)) \to \mathrm{Hom}_R(R/m, E^{\mu_d(R)}) \xrightarrow[\text{zero map by minimality of resolution}]{} \mathrm{Hom}_R(R/m, E^{\mu_{d+1}}$$

$$\therefore \; \dim \mathrm{soc}(H^d_m(R)) = \dim \mathrm{soc}(E^{\mu_d(R)})$$

$$= \mu_d(R). //$$

<u>Remark:</u> The equality $\mu(\omega_R) = \mu_d(R)$

requires only that (R, m) be CM and possessing a canonical module.

<u>Corollary</u>: (Serre) Let $R = S/I$ where (S, n) is a regular local ring and ht $I = 2$.

TFAE:

(1) R is Gorenstein

(2) R is a complete intersection

<u>Pf</u>: (2) $\Rightarrow$ (1): trivial

(1) $\Rightarrow$ (2): By Auslander-Buchsbaum,

$$\mathrm{pd}_S R = \mathrm{depth}\, S - \mathrm{depth}\, R = \mathrm{ht}\, I = 2.$$

Let
$$0 \to S^n \to S^{n+1} \to R \to 0$$
be a minimal free resolution of R.

Then $n = \mu_2(R) = 1$, so $n+1 = 2 = \mu(I)$. //

Question: Let (R, m) be a local ring of dimension d and I an ideal of R. When is $H^d_I(R) = 0$?

Certainly we need $\sqrt{I} \neq m$. Is that enough?

The Hartshorne-Lichtenbaum Vanishing theorem (HLVT) answers this.
A special case of HLVT is the following:

Let (R, m) be a complete domain of dim d.
Then $H^d_I(R) = 0 \iff \dim R/I > 0$
(i.e., $\sqrt{I} \neq m$).

We'll actually prove a more general version of for arbitrary local rings.

We begin with a very special case:

Proposition 1: Let (R, m) be a complete local Gorenstein domain of dimension d. Let $P \in \operatorname{Spec} R$ with $\dim R/P = 1$. Then $H^d_P(R) = 0$.

Proof: **Claim:** $\{P^n\}_{n \geq 1}$ and $\{P^{(n)}\}_{n \geq 1}$ are co-final.

Pf: As R is a domain, $\bigcap_{n \geq 1} P^{(n)} = 0$. (check).

By Chevalley's Theorem, $\forall\, k\ \exists\, n$ s/t $p^{(n)} \subseteq \mathfrak{m}^k$.
Now, by primary decomposition,

$$p^n = p^{(n)} \cap J_n \quad \text{where } J_n \text{ is primary to } \mathfrak{m}.$$

$\therefore$ $\mathfrak{m}^k \subseteq J_n$ for some k. $\therefore\ \exists\, t \gg 0$ s/t $p^{(t)} \subseteq \mathfrak{m}^k \subseteq J_n$. We may as well assume $t \geq n$.

Then

$$p^n = p^{(n)} \cap J_n \supseteq p^{(n)} \cap p^{(t)} = p^{(t)}.$$

$\therefore$ $\{p^n\}$ and $\{p^{(n)}\}$ are cofinal.

Note that depth $R/p^{(n)} > 0$ $\forall\, n$, as

$\mathrm{Ass}_R\, R/p^{(n)} = \{p\}$.

Now, $H^d_p(R) = \varinjlim \mathrm{Ext}^d_R(R/p^{(n)}, R)$

But by local duality, $\mathrm{Ext}^d_R(R/p^{(n)}, R) = H^0_{\mathfrak{m}}(R/p^{(n)})^\vee$

$$= 0.$$

$\therefore$ $H^d_p(R) = 0$. //

Lemma: Let R be a Noetherian ring, I an ideal, $x \in R$, M an R-module. Then $\exists$ a l.e.s.,

$$\cdots \to H^i_{(I,x)}(M) \to H^i_I(M) \to H^i_{I_x}(M_x) \to H^{i+1}_{(I,x)}(M) \to \cdots$$

Proof: We've already proved this for Čech cohomology on 6/16.

Prop ~~3~~ 2: Let (R,m) be a ~~total ring~~ local ring of dimension d

TFAE: (1) $H^d_I(R) = 0$ for ~~some~~ all ideals I s/t $\dim R/I > 0$.

(2) $H^d_p(R) = 0$ for all $p \in \operatorname{Spec} R$ s/t $\dim R/p = 1$.

Proof: Clearly (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Suppose $\exists$ an ideal I s/t $\dim R/I > 0$ and ~~$\dim R/I$~~ $H^d_I(R) \neq 0$. ~~Then~~ Let I be maximal wrt this property.
By hypothesis, I is not prime of dim 1.
$\therefore$ $\exists$ $x \in R \setminus I$ s/t $\dim R/(I,x) > 0$.

By the l.e.s.

$$H^d_{(I,x)}(R) \to \underset{\neq 0}{H^d_I(R)} \to \underset{= 0}{H^d_{I_x}(R_x)}$$

as $\dim R_x < d$.

$\therefore$ $H^d_{(I,x)}(R) \neq 0$, $*$. //

Prop 3:

6/29 Let (R, m) be a local ring of dim d, $I \subseteq R$ and M an R-module.

Then $H^d_I(M) \cong H^d_I(R) \otimes_R M$

Hence, if $H^d_I(R) = 0$ then $H^d_I(M) = 0$ for all R-modules M.

Proof: As $\operatorname{ara}(I) \leq d$, let $I = \sqrt{(x_1, \dots, x_d)}$ for some $x_1, \dots, x_d \in R$.

Then

$$\bigoplus_i R_{x_1 \cdots \hat{x}_i \cdots x_d} \longrightarrow R_{x_1 \cdots x_d} \longrightarrow H^d_I(R) \longrightarrow 0 \text{ is exact}$$

$\otimes_R M$:

$$\bigoplus_i M_{x_1 \cdots \hat{x}_i \cdots x_d} \longrightarrow M_{x_1 \cdots x_d} \longrightarrow H^d_I(R) \otimes_R M \to 0 \text{ exact.}$$

But this must mean $H^d_I(M) \cong H^d_I(R) \otimes_R M$. //

Corollary: Let (R, m) be a local ring of dim d.

TFAE:

(1) $H^d_I(R) = 0$

(2) $H^d_I(M) = 0$ for all R-modules M.

Let (R, m) be a local ring. Then one of the following holds:

(i) $\operatorname{Char} R = 0$ and $\operatorname{Char} R/m = 0$
(ii) $\operatorname{Char} R = p$ and $\operatorname{Char} R/m = p$
(iii) $\operatorname{Char} R = 0$ and $\operatorname{Char} R/m = p$
(iv) $\operatorname{Char} R = p^n, n > 1$ and $\operatorname{Char} R/m = p$

If (i) or (ii) holds, R is said to have equal characteristic; otherwise, R has unequal characteristic

Note also that (i) holds $\Leftrightarrow \mathbb{Q} \subseteq R$
(ii) holds $\Leftrightarrow \mathbb{Z}_p \subseteq R$

$\therefore$ R has equal characteristic $\Leftrightarrow$ R contains a field.

Defn: Let (R, m) be a complete local ring. A subring $K \subseteq R$ is called a coefficient ring for R if

(i) $R = K + m$
(ii) If R has equal characteristic, then K is a field. Otherwise, (K, n) is a complete local ring s/t $n = pK$, where $p = \operatorname{char} R/m$.

$R/m \cong K/n$.
Note: If K is a field, $K \cong R/m$

Remark: If R is a domain then K is a domain. Hence, K is a field or a complete DVR. In any case, K is a quotient of a ~~DVR~~ complete DVR.

Theorem: (Cohen) Every complete local ring has a coefficient ring.

Pf: Matsumura.

Lemma: Let (R, m) be a complete local ring, K a coefficient ring for R, and $y_1, \ldots, y_d$ a s.o.p. for R. Let $A = K[[y_1, \ldots, y_d]]$. Then R is a finite A-module.

Proof: First note that A is the image of the ring map

$$\phi: K[[T_1, \ldots, T_d]] \longrightarrow R$$
$$T_i \longrightarrow y_i$$

$\therefore$ As $K[[T_1, \ldots, T_d]]$ is complete, local, so is A.

Let n be the maximal ideal of A. Then $n = (p, y_1, \ldots, y_d)A$ where $p = \text{char } R/m$ (zero or a prime).
Clearly, $n \subseteq m$.

By definition of coefficient ring, $A/n \cong R/m$.
∴ Every R-module of finite length has finite length as an A-module.
In particular, $\lambda_A(R/nR) < \infty$. (as n contains an s.o.p. for R).

Choose $x_1, \ldots, x_r \in R$ s/t

$$R/nR = A\bar{x}_1 + \cdots + A\bar{x}_r .$$

<u>Claim:</u> $R = Ax_1 + \cdots + Ax_r$

<u>Pf:</u> we have $R = \sum Ax_i + nR$

$$\cancel{= \sum Ax_i + n(\sum Ax_i + nR)}$$
$$\cancel{= \sum Ax_i + n^2 R}$$

~~Inductively, we get $R = \sum Ax_i + n^k R$ for any $k \geq 1$.~~

Let $u \in R$.

Write $u = \sum a_{i,0} x_i + u_1$, $a_{i,0} \in A$, $u_1 \in nR$.

then $u_1 = \sum a_{i1} x_i + u_2$, $a_{i1} \in n$, $u_2 \in n^2 R$

$\vdots$

$u_k = \sum a_{ik} x_i + u_{k+1}$, $a_{ik} \in n^k$, $u_{k+1} \in n^{k+1} R$.

Now, for each i,

$a_i = a_{i,0} + a_{i1} + a_{i2} + \cdots$ converges in A.

Then $u - \sum_{i=1}^{r} a_i x_i \in \bigcap_k n^k R \subseteq \bigcap m^k = 0.$ //

Prop 4: Let (R, m) be a complete local domain of dimension d and I an ideal of R. TFAE:

(1) $H^d_I(R) \neq 0$

(2) $\dim R/I = 0$.

Proof: (due to Huneke and Bredmann, independently). (1994)
The content is (2) ⇒ (1).
By Prop 2, ETS $H^d_P(R) = 0$ for any $P \in \operatorname{spec} R$ s/t $\dim R/P = 1$.

Let K be a coefficient ring for R. As R is a domain, K is a field or a complete DVR with uniformizing parameter q, where $q = \operatorname{char} R/m$.

Let $P \in \operatorname{spec} R$, $\dim R/P = 1$. As $\operatorname{ara}(I) \leq d$, we know $\exists\ x_1, \ldots, x_d \in R$ s/t $P = \sqrt{(x_1, \ldots, x_d)}$

Furthermore, we may choose $x_1, \dots, x_d$ with the following properties

a) $x_1, \dots, x_{d-1}$ form part of an s.o.p. for R (as $\operatorname{ht} P = d-1$).

b) If K is not a field, ~~$x_1 = q$~~ and $q \in P$ then $x_1 = q$. (R is a domain).

c) If K is not a field and $q \notin P$ then $x_1, \dots, x_{d-1}, q$ is an s.o.p. for R.
($\sqrt{(P,q)} = m$ so choose $\bar{x}_1, \dots, \bar{x}_{d-1} \in \bar{P} = (P+q)/(q)$ to form an s.o.p. for $R/(q)$.)

If K is either a field or $q \in P$, choose $y \in R$ s/t $x_1, \dots, x_{d-1}, y$ is an s.o.p. for R.
If ~~K is not a field~~ $q \notin P$, let $y = q$. By c), $x_1, \dots, x_{d-1}, y$ is an s.o.p. for R.

Let $A = K[[x_1, \dots, x_{d-1}, y]]$. Then (as remarked in the previous lemma) A is a complete local domain (as R is a domain) and R is a finite A-module.

Thus $\dim A = \dim R = d$.

Claim 1: A is a complete RLR

Pf. case 1: K is a field

Then $A \cong K[[T_1, .., T_d]]/_I$ where $T_1, .., T_d$ are indet. As $K[[T_1, .., T_d]]$ is a d-dim'l complete RLR and $\dim A = d$, $I = 0$.

case 2: K is not a field.

Then $q \in A$.

Hence, $A = K[[x_2, .., x_{d-1}, y]]$ if $x_1 = q$

or $A = K[[x_1, .., x_{d-1}]]$ if $y = q$.

In either case, $A \cong K[[T_1, .., T_{d-1}]]/_I$

Again, $K[[T_1, .., T_{d-1}]]$ is a complete RLR of dim d. $\therefore I = 0$.

Now let $B = A[x_d]$. Then $A \subseteq B \subseteq R$.

Claim 2: B is a complete local Gorenstein domain and R is a finite B-module.

Pf: As R is a finite A-module, R is certainly a finite B-module.

Clearly, B is Noetherian (as A is).

Since R is a domain, B is also.
As R is integral over B, any maximal ideal of B is contracted from R.
As R is local, B must be also.

To see that B is complete, first note that, as B is a finite A-module and A is complete, B is complete as an A-module. Let m_A, m_B represent the max'l ideals of A and B, resp. As B/A is integral, $\sqrt{m_A B} = m_B$. ∴ $m_B^n \subseteq m_A B$ for some n. Hence, the m_A and m_B-adic topologies on B are equivalent. So B is complete

Finally, ~~[illegible]~~

$$B = A[x_d] \cong A[T]/_I$$ where T is an indet and I a prime ideal. Since we know B is local,

$$B \cong A[T]_M / I_M$$ where $M = (m_A, T)A[T]$.

A is a RLR of dim d $\Rightarrow$ $A[T]_M$ is a RLR of dim d+1.

Since B is a domain of dim d, I_M is a ht 1 prime of $A[T]_M$, hence principal (as RLR ⇒ UFD). //

Now let $Q = P \cap B$. Since R/P is int over B/Q, $\dim B/Q = 1$. By Prop 1, $H^d_Q(B) = 0$.

Since $P = \sqrt{(x_1, \dots, x_d)}$ and $x_1, \dots, x_d \in B$,

$Q = \sqrt{(x_1, \dots, x_d)B}$ (by lying over).

Thus

$$\begin{aligned} H^d_P(R) &= H^d_{(x_1,\dots,x_d)R}(R) \\ &= H^d_{(x_1,\dots,x_d)B}(R) \quad \text{(by change of rings)} \\ &= H^d_{(x_1,\dots,x_d)B}(B) \otimes_B R \quad \text{by Prop 3} \\ &= H^d_Q(B) \otimes_B R \\ &= 0 \;/\!/ \end{aligned}$$

<u>Remark</u>: The proof given also shows that if (R, m) is a complete local domain of dim d then $\exists$ a complete RLR A of dim d s/t R is a finite A-module.

①

7/1

Theorem (Hartshorne-Lichtenbaum Vanishing theorem, 1968)
Let (R, m) be a local ring of dim d and I an ideal of R. TFAE:

(1) $H^d_I(R) = 0$

(2) $\dim \hat{R}/_{I\hat{R}+p} > 0 \ \forall \, p \in \operatorname{Spec} \hat{R}$ s/t $\dim \hat{R}/p = d$

(3) $H^d_I(M) = 0$ for all R-modules M.

Proof: We've already shown the equivalence of (1) and (3) (as a corollary to Prop 3).

(1)⇒(2): Let $p \in \operatorname{Spec} \hat{R}$ s/t $\dim \hat{R}/p = d$.

Then $H^d_{\frac{I\hat{R}+p}{p}}(\hat{R}/p) \cong H^d_I(R) \otimes_R \hat{R}/p = 0$

By Prop 4, $\dim \hat{R}/_{I\hat{R}+p} > 0$.

(2)⇒(1): Suppose $H^d_I(R) \neq 0$. Then $H^d_{I\hat{R}}(\hat{R}) \neq 0$ (as $\hat{R}$ is faithfully flat R-module).
Let J be an ideal of $\hat{R}$ maxl w.r.t the property that $H^d_{I\hat{R}}(\hat{R}/J) \neq 0$.

Then necessarily $\dim \hat{R}/J = d$. Let $p \in \mathrm{Ass}_{\hat{R}}(\hat{R}/J)$ s/t $\dim \hat{R}/p = d$. Then we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \hat{R}/p \arrow[r] & \hat{R}/J \arrow[r] & \hat{R}/(J,x) \arrow[r] & 0 \\
& \bar{1} \arrow[r] & \bar{x} \neq 0 & &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
H^d_{I\hat{R}}(\hat{R}/p) \arrow[r] & H^d_{I\hat{R}}(\hat{R}/J) \arrow[r] & H^d_{I\hat{R}}(\hat{R}/(J,x)) \\
& \neq 0 & = 0
\end{tikzcd}$$

by maximality of J

$\therefore H^d_{I\hat{R}}(\hat{R}/p) \neq 0$, $*$ .//

History: Originally, Lichtenbaum conjectured a geometric analogue of this vanishing theorem for sheaf cohomology. Grothendieck proved this conjecture in 1961. (Nevertheless, it became known as "Lichtenbaum's Theorem".) Hartshorne proved this local ~~version~~ vanishing theorem in 1968. Lichtenbaum's theorem follows readily from Hartshorne's.

Theorem: (Faltings, 1979) Let (R, m) be a complete local domain of dim d and I an ideal s/t $\text{ara}(I) \leq d-2$. Then

$$\text{Spec}(R/I) - \{m/I\} \text{ is connected}$$

Pf: (due to J. Rung)

Let $\mathcal{U} = \text{Spec}(R/I) - \{m/I\}$
$\quad\triangleq V(I) - \{m\}$.

Suppose $\mathcal{U}$ is disconnected.
This means $\exists$ ideals $J, K^{\supseteq I}$ in R s/t

(i) $J \cap K \subseteq \sqrt{I}$, so $\sqrt{J \cap K} = \sqrt{I}$
(ii) $\sqrt{J+K} = m$
(iii) $\sqrt{J} \neq m$, $\sqrt{K} \neq m$ (i.e., $\dim R/J > 0$, $\dim R/K > 0$)

By the Mayer-Vietoris sequence, we have

$$H^{d-1}_{J+K}(R) \longrightarrow H^{d-1}_{J}(R) \oplus H^{d-1}_{K}(R) \longrightarrow H^{d-1}_{J\cap K}(R)$$

$$\longrightarrow H^{d}_{J+K}(R) \longrightarrow H^{d}_{J}(R) \oplus H^{d}_{K}(R)$$

Now, $H^{d-1}_{J\cap K}(R) = 0$ as $\sqrt{J\cap K} = \sqrt{I}$ and $\text{ara}(I) \leq d-2$.

$$\therefore \quad 0 \longrightarrow \underset{\neq 0}{H^d_{m}(R)} \longrightarrow H^d_J(R) \oplus H^d_K(R) \qquad \text{exact}$$

So $H^d_J(R) \neq 0$ or $H^d_K(R) \neq 0$.

But $\dim R/J > 0$, $\dim R/K > 0$, $\ast$ HLVT. //

This theorem has the following geometric consequence:

<u>Theorem</u>: (Fulton-Hansen, 1979) Let K be an alg. closed field and X, Y irreducible projective varieties in $\mathbb{P}^n_K$. Suppose $\dim X + \dim Y > n$. Then $X \cap Y$ is connected.

(Proof uses reduction to the diagonal:

$$k(X\times Y) = k(X) \otimes_k k(Y) \cong k[x_0, \dots, x_n, y_0, \dots, y_n] \big/ I(X) + I(Y)$$

has $\dim > n+2$.

Mod out by $\{x_i - y_i\}_{i=0}^{n}$ and use Faltings' result.)

~~Pf: $0 \to H^d_m(R) \to H^d_J(R) \oplus H^d_K(R)$~~

~~$\neq 0$~~

~~$\Rightarrow H^d_J(R) \neq 0$ or $H^d_K(R) \neq 0$, * HLVT.~~ //

Q: Let (R, m) be a complete local domain, $I \subseteq R$.

When is $H^{d-1}_I(R) = 0$? $(d = \dim R)$.
and $H^d_I(R) = 0$

One might guess $\iff \dim R/I > 1$

But this is false, as shown by the following example of Hartshorne:

Example: Let $R = K[[x, y, u, v]] / (xu - yv)$, K a field

Then R is a 3-dim'l complete Gorenstein (in fact, a hypersurface) domain.

Let $I = (x, y)R$. $R/I \cong K[[u, v]]$, so

I is a prime of dim 2. If the conjecture were true, then $H^2_I(R) = 0$.

We know $H^3_I(R) = 0$ as $\mu(I) = 2$. (Also by HLVT).

Let $J = (u,v)R$.

Consider the s.e.s.

$$0 \to J \to R \to R/J \to 0$$

Then

$$\cdots \to H^2_I(R) \to H^2_I(R/J) \to H^3_I(J)$$

$$\simeq 0$$

as $\mu(I)=2$

$$H^2_I(R/J) = H^2_{\frac{I+J}{J}}(R/J) = H^2_{m/J}(R/J) \neq 0$$

as $\dim R/J = 2$.

$\therefore$ $H^2_I(R) \neq 0$. //

Note that in this example, $\operatorname{ht} I = \operatorname{ht} J = 1$
but $\operatorname{ht}(I+J) = \operatorname{ht}(m) = 3$.
If R is a RLR, we always have
$\operatorname{ht}(p+q) \leq \operatorname{ht} p + \operatorname{ht} q$ $\forall$ $p,q \in \operatorname{Spec} R$.
$\therefore$ There is reason to believe ~~this~~ the conjecture might hold for RLRs.

Theorem: (Peskine-Szpiro in char $p > 0$, 1973
Ogus in char 0, 1973)

Let (R, m) be a complete RLR containing a field. Suppose R/m is alg. closed. Let I be an ideal of R.

TFAE: (1) $H^{d-1}_I(R) = H^d_I(R) = 0$

(2) $\dim R/p > 1 \ \forall \ p \in \operatorname{Min} R/I$

and $\operatorname{Spec}(R/I) - \{m/I\}$ is connected.

Further improvements and generalization have been given by Huneke and Lyubeznik.

Theorem (Sharp, 1981) Let (R, m) be a local ring ~~and~~ I an ideal of R and M a finite R-module of dim n. Then $H^n_I(M)$ is Artinian

Proof: As $R \to \hat{R}$ is faithfully flat, if

$$H^n_{I\hat{R}}(\hat{M}) = H^n_I(M) \otimes_R \hat{R} \text{ has DCC,}$$

then $H^n_I(M)$ has DCC. ∴ We may assume R is complete.

By the change of rings principal, we may pass to the ring $R/\mathrm{Ann}_R M$ and so assume $\mathrm{Ann}_R M = 0$ and $\dim R = \dim M = n$.

Let $R = S/L$ where S is a complete RLR. Let $g = \mathrm{ht}\, L$ and $x_1, \ldots, x_g \in L$ an S-sequence. Let $B = S/(\underline{x})$ and $J = L/(x)$. Then

$$R = B/J \quad , \quad \dim R = \dim B = n \text{ and } B \text{ is}$$

a complete Gorenstein ring.

M can be considered as a B-module.

$\therefore$ ETS $H^n_{IB}(M)$ is Artinian.

<u>Claim:</u> $H^n_J(B)$ is Artinian for any ideal J.

<u>Pf:</u> An injective resolution for B looks like

$$0 \to B \to \bigoplus_{\mathrm{ht}\, p = 0} E_B(B/p) \to \cdots \to E_B(B/m) \to 0$$

with 0 above the first injective term and n above the last.

we know $E_B(B/m)$ is Artinian.

~~$H^n_J(B)$ is a subs~~ ∴ $\mathrm{Hom}_B(B/J, E)$ is Artinian. $H^n_J(B)$ is a quotient of this module, and hence is Artinian.

Now, we've seen $H^n_J(M) \cong H^n_J(B) \otimes_B M$

as $n = \dim B$.

As $H^n_J(B)$ is Artinian, ETS $N \otimes_B M$ is Artinian if N is Artinian and M is f.g.

By Matlis Duality, ETS $(N \otimes_B M)^\vee$ is f.g.

$$\begin{aligned} \text{But } (N \otimes_B M)^\vee &= \operatorname{Hom}_B(N \otimes_B M, E) \\ &= \operatorname{Hom}_B(M, N^\vee) \quad \text{is f.g.} // \end{aligned}$$

$\uparrow$

$N^\vee$ is f.g.

V. Sapko

7/2

An application of HLVT

Defn: Let (R, m) be a local ring, M an R-module and $E = E_R(R/m)$. A coassociated prime of M is an associated prime of $M^\vee = \mathrm{Hom}_R(M, E)$. ~~Denote~~ That is,

$$\mathrm{Coass}(M) = \mathrm{Ass}(M^\vee).$$

Remark ⓪ Let (R, m) be a local ring, M a f.g. R-module, N any R-module. Then

$$\mathrm{Ass}\,\mathrm{Hom}_R(M, N) = \mathrm{Supp}\, M \cap \mathrm{Ass}\, N$$

Pf: Recall that

$$p \in \mathrm{Ass}\,\mathrm{Hom}_R(M, N)$$

$$\Leftrightarrow \mathrm{Hom}_{R_p}(k(p), \mathrm{Hom}_R(M, N)_p) \neq 0$$

$$\Leftrightarrow \mathrm{Hom}_{R_p}(k(p), \mathrm{Hom}_{R_p}(\overset{M_p, N_p}{\cancel{N_p, M_p}})) \neq 0$$

$$\Leftrightarrow \mathrm{Hom}_{R_p}(k(p) \otimes_{R_p} M_p, N_p) \neq 0$$

$$\Leftrightarrow \mathrm{Hom}_{R_p}(k(p)^{\mu(M_p)}, N_p) \neq 0$$

$$\Leftrightarrow \mathrm{Hom}_{R_p}(k(p), N_p)^{\mu(M_p)} \neq 0$$

$$\Leftrightarrow p \in \mathrm{Ass}\, N \text{ and } \mu(M_p) \neq 0 \,. //$$

Remark ①: Let (R, m) be a Noeth local ring, M a f.g. R-module, N any R-module. Then

$$\operatorname{Coass}(M \otimes_R N) = \operatorname{Supp} M \cap \operatorname{Coass} N$$

Pf: $$\operatorname{Coass}(M \otimes_R N) = \operatorname{Ass}((M \otimes_R N)^\vee)$$

$$= \operatorname{Ass} \operatorname{Hom}_R(M \otimes_R N, E)$$

$$= \operatorname{Ass} \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, E))$$

$$= \operatorname{Ass} \operatorname{Hom}_R(M, N^\vee)$$

$$= \operatorname{Supp} M \cap \operatorname{Ass} N^\vee = \operatorname{Supp} M \cap \operatorname{Coass} N \quad //$$

Recall: Let R be a local ring of dim d, $I \subseteq R$, M an R-module. Then

$$H^d_I(M) = M \otimes_R H^d_I(R).$$

HLVT: If (R, m) is a complete local ring of dim d, $I \subseteq R$, then $H^d_I(R) \neq 0$

$\Longleftrightarrow$ $\sqrt{I + p} = m$ for some $p \in \operatorname{Spec} R$ s/t $\dim R/p = d$.

Lemma 3: Let (R, m) be a complete local ring, $I \subseteq R$, M a f.g. R-module of dim n. Then

$$\operatorname{Coass} H^n_I(M) = \left\{ p \supseteq \operatorname{Ann}_R M \;\middle|\; \begin{array}{l} \dim R/p = n \text{ and} \\ \sqrt{I+p} = m \end{array} \right\}.$$

Pf. By the change of rings principle, we may assume $\dim M = \dim R$ and $\operatorname{Ann}_R M = 0$.

$$\operatorname{Coass} H^n_I(M) = \operatorname{Coass}(M \otimes_R H^n_I(R))$$

$$= \operatorname{Supp} M \cap \operatorname{Coass} H^n_I(R)$$

$$= \operatorname{Coass} H^n_I(R) \quad (\text{as } \operatorname{Ann}_R M = 0.)$$

By ~~HLVT~~ We may assume $H^n_I(R) \neq 0$. (Otherwise, both sets in the theorem are empty, by HLVT.)

Let $q \in \operatorname{Coass} H^n_I(R)$

Then $q \in \operatorname{Coass}(R/q \otimes H^n_I(R)) = \operatorname{Supp} R/q \cap H^n_I(R)$.

$\therefore$ $R/q \otimes_R H^n_I(R) = H^n_I(R/q) \neq 0$ so

$\dim R/q = n$ and $\sqrt{I+q} = m$ by HLVT.

Let $q \in \operatorname{Spec} R$ s/t $\dim R/q = n$ and $\sqrt{I+q} = m$.

Hence, [illegible]

$$R/q \otimes_R H^n_I(R) \cong H^n_{\frac{I+q}{q}}(R/q) \neq 0 \quad \text{by HLVT.}$$

Let $p \in \operatorname{Coass}(R/q \otimes H^n_I(R)) = \operatorname{Supp} R/q \cap \operatorname{Coass} H^n_I(R)$.

So $p \geqq q$ and $p \in \operatorname{Coass} H^n_I(R)$.

But we've shown that if $p \in \operatorname{Coass} H^n_I(R)$ then p is minimal.

$\therefore p = q$. //

Remark: Let (R, m) be a complete local ring, M, N R-modules, M f.g. and N is Artinian.
Then
$$\operatorname{Ext}^i_R(M,N)^\vee \cong \operatorname{Tor}^R_i(M, N^\vee).$$

Pf: If $F_\circ$ is a free resolution of $N^\vee$, then

$F_\circ^\vee$ is an injective resolution of $N^{\vee\vee} \cong N$.

$$\begin{aligned}
\mathrm{Tor}_i^R(M,N^\vee)^\vee &= H_i(M\otimes_R F_\circ)^\vee \\
&= H^i((M\otimes_R F_\bullet)^\vee) \\
&= H^i(\mathrm{Hom}_R(M\otimes_R F_\bullet, E)) \\
&\cong H^i(\mathrm{Hom}_R(M, F_\bullet^\vee)) \\
&\cong \mathrm{Ext}^i_R(M,N) \,//
\end{aligned}$$

<u>Defn</u>: Let (R,m) be a local ring, $I \subseteq R$, N an R-module. N is <u>I-cofinite</u> if $\mathrm{Supp}\, N \subseteq V(I)$ and

$$\mathrm{Ext}^i_R(R/I, N) \text{ is f.g. } \forall i.$$

<u>Lemma 2</u>: Let (R,m) be a local ring and $\hat{R}$ the m-adic completion of R, $I \subseteq R$, M an R-module. Then $H^i_I(M)$ is I-cofinite $\Leftrightarrow$ $H^i_{I\hat{R}}(M\otimes_R \hat{R})$ is $I\hat{R}$-cofinite.

<u>Pf</u>: $\mathrm{Ext}^i_R(R/I, H^i_I(M)) \otimes_R \hat{R} \cong$

$$\mathrm{Ext}^i_{\hat{R}}(\hat{R}/I\hat{R}, H^i_{I\hat{R}}(M\otimes_R \hat{R})).$$

ETS $N\otimes_R \hat{R}$ is f.g. $\Leftrightarrow$ N is f.g. (Shown yesterday).

__Theorem__ (Delfino-Marley, 1997) Let (R, m) be a Noetherian local ring, $I \subseteq R$, M a f.g. R-module of dim n.
Then $H^n_I(M)$ is I-cofinite. In fact, $\mathrm{Ext}^i_R(R/I, H^n_I(M))$ has finite length $\forall i$.

__Proof:__ By lemma 2, we may assume (R, m) is complete.
As $H^n_I(M)$ is Artinian, $H^n_I(M)^\vee$ is f.g.
$\therefore$

Coass $H^n_I(M)$ is a finite set; say

Coass $H^n_I(M) = \{p_1, \ldots, p_k\}$.

$\therefore$ Supp $H^n_I(M) = V(p_1 \cap \cdots \cap p_k)$.

Now, $\mathrm{Ext}^i_R(R/I, H^n_I(M))$ has finite length

$\iff \mathrm{Ext}^i_R(R/I, H^n_I(M))^\vee$ has finite length

$\iff \mathrm{Tor}^R_i(R/I, H^n_I(M)^\vee)$ has finite length.

As $\mathrm{Tor}^R_i(R/I, H^n_I(M)^\vee)$ is a f.g. R-module,

it's enough to show it's support is $\{m\}$.

Now, $\operatorname{Supp} \operatorname{Tor}_i^R(R/I, H_I^n(m)^\vee) \subseteq V(I) \cap \operatorname{Supp} H_I^n(m)^\vee$

$$= V(I) \cap V(p_1 \cap \cdots \cap p_k)$$

$$= V(I + p_1 \cap \cdots \cap p_k)$$

$$= \{m\}$$

as $\sqrt{I + p_i} = m \quad \forall i$. //

7/6 <u>Remarks on Matlis Duality:</u>

Recall that we "proved" that if (R, m) is a local ring, $E = E_R(R/m)$, then the functor $(\ )^{\vee} = \operatorname{Hom}_R(-, E)$ gives an antiequivalence of the categories

$$\langle\langle \text{Noeth } \hat{R}\text{-modules} \rangle\rangle \longleftrightarrow \langle\langle \text{Artinian } R\text{-modules} \rangle\rangle.$$

$$M \longleftrightarrow M^{\vee}.$$

This is <u>false</u>. The functor should be $\operatorname{Hom}_{\hat{R}}(-, E)$.

To see the functor $\operatorname{Hom}_R(-, E)$ does not work, we prove the following:

<u>Prop:</u> Let (R, m) be a local ring which is <u>not</u> complete. Then $\operatorname{Hom}_R(\hat{R}, E)$ is not Artinian.

<u>Proof:</u> Suppose $\operatorname{Hom}_R(\hat{R}, E)$ is an Artinian R-module. Then it is an Artinian $\hat{R}$-module. As $\operatorname{Hom}_R(\hat{R}, E)$ is an injective $\hat{R}$-module, we must have

$$\operatorname{Hom}_R(\hat{R}, E) \cong E^n \quad \text{for some } n.$$

Then $\operatorname{Hom}_R(R/m, \operatorname{Hom}_R(\hat{R}, E)) \cong \operatorname{Hom}_R(R/m, E^n)$

So

$$\operatorname{Hom}_R(R/m \otimes_R \hat{R}, E) \cong ~~\operatorname{Hom}_R(~~ (R/m)^n$$

$\Rightarrow$ $R/m \cong \operatorname{Hom}_R(R/m, E) \cong (R/m)^n$

$\therefore$ $n = 1$. Hence, $\operatorname{Hom}_R(\hat{R}, E) \cong E$.

Now consider the natural injection

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\hat{R}}(\hat{R}, E) \arrow[r, "\phi"] & \operatorname{Hom}_R(\hat{R}, E) \arrow[r] & C \arrow[r] & 0 \\ & f \arrow[r] & f & & \end{tikzcd}$$

As $\operatorname{Hom}_{\hat{R}}(\hat{R}, E) \cong E$ and $\operatorname{Hom}_R(\hat{R}, E) \cong E$, and E is indecomposable, $C = 0$.
$\therefore$ ϕ is surjective.

Hence, $\operatorname{Hom}_R(\hat{R}, E) = \operatorname{Hom}_{\hat{R}}(\hat{R}, E)$.

~~But~~ Now consider R as an R-submodule of $\hat{R}$.
(The map $R \hookrightarrow \hat{R}$ is always 1-1).
Then $\hat{R}/R \neq 0$. Hence, $\operatorname{Hom}_R(\hat{R}/R, E) \neq 0$.
Let $g : \hat{R}/R \to E$ be a nonzero map.
Then the map
$f : \hat{R} \to \hat{R}/R \to E$ is a nonzero R-homomorphism

f is not an $\hat{R}$-homomorphism, else
$f(\hat{r}) = \hat{r} f(1) = \hat{r} \cdot 0 = 0 \quad \forall\, \hat{r} \in \hat{R}$, $\times$.
$\therefore$ $\operatorname{Hom}_{\hat{R}}(\hat{R}, E) \subsetneq \operatorname{Hom}_R(\hat{R}, E)$, so $\operatorname{Hom}_R(\hat{R}, E)$ is not Art. //

Graded local cohomology

Let $R = \oplus R_n$ be a $\mathbb{Z}$-graded ring, $x \in R$ a homogeneous element and M a graded R-module. Note that M_x is a graded R- (and R_x-) module, where

$$\deg \frac{m}{x^n} = \deg m - \deg x^n = \deg m - n \deg x.$$

A R-homomorphism $f : M \to N$ of graded R-modules is said to be (homogeneous) of <u>degree</u> 0 if $f(M_n) \subseteq N_n$ for all n. The kernel and image of degree 0 homomorphisms are graded submodules of M and N, resp.

Now, if M is a graded R-module and $\underline{x} = x_1, \ldots, x_n \in R$ is a sequence of homogeneous elements, then it is easy to see that ~~the Čech complex~~ all the maps in the Čech complex $C^{\cdot}(\underline{x}; M)$ are degree 0.

(In the case $n=1$, we have $0 \to M \to M_x \to 0$.
$$m \to \frac{m}{1}$$

Now use induction.)

$\therefore$ The homology modules $H^i_{\underline{x}}(M)$ are graded R-modules.

Since every homogeneous ideal has a homogeneous set of generators, we get that $\forall i$,

$H^i_I(M)$ is a graded R-module for every homogeneous ideal I of R and graded R-module M.

~~Now let R be [illegible] $\mathbb{N}$-graded ring~~

From now on, when we say R is a "graded ring", let's assume R is $\mathbb{N}$-graded.

Then R is a Noetherian graded ring $\Leftrightarrow$ R_0 is Noetherian and $R = R_0[x_1, \ldots, x_n]$ where $x_1, \ldots, x_n$ are homogeneous elements in $R_+ = \bigoplus_{n>0} R_n$. If the x_i can be chosen s/t $\deg x_i = 1 \;\forall i$, we say that R is a ~~homogeneous~~ <u>standard</u> graded ring.

Note that the homogeneous maximal ideals of R are of the form $(m_0, R_+)R$, where m_0 is a maximal ideal of R_0. Thus, R has a unique homogeneous maximal ideal $\Leftrightarrow$ R_0 is local. We'll call such graded rings <u>*local</u> (a la [B-H]). (We'll assume Noetherian in *local.)
We'll say (R, m) is *local, where m is the homog. max'l ideal.

Proposition: Let (R, m) be a *local ring and M a finitely generated graded R-module. Then

(1) $H^i_m(M)_n = 0 \quad \forall\, n \gg 0, \ \forall\, i.$

(2) $H^i_m(M)_n$ is an Artinian R_0-module, $\forall\, i, n$.

Proof:

Note that as every element of $H^i_m(M)$ is annihited by a power of m,

$$H^i_m(M) \cong H^i_{mR_m}(M_m) \quad \forall\, i.$$

In the local case, we showed $H^i_{mR_m}(M_m)$ is Artinian. $\therefore$ $H^i_m(M)$ is an Artinian R-module.

Let $H^i_m(M)_{\geq t} := \bigoplus_{n \geq t} H^i_m(M)_n$.

Then $H^i_m(M)_{\geq t}$ is a graded R-module (as R is $\mathbb{N}$-graded) and

$$H^i_m(M)_{\geq t} \supseteq H^i_m(M)_{\geq t+1} \supseteq \cdots$$

By DCC, $H^i_m(M)_{\geq t} = H^i_m(M)_{\geq t+1} \quad \forall\, t \gg 0$

$\Rightarrow H^i_m(M)_t = 0 \quad \forall\, t \gg 0.$

For (2), suppose

$$H^i_m(M)_n = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

is a descending chain of R_0-submodules of $H^i_m(M)_n$.

Then $RN_0 \supseteq RN_1 \supseteq RN_2 \supseteq \cdots$

is a descending chain of R-submodules of $H^i_m(M)$.

Hence, $RN_t = RN_{t+1}$ for ~~some~~ $t_0 \gg 0$

$$\therefore \quad N_t = RN_t \cap H^i_m(M)_n = RN_{t+1} \cap H^i_m(M)_n$$

$$= N_{t+1} \quad \text{for } t \gg 0.$$

Hence, $H^i_m(M)_n$ is an Artinian R_0-module.

<u>Corollary</u>: Suppose in the above proposition that R_0 is Artinian. Then $\lambda_{R_0}(H^i_m(M)_n) < \infty$ $\forall\, i, n$.

<u>Proof</u>: An Artinian module over an Artinian ring has finite length.

Defn: Let $(R, \mathfrak{m})$ be a $*$local CM standard graded ring. The *a-invariant* of R is defined by

$$a(R) = \sup \{ n \mid H^d_{\mathfrak{m}}(R)_n \neq 0 \} \quad (d = \dim R).$$

Example: Let $R = K[x_1, \dots, x_d]$, K a field.

Then we've seen

$$H^d_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m}) \cong R_{x_1 \cdots x_d} \Big/ \sum R_{x_1 \cdots \hat{x}_i \cdots x_d}$$

$$\cong \bigoplus_{i_j < 0 \ \forall j} K x_1^{i_1} \cdots x_d^{i_d}$$

$\therefore \ a(R) = -d.$

7/7

Proposition: Let $(R,\mathfrak{m})$ be a *local CM standard graded ring. Suppose $x \in R$ is a homogeneous NZD on R. Then

$$a(R/(x)) = a(R) + \deg x$$

Proof: Consider the exact seq (let $k = \deg x$)

$$0 \to R(-k) \xrightarrow{x} R \to R/(x) \to 0$$

Then we have

$$\cdots 0 \to H^{d-1}_{\mathfrak{m}}(R/(x)) \to H^{d}_{\mathfrak{m}}(R(-k)) \xrightarrow{x} H^{d}_{\mathfrak{m}}(R) \to 0.$$

is exact

These are degree 0 maps, so

$$0 \to H^{d-1}_{\mathfrak{m}}(R/(x))_n \to H^{d}_{\mathfrak{m}}(R)_{n-k} \xrightarrow{x} H^{d}_{\mathfrak{m}}(R)_n \to 0.$$

Now, $H^{d-1}_{\mathfrak{m}}(R/(x))_n \neq 0$ if $n = a(R/(x))$.

$\therefore$ $H^{d}_{\mathfrak{m}}(R)_{a(R/(x))-k} \neq 0$. $\therefore$ $a(R) \geq a(R/(x)) - k$.

As $H^{d-1}_{m}(R/(x))_n = 0$ for $n > a(R/(x))$,

$$H^d_m(R)_{n-k} \xrightarrow{x} H^d_m(R)_n \text{ is } 1\text{-}1 \ \forall\, n > a(R/(x)).$$

But every elt in $H^d_m(R)$ is annihilated by a power of x. $\therefore$ $H^d_m(R)_n = 0 \ \forall\, n > a(R/(x)) - k$.

$$\therefore \ a(R) = a(R/(x)) - k.$$

<u>Theorem</u>: Let (R, m) be a CM *local standard graded ring s/t R_0 is Artinian. Then

$a(R) \geq -\dim R$ with equality $\iff R \cong R_0[T_1, \dots, T_d]$ (poly ring)

<u>Proof</u>: Assume R/m is infinite (else ⊗ with $R[T]_{mR[T]}$)
Note that as R_0 is Artinian,
$m = \sqrt{R_+} = \sqrt{R_1 R}$.

Let $n = \mu_{R_0}(R_1)$

Choose minimal generators $x_1, \dots, x_n$ for R_1 s/t $x_1, \dots, x_d$ is an R-regular sequence.
(we can do this since R is CM.
Choose $x_1 \in R_1 - m_0R_1 \cup p_1 \cup \dots \cup p_r$, where $\{p_1, \dots, p_r\} = \mathrm{Ass}(R)$)

Induct on d.

<u>d=0:</u> $H^0_m(R) = R$. ∴ $a(R) \geq 0$.

~~If~~ $a(R) = 0$ ~~then~~ $\iff$ $R = R_0$.

<u>d>0:</u> $a(R) = a(R/(x_1)) - 1 \geq -d+1-1 = -d$.

~~Step~~ Write $R = R_0[T_1, \dots, T_n]\big/ I$ $T_1, \dots, T_n$ indet.

$n = \mu_{R_0}(R_1)$.

Now, $a(R/(\bar{T}_1)) = a(R) + 1 = -d + 1$

∴ $R/(\bar{T}_1) = R/(I, T_1) \cong R_0[\bar{T}_2, \dots, \bar{T}_n]$

∴ $n-1 = d-1$ (by induction).

We need to show $I = 0$.

We have (as $\bar{T}_1, \dots, \bar{T}_n$ are indet.),

$I \subseteq (T_1)$. If $I \neq 0$, Then $\exists$ $f \notin (T_1)$ s/t $f \cdot T_1 \in I$. (else $T_1^r \subseteq I$).

But this means T_1 is a zero-divisor in R, ✱.

∴ $I = 0$ .//

The a-invariant is closely related to the Castelnuovo - Mumford regularity of R.

<u>Defn:</u> Let (R, m) be a * local standard graded ring of dim d s/t R_0 is Artinian. Define

$$a_i(R) := \sup\{ n \mid H^i_m(R)_n \neq 0 \} \quad \text{for } i=0,\dots,d.$$

(Set $a_i(R) = -\infty$ if $H^i_m(R) = 0$.)

The Castelnuovo - Mumford regularity of R is

$$\operatorname{reg}(R) := \max\{ a_i(R) + i \mid i = 0, \dots, d \}$$

One can prove that $\operatorname{reg}(R) \geq 0$ with equality $\iff R \cong R_0[T_1, \dots, T_d]$.

Defn: Let R be a *local standard graded ring such that R_0 is Artinian and M a f.g. graded R-module. As each M_n is a f.g. R_0-module, $\lambda_{R_0}(M_n) < \infty$ $\forall n$. Define the Hilbert function of M by

$$H_M(n) := \lambda_{R_0}(M_n).$$

Example: Let $R = K[x_1, \dots, x_d]$, K a field.

$$\text{Then } H_R(n) = \binom{n+d-1}{d-1}$$

$$= \left(\#\text{ of monomials of degree } n \text{ in } x_1, \dots, x_d\right).$$

Example: $R = k[x,y]/(x^3, xy)$

$H_R(0) = 1$

$H_R(1) = 2$

$H_R(2) = 2$

$H_R(3) = 1$ $\qquad H_R(n) = 1$ $\forall n \geq 3$.

Theorem: Let (R, m) be a $*$local standard ring s/t R_0 is Artinian and M a f.g. graded R-module of dim n. Then $\exists !$ a poly $P_M(x) \in \mathbb{Q}[x]$ s/t $P_M(n) = H_M(n)$ for $n \gg 0$. $P_M(x)$ is the Hilbert poly of M.

Pf: Atiyah-McDonald.

Defn: Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function. Define $\Delta(f) : \mathbb{Z} \to \mathbb{Z}$ by $\Delta(f)(n) = f(n) - f(n-1)$.

Remark: Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be function. Then $\Delta(f) = \Delta(g) \iff f - g$ is a constant.

Pf. easy.

Defn: Let (R, m) be a $*$local standard ring s/t R_0 is Artinian and M a f.g. graded R-module. Define

$$\chi_M(n) := \sum_{i=0}^{\infty} (-1)^i \lambda(H^i_m(M)_n).$$

(Note the sum is actually finite).

Note $\chi_M(n) = 0$ for $n \gg 0$.

If fact, $\chi_M(n) = 0$ for $n > \max\{a_0(M), \ldots, a_d(M)\}$

d = dim M.

Lemma: Let (R, m) be a $*$local standard graded ring ~~and~~ s/t R_0 is Artinian and

$$0 \to A \to B \to C \to 0$$ a s.e.s.

of f.g. graded R-modules with degree 0 maps.

Then (1) $H_B(n) = H_A(n) + H_C(n) \quad \forall n$

(2) $P_B(x) = P_A(x) + P_C(x)$

(3) $\chi_B(n) = \chi_A(n) + \chi_C(n) \quad \forall n.$

Pf: (1) follows from the exactness of

$$0 \to A_n \to B_n \to C_n \to 0 \quad \forall n.$$

(2) immediate from (1).

(3) We have a l.e.s. with degree 0 maps

$$\cdots \to H^i_m(A) \to H^i_m(B) \to H^i_m(C) \to \cdots$$

So

$$\cdots \to H^i_m(A)_n \to H^i_m(B)_n \to H^i_m(C)_n \to \cdots$$

is exact $\forall n$. Use the additivity of λ.

<u>Theorem</u>: Let $(R, \mathfrak{m})$ be a $*$local standard graded ring s/t R_0 is Artinian and M a f.g. graded R-module. Then

$$H_M(n) - P_M(n) = \chi_M(n) \quad \forall\, n.$$

<u>Pf.</u> Let $R = R_0[x_1, \ldots, x_s]$, where $x_1, \ldots, x_s \in R_1$. Induct on s.

<u>$s=0$:</u> Then $R = R_0$ and $\lambda(M) < \infty$.

$\therefore$ $M_n = 0$ for $n \gg 0 \Rightarrow P_M(n) = 0 \ \forall n$.

$H^0_{\mathfrak{m}}(M) = M$ and $H^i_{\mathfrak{m}}(M) = 0 \ \forall\, i > 0$.

$\therefore$ $\chi_M(n) = \lambda(M_n) = H_M(n)$. ✓

<u>$s>0$:</u> Consider the exact seq

$$0 \to K \to M(-1) \xrightarrow{x_s} M \to C \to 0$$

of graded R-modules and degree 0 maps.

By the lemma,

$$\begin{aligned}\Delta(H_M(n) - P_M(n)) &= H_M(n) - H_M(n-1) - P_M(n) + P_M(n-1) \\ &= H_C(n) - P_C(n) - (H_K(n) - P_K(n))\end{aligned}$$

Now, $x_r K = 0 = x_r C$, so K and C are

R/x_sR - modules. By induction on s,

$$\Delta(H_M(n) - P_M(n)) = \chi_C(n) - \chi_K(n)$$

$$= \chi_M(n) - \chi_M(n-1)$$

$$= \Delta(\chi_M(n))$$

By the remark,

$$H_M(n) - P_M(n) = \chi_M(n) + C$$

But $\chi_M(n) = 0$ for $n \gg 0$ and $H_M(n) - P_M(n) = 0$ for $n \gg 0$. $\therefore$ $C = 0$. //

<u>Corollary</u>: Let $(R, \mathfrak{m})$ be a CM *local standard graded ring s/t R_0 is Artinian. Then

$$a(R) = \min\{n \in \mathbb{Z} \mid P_R(n) \neq H_R(n)\}.$$

<u>Pf</u>:

$$H_R(n) - P_R(n) = (-1)^d \lambda(H^d_{\mathfrak{m}}(R)_n) .//$$

7/8 <u>Question:</u> Let (R,m) be a local ring, M a f.g. R-module and $I \subseteq R$. When is $H^i_I(M)$ f.g.?

Certainly when $i=0$. However, not always:

<u>Remark:</u> $H^i_I(M)$ is a f.g. R-module $\iff$ $H^i_{I\hat{R}}(\hat{M})$ is a f.g. $\hat{R}$-module.

<u>Proposition:</u> Let (R,m) be a local ring and M a f.g. R-module of dim $n>0$. Then $H^n_m(M)$ is not f.g.

<u>Proof:</u> WLOG, we may assume R is complete. By change of rings, we may assume $\operatorname{ann}_R M = 0$, so $\dim M = \dim R = n > 0$.

Write $R = T/I$ where T is a complete Gorenstein local ring of dim n. By change of rings, assume $R=T$.

Then $H^n_m(M) \cong \operatorname{Hom}_T(M,T)^\vee$

Easier proof: If $H^n_m(M)$ is f.g then $H^n_m(M)\otimes R/m \neq 0$
$\cong$
But $H^n_m(M/mM) = 0$
as $\dim M/mM = 0$.

If $H^n_m(M)$ has finite length then $\operatorname{Hom}_T(M,T)$ has finite length. Choose $p \supseteq I$

s.t. Let $p=0$. Then

$0 = \operatorname{Hom}_T(M,T)_p = \operatorname{Hom}_{T_p}(M_p,T_p) = (M_p)^\vee \neq 0$, $*$. (as $M_p \neq 0$)

Proposition: Let R be a Noeth ring, $I \subseteq R$, M a f.g. R-module. TFAE:

(1) $H^i_I(M)$ is f.g. $\forall \ i \le t$

(2) $I \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)} \ \ \forall \ i \le t$

(i.e., $\exists$ k s/t $I^k H^i_I(M) = 0 \ \ \forall \ i < t$)

Proof: (1) $\Rightarrow$ (2): Clear, as every element in $H^i_I(M)$ is killed by a power of I.

(2) $\Rightarrow$ (1): Induct on t.

$t = 0$: OK.

Suppose $t > 0$. Let $L = H^0_I(M)$ and $N = M/L$.

Then $H^0_I(L) = L$ and $H^i_I(L) = 0 \ \ \forall \ i \geq 1$.

∴ From the l.e.s.

$$\cdots \to H^i_I(L) \to H^i_I(M) \to H^i_I(N) \to \cdots$$

we get $H^0_I(N) = 0$ and $H^i_I(N) \cong H^i_I(M) \ \ \forall \ i \geq 1$.

Hence, we may assume $\operatorname{depth}_I M > 0$.

③

Let $x \in I$ s/t $x^{\in I}$ is a NZD on M.
By assumption, $\exists\ k$ s/t $x^k H^i_I(M) = 0 \ \forall\ i \leq t$.
As x^k is a NZD on M, replace x^k by x.

From $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get

$$\cdots \xrightarrow{0} H^{t-1}_I(M) \to H^{t-1}_I(M/xM) \to H^t_I(M) \xrightarrow[0]{x} H^t_I(M)$$

$\cdots$ ~~$0 \to H^{t-2}_I(M) \to H^{t-2}$~~

By induction, $H^i_I(M)$ is f.g. $\forall\ i \leq t-1$.

Also, as $I^k H^i_I(M) = 0 \ \forall\ i \leq t$ and

$$0 \to H^{i-1}_I(M) \to H^{i-1}_I(M/xM) \to H^i_I(M) \to 0 \text{ is exact } \forall\ i \leq t$$

$I^{2k} H^{i-1}_I(M/xM) = 0 \ \forall\ i \leq t$. $\therefore$ ~~$H^{t-1}_I(M)$~~

$\therefore\ H^{t-1}_I(M/xM)$ is f.g. $\Rightarrow H^t_I(M)$ is f.g.//

Thus, the finite generation of $H^i_I(M)$ is related to the annihilation of $H^i_I(M)$.

<u>Theorem</u>: (Faltings, 1978) Let (R, m) be a local ring which is the homomorphic image of a RLR. Let M be a f.g. R-module and $J \subseteq I$ two ideals of R.

$$\text{Set} \quad s = \min_{p \not\supseteq J} \left\{ \operatorname{depth} M_p + \operatorname{ht} {}^{I+p}/_{p} \right\}$$

$$\text{Then} \quad (1) \quad J \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)} \quad \forall\, i < s$$

$$(2) \quad J \not\subseteq \sqrt{\operatorname{Ann}_R H^s_I(M)}$$

(<u>note</u>: we define $\operatorname{depth} M_p = \infty$ if $M_p = 0$.)
Also, $\min \phi = \infty$.

As a corollary, we get

<u>Theorem</u>: (Grothendieck, SGA II, 1968) Let (R, m) be a local ring which is the quotient of a RLR. Let M be a f.g. R-mod and $I \subseteq R$.

$$\text{Set} \quad s = \min_{p \not\supseteq I} \left\{ \operatorname{depth} M_p + \operatorname{ht} {}^{I+p}/_{p} \right\}$$

$$\text{Then} \quad H^i_I(M) \text{ is f.g. } \forall\, i < s$$

$$H^s_I(M) \text{ is not f.g.}$$

Pf: Set $J = I$ in Faltings Thm and use the Prop.

⑤

<u>Lemma</u>: Let (R, m) be a local ring which is the quotient of a Gorenstein ring. Let M be a f.g. R-module and $J \subseteq R$ an ideal. Then

$$J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)} \iff \forall\ p \not\supseteq J \qquad H^{i-\dim R/p}_{pR_p}(M_p) = 0.$$

<u>Pf</u>: Let $R = T/I$ where (T,n) is a Gorenstein local ring. Let $K \subseteq T$ s/t $K/I = J$.

Then, by change of rings,

$$J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)} \iff K \subseteq \sqrt{\operatorname{Ann}_T H^i_n(M)}.$$

Also, if $q \supseteq I$, $q \not\supseteq K$ then $H^{i-\dim \cancel{R}^{T}/q}_{qT_q}(M_q) \cong H^{i-\dim R/q}_{pR_p}(M_p)$

where $p = q/I$.

If $q \not\supseteq I$ then $M_q = 0$, which is OK.

Hence, we may assume (R, m) is a Gorenstein local ring.

Now, $J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$ ~~$\iff J \subseteq \sqrt{\operatorname{Ann}_R H^i_m(M)}$~~

$$\iff J \subseteq \sqrt{\operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M,R)^{\vee}}$$

$$\iff J \subseteq \sqrt{\operatorname{Ann}_R \operatorname{Ext}^{d-i}_R(M,R)}$$

$\iff \forall\ p \not\supseteq J\ ,\ \mathrm{Ext}^{d-i}_{R_p}(M_p, R_p) = 0$

$\iff \forall\ p \not\supseteq J\ ,\ H^{\dim R_p - d + i}_{pR_p}(M_p) = 0$

and $d - \dim R_p = \dim R/p$. //

<u>Proposition</u>: Let (R, m) be a local ring which is the quotient of a Gorenstein ring. Let M be a f.g. R-module and $J \subseteq R$ an ideal. Let

$$s = \min_{p \not\supseteq J} \{\operatorname{depth} M_p + \dim R/p\}.$$

Then $J \subseteq \sqrt{\mathrm{Ann}_R H^i_m(M)} \quad \forall\ i < s$

and $J \not\subseteq \sqrt{\mathrm{Ann}_R H^s_m(M)}$.

<u>Proof</u>: By the lemma,

$J \subseteq \sqrt{\mathrm{Ann}_R H^i_m(M)} \quad \forall\ i < t$

$\iff H^{i - \dim R/p}_{pR_p}(M_p) = 0 \quad \forall\ p \not\supseteq J,\ i < t$

$\iff \forall\ p \not\supseteq J,\ t - \dim R/p \leq \operatorname{depth} M_p$

$\iff t \leq s$. //

<u>Lemma 1</u>: Let (R, m) be a CM local ring, M a f.g. R-module, $I \subseteq R$. Suppose $\exists$ $p \in \operatorname{Spec} R$ s/t M_p is free. Then $\exists$ $s \in R - p$ s/t $sH^i_I(M) = 0$ $\forall$ $i < \operatorname{ht} I$.

<u>Proof</u>: There exists exact seq's

$$0 \to C \to F \to T \to 0$$

$$0 \to T \to M \to D \to 0$$

s/t F is a f.g. free R-module and $C_p = D_p = 0$. Choose $s \notin p$ s/t $sC = sD = 0$.

Then $sH^i_I(C) = sH^i_I(D) = 0$ $\forall i$.

Now we have

$$\cdots \to H^i_I(T) \to H^i_I(M) \to H^i_I(D) \to \cdots$$

$$\cdots \to H^i_I(F) \to H^i_I(T) \to H^{i+1}_I(C) \to \cdots$$

As R is CM, $H^i_I(F) = \oplus H^i_I(R) = 0$ $\forall$ $i < \operatorname{ht} I$.
$\therefore$ $sH^i_I(T) = 0$ $\forall$ $i < \operatorname{ht} I$.

Hence, $s^2 H^i_I(M) = 0$ $\forall$ $i < \operatorname{ht} I$. //

Proof of part (i) of Faltings Theorem (due to M. Brodmann) 1983

Set $s(J, I, M) := \min_{p \not\supseteq J} \left\{ \operatorname{depth} M_p + \operatorname{ht} \frac{I+p}{p} \right\}$.

We use induction on $\dim R/I$ to prove $\exists\, k$ s/t

$$J^k H^i_I(M) = 0 \quad \forall \quad i < s = s(J, I, M).$$

The case $\dim R/I = 0$ is taken care of by Prop 1.
So assume $\dim R/I > 0$.
We make a series of reductions.

Reduction 1: We may assume R is a RLR.

Pf: Write $R = T/L$ where T is a RLR.

Let I', J' be ideals of T s/t $I'/L = I$, $J'/L = J$.
Then, as noted in the lemma preceding Prop 1, $s(J', I', M) = s(J, I, M)$, and $H^i_{I'}(M) \cong H^i_I(M) \quad \forall\, i$.

Reduction 2: We may assume $s(J, I, M) < \infty$

Pf: $s(J, I, M) = \infty \iff M_p = 0 \;\; \forall\, p \not\supseteq J$

$\iff J \subseteq \sqrt{\operatorname{Ann}_R M}$

$\overset{\exists k,}{\Longrightarrow} J^k H^i_I(M) = 0 \quad \forall\, i$ //

<u>Reduction ^{3}</u>: we may assume $\operatorname{depth}_J M > 0$.

Pf. Let $N = M/H^0_J(M)$. $N \neq 0$ else $J^k M = 0$ for some k $\Rightarrow s(J, I, M) = \infty$.

Then, as $H^0_J(M)_p = 0$ $\forall\, p \not\supseteq J$, $M_p \cong N_p$ $\forall\, p \not\supseteq J$. $\therefore s(J,I,M) = s(J,I,N)$. Furthermore, as remarked before, ~~$H^0_J(N) = 0$ and the~~ $\operatorname{depth}_J N > 0$.

From $0 \to H^0_J(M) \to M \to N \to 0$, we get

$$\cdots \to H^i_I(H^0_J(M)) \to H^i_I(M) \to H^i_I(N) \to \cdots$$

If we know the theorem for N, then

$$J^k H^i_I(N) = 0 \quad \forall\ i < s = s(J,I,M).$$

As $J^\ell H^0_J(M) = 0$, some ℓ, $J^\ell H^i_I(H^0_J(M)) = 0$ $\forall i$.

$\therefore$ $J^{\ell+k} H^i_I(M) = 0$ $\forall\ i < s$. //

Reduction ~~3~~4: We may assume $J \supseteq \mathrm{ann}_R M$.

Pf: By change of rings,

$$H^i_I(M) \cong H^i_{I\cdot R/\mathrm{Ann}_R M}(M) \cong H^i_{I+\mathrm{Ann}_R M}(M) \quad \forall i.$$

Also, as $\mathrm{Ann}_R M \subseteq \sqrt{\mathrm{Ann}_R H^i_I(M)}$ $\forall i$,

$$J \subseteq \sqrt{\mathrm{Ann}_R H^i_I(M)} \iff J + \mathrm{Ann}_R M \subseteq \sqrt{\mathrm{Ann}_R H^i_I(M)}.$$

Finally, if $p \not\supseteq \mathrm{ann}_R M$ then $\operatorname{depth} M_p = \infty$.

Hence, $s(J+\mathrm{ann}_R M, I+\mathrm{ann}_R M, M) = s(J,I,M)$. //

~~Reduction~~ [illegible]

Claim 1: $s(J,I,M) \le \operatorname{ht} I$. Furthermore, if $s(J,I,M) = \operatorname{ht} I$ then $\mathrm{Ann}_R M = 0$.

Pf: Let $h = \operatorname{ht} I$ and ~~$q_1, \dots, q_t$ the minimal primes of I of height h.~~ q a prime minimal over I of ht h. ~~Let $p \subseteq q$ be~~ Let p be a prime minimal over $\mathrm{Ann}_R M$ contained in q. Then $p \not\supseteq J$ as $\operatorname{depth}_J M > 0$.

$\therefore$ $s(J,I,M) \le \operatorname{depth} M_p + \operatorname{ht}\left(\frac{I+p}{p}\right) \le \operatorname{ht}(q/p) \le h$.

Furthermore, as R is a domain, we get $= \iff p = 0 \iff \mathrm{Ann}_R M = 0$. //

①

7/9 Continuation of the proof of part (i) of Faltings' Thm:

(R, m) local ring which is the quotient of a RLR.
M f.g. R-module, $J \subseteq I$ ideals.
$s = s(J,I,M) = \min_{p \not\supseteq J} \{ \operatorname{depth} M_p + \operatorname{ht} \frac{I+p}{p} \}$.

Then $\exists\, k$ s/t $J^k H^i_I(M) = 0 \quad \forall\, i < s$.

Induction on dim R/I. The case dim $R/I = 0$ is done.
We've reduced to the case

- R is a RLR
- $s(J,I,M) < \infty$
- $\operatorname{depth}_J M > 0$
- $J \supseteq \operatorname{Ann}_R M$.

Claim 1: $s(J,I,M) \leq \operatorname{ht} I$. Furthermore, if $s(J,I,M) = \operatorname{ht} I$ then $\operatorname{Ann}_R M = 0$.

Pf: Let q be a prime minimal over I s/t $\operatorname{ht} q = \operatorname{ht} I = h$. As $I \supseteq J \supseteq \operatorname{Ann}_R M$, q contains a prime p which is minimal over $\operatorname{Ann}_R M$. Then $p \in \operatorname{Ass}_R M$, so $p \not\supseteq J$ as $\operatorname{depth}_J M > 0$.

$\therefore\ s(J,I,M) \leq \operatorname{depth} M_p + \operatorname{ht}\left(\frac{I+p}{p}\right) \leq \operatorname{ht} q/p \leq h$.

If we have equality, then (as R is a domain) $p = 0$. $\therefore \operatorname{Ann}_R M = 0$.

<u>Case 1</u>: $s := s(J, I, M) = \operatorname{ht} I =: h$.

By the claim, $\operatorname{Ann}_R M = 0$.

Let $\mathcal{U} = \{ p \in \operatorname{Spec} R \mid M_p \text{ is free} \}$.

$\mathcal{U} \neq \phi$ as $M_{(0)}$ is free and $\mathcal{U}$ is open (902 exercise). Let $\mathcal{U} = \operatorname{Spec} R - V(L)$, $L \subseteq R$.

Let $\Lambda := \{ p \in \operatorname{Min} R/L \mid p \not\supseteq J \}$.

<u>Case 1(a)</u>: $\Lambda = \phi$

Then $p \not\supseteq J \Rightarrow p \not\supseteq L$

$\Rightarrow M_p$ is free.

By Lemma 1, $\forall\ p \not\supseteq J\ \exists\ s_p \notin p$ s/t

$$s_p H^i_I(M) = 0 \quad \forall\ i < h = s.$$

Let $A = (\{s_p\}_{p \not\supseteq J})R$. Then $A \cdot H^i_I(M) = 0\ \forall\, i < s$.

Furthermore, $J \subseteq \sqrt{A}$. For if $q \in \operatorname{Spec} R$, $q \supseteq A$ then $q \supseteq J$ (else $\exists\ s_q \in A$, $s_q \notin q$).

$\therefore\ \exists\ k$ s/t $J^k H^i_I(M) = 0\ \forall\, i < s$. DONE!

Case 1(b): $\Lambda \neq \phi$.

Let $\Lambda = \{p_1, \dots, p_s\}$.

Let $\{q_1, \dots, q_t\}$ be the minimal primes of ht h.

Claim 2: $\bigcap_{i=1}^{s} p_i \not\subseteq \bigcup_{i=1}^{t} q_i$

Pf: Suppose not. Then $p_i \subseteq q_j$ for some j.

Then M_{p_i} is not free as $p_i \notin U$.

By Auslander-Buchsbaum, this means

$$\operatorname{depth} M_{p_i} < \dim R_{p_i}$$

$$\therefore \quad s \leq \operatorname{depth} M_{p_i} + \operatorname{ht}\left(\frac{I+p_i}{p_i}\right) \quad (\text{as } p_i \not\supseteq J)$$

$$< \dim R_{p_i} + \operatorname{ht} q_j/p_i = \operatorname{ht}(q_j) = h, \; * //$$

So choose $x \in \bigcap_{i=1}^{s} p_i \setminus \bigcup_{i=1}^{t} q_i$

Note that

- $\dim R/(I,x) < \dim R/I$ as $x \notin \bigcup_{i=1}^{t} q_i$
- If $p \not\supseteq J$ and $x \notin p$ then M_p is free.

(Else, $p \supseteq L \Rightarrow p \supseteq p_i$, some i., $*$ as $x \in p_i$).

Claim 3: $J \subseteq \sqrt{\operatorname{Ann}_R H^i_{I_x}(M_x)}$ $\forall$ $i < s = h$.

Pf: ETS $J_x \subseteq \sqrt{\operatorname{Ann}_{R_x} H^i_{I_x}(M_x)}$ $\forall$ $i < h$.

$\forall$ $p_x \in \operatorname{Spec}(R_x)$, $p_x \not\supseteq J_x$, $(M_x)_{p_x} \cong M_p$ is free.

$\therefore$ By the same argument appearing in case 1(a),

$\exists$ k s/t $J_x^k H^i_{I_x}(M_x) = 0$ $\forall$ $i < ht(I_x) = h$. //

Claim 4: $J \subseteq \sqrt{\operatorname{Ann}_R H^i_{(I,x)}(M)}$ $\forall$ $i < s$.

Pf. Note that as $ht\left(\frac{(I,x)+p}{p}\right) \geq ht\left(\frac{I+p}{p}\right)$ $\forall$ p,

$s' = s(J,(I,x),M) \geq s$. As $\dim R/(I,x) < \dim R/I$,
we have by induction the claim by induction

Now, we have the l.e.s.

$$\cdots \longrightarrow H^i_{(I,x)}(M) \longrightarrow H^i_I(M) \longrightarrow H^i_{I_x}(M_x) \longrightarrow \cdots$$

So Case 1 follows from Claim 3 and 4.

Case 2: $s < h$.

Use induction on $s - h \geq 0$. (The case $s-h=0$ is case 1.)

Let F be a f.g. free R-module s/t

$0 \to K \to F \to M \to 0$ is exact.

Claim 5: $s' := s(J, I, K) > s$.

Pf: Let $p \in \operatorname{Spec} R$, $p \not\supseteq J$.

If M_p is free: then K_p is free

$$\text{Then, } \operatorname{depth} K_p + ht\left(\frac{I+p}{p}\right) = \dim R_p + ht\left(\frac{I+p}{p}\right)$$

$$= ht(I+p) \geq ht\, I > s.$$

If M_p is not free:

Then $pd\, K_p = pd\, M_p - 1$.

By A-B, $\operatorname{depth} K_p = \operatorname{depth} M_p + 1$

$$\therefore \operatorname{depth} K_p + ht\left(\frac{I+p}{p}\right) > \operatorname{depth} M_p + ht\left(\frac{I+p}{p}\right) \geq s. //$$

Thus, $h - s' < h - s$. (note that $\operatorname{depth}_J K > 0$ and $\operatorname{Ann}_R K = 0$, as $K \subseteq F$ and R is a domain. Thus, claim 1 ($s' \leq h$) still holds.)

By induction,

$$J \subseteq \sqrt{\operatorname{Ann}_R H^i_I(K)} \quad \forall \; i < s' \text{ (hence for } i+1 < s)$$

As R is a RLR, $H^i_I(F) = 0 \;\; \forall \; i < h \, (> s)$.

From the l.e.s.

$$\cdots \longrightarrow H^i_I(F) \longrightarrow H^i_I(M) \longrightarrow H^{i+1}_I(K)$$

we get

$$J \subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)} \quad \forall \; i < s.$$

QED.//

Proof of part (2) of Faltings Theorem:

Set-up: $(R, \mathfrak{m})$ local ring
M f.g. R-module
$J \subseteq I$ ideals of R

$$s(J,I,M) = \min_{p \not\supseteq J} \left\{ \operatorname{depth} M_p + \operatorname{ht}\left(\frac{I+p}{p}\right) \right\}.$$

We'll show that if $s = s(J,I,M) < \infty$ then

$$J \not\subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)} \text{ for some } i \leq s.$$

As in the proof of part (1), we may replace M by $M/H^0_J(M)$ and assume $\operatorname{depth}_J M > 0$.

We induct on s.

Note that if $p \not\supseteq J$ then $\operatorname{ht}\left(\frac{I+p}{p}\right) \geqslant 1$.
Thus, $s \geqslant 1$.

$\underline{s=1:}$ Need to show $J \not\subseteq \sqrt{\operatorname{Ann} H^1_I(M)}$.

Choose $p \not\supseteq J$ s/t $1 = \operatorname{depth} M_p + \operatorname{ht}\left(\frac{I+p}{p}\right)$.

Then $\operatorname{depth} M_p = 0$ and $\operatorname{ht}\left(\frac{I+p}{p}\right) = 1$.

Then $p \in \mathrm{Ass}_R M$, so $\exists$ an exact seq

$$0 \to R/p \to M \to N \to 0$$

$\therefore \quad 0 \to H^0_I(N) \to H^1_I(R/p) \to H^1_I(M)$ is exact.

Suppose $J \subset \sqrt{\mathrm{Ann}_R H^1_I(M)}$.

As $H^0_I(N)$ is f.g., $J \subseteq I \subseteq \sqrt{\mathrm{Ann}_R H^0_I(N)}$.

Thus, $J \subseteq \sqrt{\mathrm{Ann}_R H^1_I(R/p)}$.

As $\mathrm{ht}\left(\frac{I+p}{p}\right) = 1$, choose $q \supseteq I + p$ s/t $\mathrm{ht}(q/p) = 1$.

Then $J_q \subseteq \sqrt{\mathrm{Ann}_R H^1_{I_q}(R_q/p_q)}$

~~Now $p \not\supseteq J$ so $\sqrt{m}$~~

Let $A = R_q/p_q$ with maximal ideal $n = (q/p)_q$.

Then A is a 1-dim'l local domain.

As $p \not\supseteq J$,

Hence $n = \sqrt{\mathrm{Ann}_R H^1_n(A)}$

$\sqrt{J_q A} = \sqrt{I_q A} = n$. $\Rightarrow H^1_n(A)$ is f.g., $\divideontimes$.

Now suppose $s > 1$:

Choose $p \nsupseteq J$ s/t $s = \text{depth}\, M_p + \text{ht}\left(\frac{I+p}{p}\right)$.

Let q be a prime $\supseteq I+p$ s/t $\text{ht}(q/p) = \text{ht}(I+p/p)$.

Let $y \in J \sim p$. and consider the set

$$\Lambda = \{ Q \in \text{Spec}\, R \mid p \subseteq Q \subseteq q \ , \ y \notin Q \}.$$

$p \in \Lambda$ so $\Lambda \neq \emptyset$.

Choose $Q \in \Lambda$ maximal. Clearly, $Q \nsupseteq J$.

<u>Claim</u>: $\text{ht}\, q/Q = 1$.

<u>Pf:</u> ~~suppose~~ Clearly, $q \subsetneq Q$ as $y \in J \subseteq I \subseteq q \}$.

Suppose $\text{ht}(q/Q) > 1$.

By Prime avoidance and KPIT, $\exists\ Q_1 \subseteq q$ s/t $y \notin Q_1$ and $\text{ht}(Q_1/Q) > 0$. But then $Q_1 \in \Lambda$, $*$ maximality of Q. //

Claim 2: $s = \text{depth}\, M_Q + \text{ht}(I+Q/Q)$.

Pf: $s = \text{depth}\, M_p + \text{ht}\left(\frac{I+p}{p}\right) \le \text{depth}\, M_Q + \text{ht}\left(\frac{I+Q}{Q}\right)$

by definition of s. Also

$$\text{depth}\, M_Q + \text{ht}\left(\frac{I+Q}{Q}\right) \le \text{depth}\, M_Q + \text{ht}(q/Q) \quad (\text{as } q \supseteq I+Q)$$

$$*(\text{see below}) \longrightarrow \quad \le \text{depth}\, M_p + \text{ht}(Q/p) + \text{ht}(q/Q)$$

$$\le \text{depth}\, M_p + \text{ht}(q/p)$$

$$= \text{depth}\, M_p + \text{ht}\left(\frac{I+p}{p}\right)$$

To see the inequality (*), we need to show that if (R, m) is local, M a f.g. R-module and $p \in \text{Spec}\, R$ then

$$\text{depth}\, M \le \text{depth}\, M_p + \dim R/p.$$

But this follows from Ischebeck's Theorem (Mat's, Theorem 17.1).

This proves Claim 2. //

By Claim 1, q is minimal over $I+Q$
and $\operatorname{ht}(q/Q) = 1$.

Replace Q by p (so we assume $\operatorname{ht}\left(\frac{I+p}{p}\right)=1$).

It is enough to show

$$J_q \not\subseteq \sqrt{\operatorname{Ann} H^i_{I_q}(M_q)} \quad \text{for some } i \le s.$$

∴ Localize at q and assume $q = m$.

Hence, $s = \operatorname{depth} M_p + \dim R/p$
$= \operatorname{depth} M_p + 1$.

Claim 3: p contains a NZD.

Pf: If not p is contained in an associated prime of M. As $\dim R/p = 1$, and $\operatorname{depth}_J M > 0$, $p \in \operatorname{Ass}_R M$.

Then $\operatorname{depth} M_p = 0$ and $s = 1$, $\ast$ $(s>1)$.//

Now, let $x \in p$ be a NZD on M.

Then $0 \to M \xrightarrow{x} M \to M/xM \to 0$ exact.

Note that

$$s' = s(J, I, M/xM) \leq s-1$$

as $\operatorname{depth}(M/xM)_p = \operatorname{depth} M_p - 1$.

$\therefore$ For some $i \leq s-1$, $J \not\subseteq \sqrt{\operatorname{Ann}_R H^i_I(M/xM)}$.

From $\cdots \to H^i_I(M) \to H^i_I(M/xM) \to H^{i+1}_I(M) \to \cdots$

we see that

$$J \not\subseteq \sqrt{\operatorname{Ann}_R H^i_I(M)} \text{ for some } i \leq s.$$

Q.E.D!