Math 906: Commutative Algebra II

These notes are based on a graduate course I gave during the 2024 spring semester at UNL. This course is the third in a three-semester sequence, following Math 905: *Commutative Algebra I* and Math 915: *Homological Algebra*, both taught by my colleague Eloísa Grifo. In these notes, I frequently cite results from her lecture notes for those courses, which can be found at `https://eloisagrifo.github.io/teaching.html`. I am grateful to the students for catching many typos and other errors (some minor, some not so minor) in previous drafts of this document.

I also want to emphasize that, while the material is arranged and presented in my own fashion, none of the results presented here (save the occasional odd lemma) are original, and I have borrowed liberally from the standard references for commutative algebra, particularly *Introduction to Commutative Algebra* by Atiyah-Macdonald, *Commutative Ring Theory* by H. Matsumura, and *Cohen-Macaulay Rings* by Bruns-Herzog.

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Contents

1 Injective modules over Noetherian rings

Throughout these notes, unless otherwise stated, all rings are commutative with an identity. Similarly, unless less otherwise stated, by an R-algebra we mean a commutative ring S equipped with a ring homomorphism $R \to S$, giving a natural R-module structure on S.

Let R be a ring. Recall that an R-module E is injective if the functor $\operatorname{Hom}_R(-, E)$ is exact.

Remark 1.1. Let R be a ring. The following facts were proved in Math 915 except for (b) and (i). (See section 4.2 of Grifo's notes and the homework.)

(a) (Baer's Criterion) *An R-module E is injective if and only if for every ideal I of R the map $i^* : \operatorname{Hom}_R(R, E) \to \operatorname{Hom}_R(I, E)$ is surjective, where $i : I \to R$ is the inclusion map.*

(b) An R-module E is injective if and only if $\operatorname{Ext}^1_R(R/I, E) = 0$ for all ideals I of R.

(c) (Change of rings) *If E is an injective R-module and S an R-algebra then $\operatorname{Hom}_R(S, E)$ is an injective S-module.*

(d) Every R-module can be embedded in an injective module.

(e) An R-module E is injective if and only if every short exact sequence of R-modules of the form $0 \to E \to M \to N \to 0$ splits.

(f) Direct summands of injective modules are injective.

(g) Arbitrary products of injective modules are injective. (Thus, finite direct sums of injectives are injective.)

(h) If R is Noetherian then arbitrary direct sums of injectives are injective. (Note: The converse is also true.)

(i) If R is a domain then any torsion-free divisible module is injective.

Proof of (b): Let I be an ideal of R and consider the s.e.s $0 \to I \to R \to R/I \to 0$. Applying $\operatorname{Hom}_R(-, E)$ we obtain the exact sequence

$$\operatorname{Hom}_R(R, E) \xrightarrow{i^*} \operatorname{Hom}_R(I, R) \to \operatorname{Ext}^1_R(R/I, E) \to 0.$$

This is a portion of the corresponding long exact sequence on Ext. Note we have a zero on the right since $\operatorname{Ext}^1_R(R, E) = 0$ (since R is projective). From this exact sequence, we see that i^* is surjective if and only $\operatorname{Ext}^1_R(R/I) = 0$. The result then follows by applying Baer's criterion.

Proof of (i): Let M be a torsion-free divisible R-module. Let I be an ideal of R and $f : I \to M$ a homomorphism. We need to find a homomorphism $\tilde{f} : R \to M$ such that $\tilde{f}(i) = f(i)$ for all $i \in I$. This is trivial if $I = 0$ so assume $I \neq 0$. Let $x \in I$, $x \neq 0$. As M is divisible, there exists an element $u \in M$ such that $xu = f(x)$. Now let $i \in I$ be arbitrary. Then $xf(i) = f(xi) = if(x) = ixu$. As M is torsion-free and $x \neq 0$, we obtain $f(i) = iu$ for all $i \in I$. Now define $\tilde{f} : R \to M$ by $\tilde{f}(r) = ru$ for all $r \in R$. It is clear that $\tilde{f}$ extends f. Hence, M is injective by Baer's criterion.

Definition 1.2. A containment of R modules $L \subseteq M$ is called an *essential extension* if every nonzero submodule of M has nonzero intersection with L; equivalently, for all nonzero elements x of M, $Rx \cap L \neq 0$. The extension is called *proper* if $L \neq M$.

Example 1.3. Let R be a domain and Q its field of fractions. Then $R \subseteq Q$ is an essential extension.

Remark 1.4. Let $L \subseteq M \subseteq N$ be R-modules. Then $L \subseteq N$ is essential if and only if $L \subseteq M$ and $M \subseteq N$ are essential.

Proposition 1.5. *Let R be a ring and E an R-module. Then E is injective if and only if there does not exist a proper essential extension of E.*

Proof. Suppose E is injective and $E \subseteq M$ is an essential extension. By Remark 1.1(e), the short exact sequence $0 \to E \to M \to M/E \to 0$ splits, so $M \cong E \oplus M/E$. If $M/E \neq 0$ then $M/E \cap E \neq 0$ since $E \subseteq M$ is essential. (Here we are identifying M/E with a submodule of M.) But this is a contradiction, since M is the direct sum of E and M/E. Thus, $M/E = 0$ and $M = E$.

Conversely, suppose there does not exist a proper essential extension of E. By Remark 1.1(d), E can be embedded in an injective module I. Consider the following set of submodules of I:

$$\Lambda := \{K \subseteq I \mid K \cap E = 0\}.$$

This set is nonempty (as it contains the zero module) and partially ordered by inclusion. It is easily seen that Zorn's Lemma applies here, so let L be a maximal element in Λ. Consider the map $f : E \to I/L$ given by $f(e) = e + L$. Then f is injective since $E \cap L = 0$. Identifying E with its image $(E + L)/L$ in I/L, we have by the maximality of L that $E \subseteq I/L$ is essential. (If not, there would be a nonzero submodule N/L of I/L which intersects E trivially. But this means $L \subsetneq N$ and $N \cap E = 0$, a contradiction.) By assumption, $E = I/L$; in other words, the map $f : E \to I/L$ is an isomorphism. Hence, $I = E + L$ and $E \cap L = 0$. Thus, $I = E \oplus L$. Since I is injective, so is E.

□

Theorem 1.6. *Let R be a ring and $M \subseteq E$ an extension of R-modules. The following are equivalent:*

(a) E is a maximal essential extension of M;

(b) E is a minimal injective module containing M;

(c) E is injective and essential over M.

Moreover, for each R-module M there exists an E satisfying the equivalent conditions (a), (b), and (c). Further, any two such modules are isomorphic via an isomorphism fixing M.

Proof. $(a) \implies (c)$: It suffices to prove that E is injective. Let $E \subseteq L$ be an essential extension of R-modules. Then $M \subseteq L$ is also essential. Since E is a maximal essential extension of M, we must have $E = L$. Thus, E has no proper essential extension, so E is injective by Proposition 1.5.

$(c) \implies (b)$: Suppose E' is an injective module contained in E and containing M. Since E is essential over M, E is essential over E' as well. But as E' is injective, it has no proper essential extension by Proposition 1.5. Thus, $E' = E$ and we have proved E is a minimal injective module containing M.

$(b) \implies (a)$: Consider the set of submodules

$$\Lambda := \{L \mid M \subseteq L \subseteq E \text{ with } M \subseteq L \text{ essential}\}.$$

Then $\Lambda \neq \emptyset$ as $M \in \Lambda$ and is partially ordered by inclusion. It is easily seen that Zorn's lemma applies, so let L be a maximal element of Λ.

Claim: L is an injective R-module.

Proof of Claim: By Proposition 1.5, it suffices to show that L has no proper essential extension. So assume $L \subseteq K$ is an essential extension of R-modules. Since E is injective, there exists a homomorphism $\phi : K \to E$ such that ϕ fixes L. We claim that ϕ is injective. If $\ker \phi \neq 0$ then $\ker \phi \cap L \neq 0$ since K is essential over L. So let $x \in \ker \phi \cap L\backslash$ with $x \neq 0$. Then $0 = \phi(x) = x$ since ϕ fixes L, a contradiction. Thus, ϕ is injective. Now consider the extensions $M \subseteq L \subseteq \phi(K) \subseteq E$. As $L \subseteq K$ is essential, so is $L \subseteq \phi(K)$. (Let $\phi(k) \neq 0$ for some $k \in K$. Then certainly $k \neq 0$, so there exists $r \in R$ such that $rk \in L$ with $rk \neq 0$. Hence $r\phi(k) = \phi(rk) = rk \neq 0$, as ϕ fixes L.) And since $M \subseteq L$ is essential, we have $M \subseteq \phi(K)$ is essential. By maximality of L in Λ, we conclude that $L = \phi(K)$. As ϕ is injective and fixes L, we obtain $L = \phi^{-1}(L) = \phi^{-1}(\phi(K)) = K$. Thus, we have proved L has no proper essential extensions.

Since $M \subseteq L \subseteq E$ and L is injective, we must have $E = L$ by assumption (b). Thus, $M \subseteq E$ is essential. As E is injective, E has no proper essential extensions. Hence, E is a maximal essential extension of M.

For existence, let M be an R-module and I an injective module containing M. Let E be a maximal essential extension of M inside E, which exists by Zorn's Lemma. The Claim in the proof of $(b) \implies (a)$ shows that E is injective. Thus, $M \subseteq E$ satisfies condition (c).

Now suppose $M \subseteq E$ and $M \subseteq E'$ satisfy the equivalent conditions. As E' is injective there exists a homomorphism $f : E \to E'$ which fixes M. As E is essential over M, f is injective by the same argument we used to show ϕ is injective in $(b) \implies (a)$. Hence, $f(E) \cong E$ is injective and $M \subseteq f(E) \subseteq E'$. Since E' satisfies (b), we must have $f(E) = E'$. Hence, f is an isomorphism fixing M.

□

Definition 1.7. Let M be an R-module. Any R-module satisfying the equivalent conditions of Theorem 1.6 is called an *injective hull* or *injective envelope* of M and is denoted $E_R(M)$.

Example 1.8. Let R be a domain and Q its field of fractions. Then $R \subseteq Q$ is essential and Q is an injective R-module (Remark 1.1(i)). Thus, $Q \cong E_R(R)$.

Lemma 1.9. *Let R be a ring, T an R-algebra, and L and M R-modules.*

(a) There exists a homomorphism of T-modules which is natural in L, M, and T

$$\phi_{LMT} : \mathrm{Hom}_R(L, M) \otimes_R T \longrightarrow \mathrm{Hom}_T(L \otimes_R T, M \otimes_R T)$$
$$f \otimes t \longrightarrow \widetilde{f \otimes t}$$

where $\widetilde{f \otimes t}(\ell \otimes t') = f(\ell) \otimes tt'$ for all $\ell \in L$ and $t' \in T$.

(b) If T is flat over R and L is finitely presented as an R-module then ϕ_{LMT} is an isomorphism.

Proof. The proof of part (a) is left as an exercise. One needs to check that all the maps are well-defined homomorphisms and that maps ϕ_{LMT} make all squares commute when fixing any two of the three variables.

For part (b), first note that when $L = R$ it is easy to see that ϕ_{RMT} is an isomorphism. Since $\mathrm{Hom}_R(-, M)$ and $- \otimes_R T$ commute with finite direct sums in a natural way, it follows that if ϕ_{AMT} and ϕ_{BMT} are isomorphisms, so is ϕ_{CMT} where $C = A \oplus B$. Thus, ϕ_{R^nMT} is an isomorphism for all (finite) n.

Now suppose we have an exact sequence $R^m \to R^n \to L \to 0$. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(L, M) \otimes_R T & \longrightarrow & \operatorname{Hom}_R(R^n, M) \otimes_R T & \longrightarrow & \operatorname{Hom}_R(R^m, M) \otimes_R T \\ & & \downarrow \phi_{LMT} & & \downarrow \phi_{R^nMT} & & \downarrow \phi_{R^mMT} \\ 0 & \longrightarrow & \operatorname{Hom}_T(L \otimes_R T, M \otimes_R T) & \longrightarrow & \operatorname{Hom}_T(R^n \otimes_R T, M \otimes_R T) & \longrightarrow & \operatorname{Hom}_R(R^m \otimes_R T, M \otimes_R T) \end{array}$$

The top row is exact since $\operatorname{Hom}_R(-, M)$ is left exact and $- \otimes_R T$ is exact as T is flat. The bottom row is exact for the same reason, except the functors are applied in the opposite order. Since the two vertical maps on the right are isomorphisms, so is the leftmost vertical map by the five lemma.

□

Lemma 1.10. *Let $C_\bullet$ be a complex of R-modules and* F *an exact additive functor from the category of R-modules to the category of S-modules, for some R-algebra S. Then for all i*

(a) If F is covariant then $\mathrm{F}(\mathrm{H}_i(C_\bullet)) \cong \mathrm{H}_i(\mathrm{F}(C_\bullet))$;

(b) If F is contravariant then $\mathrm{F}(\mathrm{H}_i(C_\bullet)) \cong \mathrm{H}^i(\mathrm{F}(C_\bullet))$.

Proof. Exercise. □

Remark 1.11. Let M be an R-module. Then

- $\operatorname{Hom}_R(M, -)$ is exact if and only if M is projective.
- $\operatorname{Hom}_R(-, M)$ is exact if and only if M is injective.
- $M \otimes_R -$ is exact if and only if M is flat.

Proposition 1.12. *Let R be Noetherian, M and N R-modules, and T an R-algebra. If M is finitely generated and T is flat, then*

$$\operatorname{Ext}^i_R(M, N) \otimes_R T \cong \operatorname{Ext}^i_T(M \otimes_R T, N \otimes_R T)$$

for all i.

Proof. Let $F_\bullet$ be a free resolution of M. Since M is finitely generated and R is Noetherian, we can assume that each F_i is a finitely generated, so $F_i \cong R^{n_i}$ for some n_i. Then $F_i \otimes_R T \cong R^{n_i} \otimes_R T \cong T^{n_i}$, so each module in the complex $F_\bullet \otimes T$ is a (finitely generated) free T-module. Since T is flat over R, $F_\bullet \otimes T$ is a free resolution of $M \otimes_R T$. Thus, for each i we have

$$\begin{aligned} \operatorname{Ext}^i_R(M, N) \otimes_R T &\cong \mathrm{H}^i(\operatorname{Hom}_R(F_\bullet, N)) \otimes_R T \\ &\cong \mathrm{H}^i(\operatorname{Hom}_R(F_\bullet, N) \otimes_R T) \\ &\cong \mathrm{H}^i(\operatorname{Hom}_T(F_\bullet \otimes_R T, N \otimes_R T)) \\ &\cong \operatorname{Ext}^i_T(M \otimes_R T, N \otimes_R T), \end{aligned}$$

where the second isomorphism is by Lemma 1.10 and the third is by Lemma 1.9.

□

Lemma 1.13. *Let R be a Noetherian ring and $L \subseteq M$ an essential extension. Then $\operatorname{Ass}_R L = \operatorname{Ass}_R M$.*

Proof. Clearly $\operatorname{Ass}_R L \subseteq \operatorname{Ass}_R M$. If $\operatorname{Ass}_R M = \emptyset$ then $M = L = 0$ and so $\operatorname{Ass}_R L = \emptyset$. So assume $\operatorname{Ass}_R M \neq \emptyset$ and let $p \in \operatorname{Ass}_R M$. Then $p = (0 :_R x)$ for some $x \in M$. Thus, $Rx \cong R/p$ and thus $\operatorname{Ass}_R Rx = \{p\}$. Since $L \subseteq M$ is essential, we know $Rx \cap L \neq 0$, so $\operatorname{Ass}_R(Rx \cap L) \neq \emptyset$. But as $Rx \cap L$ is a submodule of both Rx and L, $\operatorname{Ass}_R(Rx \cap L) \subseteq (\operatorname{Ass}_R Rx) \cap (\operatorname{Ass}_R L) = \{p\} \cap (\operatorname{Ass}_R L)$. Since this intersection must be nonempty, we see that $p \in \operatorname{Ass}_R L$. □

Lemma 1.14. *Let R be a Noetherian ring and $L \subseteq M$ be an essential extension of R-modules. Then for any multiplicatively closed set S of R, $L_S \subseteq M_S$ is an essential extension of R_S-modules.*

Proof. If $M_S = 0$ the statement is trivial, as the zero module is an essential extension of itself. So assume $M_S \neq 0$ and let A_S be an arbitrary nonzero submodule of M_S (where A is a submodule of M). It suffices to prove that $A_S \cap L_S = (A \cap L)_S \neq 0$. Note that for an arbitrary R-module N, $N_S \neq 0$ if and only if $\operatorname{Ass}_{R_S} N_S \neq \emptyset$, which is if and only if there exists $p \in \operatorname{Ass}_R N$ with $p \cap S \neq \emptyset$. Now as $L \subseteq M$ is essential, so is $A \cap L \subseteq A$. Thus, by Lemma 1.13, $\operatorname{Ass}_R(A \cap L) = \operatorname{Ass}_R A$. Since $A_S \neq 0$, there exists $p \in \operatorname{Ass}_R A$ such that $p \cap S = \emptyset$. Since $p \in \operatorname{Ass}_R(A \cap L)$, we conclude that $(A \cap L)_S \neq 0$. □

Proposition 1.15. *Let R be a Noetherian ring and S a multiplicatively closed set.*

(a) If E is an injective R-module then E_S is an injective R_S-module. (Note: $R_S = S^{-1}R$.)

(b) For any R-module M, $E_R(M)_S \cong E_{R_S}(M_S)$.

Proof. For part (a), let I_S be an ideal of R_S (where I is an ideal of R). Since E is an injective R-module, $\operatorname{Ext}^1_R(R/I, E) = 0$. Since R_S is a flat R-algebra, we have by Proposition 1.12

$$\operatorname{Ext}^1_{R_S}(R_S/I_S, E_S) \cong \operatorname{Ext}^1_R(R/I, E) \otimes_R R_S = 0 \otimes_R R_S = 0.$$

Hence, E_S is an injective R_S-module by Remark 1.1(b).

For (b), we have $E_R(M)_S$ is an injective R_S-module by part (a). Also, since $M \subseteq E_R(M)$ is essential, so is $M_S \subseteq E_R(M)_S$ by Lemma 1.14. Hence, $E_R(M)_S \cong E_{R_S}(M_S)$ by Theorem 1.6. □

Theorem 1.16. *Let R be a Noetherian ring and E and injective module. Then E is indecomposable if and only if $E \cong E_R(R/p)$ for some prime p of R.*

Proof. Suppose E is indecomposable and let $p \in \operatorname{Ass}_R E$. Then there exists an injective map $f : R/p \to E$. Of course, there is also an injective map $i : R/p \to E_R(R/p)$ (given by inclusion). Since E is injective there exists a map $\phi : E_R(R/p) \to E$ such that ϕ restricted to R/p is f. Since f is injective and $E_R(R/p)$ is essential over R/p, we have that ϕ is injective: If not, let x be a nonzero element in $\ker \phi$. Then there exists $r \in R$ such that $rx \neq 0$ and $rx \in R/p$. Then $f(rx) = \phi(rx) = r\phi(x) = 0$, contradicting that f is injective. Thus, we have an injective map $\phi : E_R(R/p) \to E$. As $E_R(R/p)$ is injective, ϕ splits and $E \cong E_R(R/p) \oplus F$ for some F. Finally, as E is indecomposable, F must be zero.

Conversely, let p be a prime ideal and suppose $E_R(R/p) = E_1 \oplus E_2$. Suppose both E_1 and E_2 are nonzero. Then, as $E_R(R/p)$ is an essential extension of R/p, $J_1 = E_1 \cap R/p$ and $J_2 = E_2 \cap R/p$ are nonzero submodules (i.e., ideals) of R/p. But in a domain, every pair of nonzero ideals intersect nontrivially. Thus, $J_1 \cap J_2 \neq 0$, contradicting that $E_1 \cap E_2 = 0$. □

Remark 1.17. Let R be a Noetherian ring.

(a) If E is injective and $p \in \operatorname{Ass}_R E$ then $E_R(R/p)$ is a isomorphic to a direct summand of E.

(b) For any two prime ideals p, q of R, $E_R(R/p) \cong E_R(R/q)$ if and only if $p = q$.

Proof. For part (a), this is exactly what the first part of the proof of Theorem 1.16 shows (ignoring the indecomposable hypothesis). For part (2), suppose $E_R(R/p) \cong E_R(R/q)$. Then $\operatorname{Ass}_R(R/p) = \operatorname{Ass}_R E_R(R/q)$. But by Lemma 1.13, $\operatorname{Ass}_R E_R(R/p) = \operatorname{Ass}_R R/p = \{p\}$ and $\operatorname{Ass}_R E_R(R/q) = \operatorname{Ass}_R R/q = \{q\}$. □

Theorem 1.18. *Let R be a Noetherian ring. Then any injective module is a direct sum of indecomposable injective modules.*

Proof. Let I be an injective module. If $I = 0$ then the statement is trivially true (as I is the sum of an empty set of indecomposable injectives). So assume $I \neq 0$. Let X_I be the set of indecomposable injective submodules of I and $\mathcal{F}$ a subset of X_I. We let $M_{\mathcal{F}}$ denote the submodule of I generated by all the submodules in $\mathcal{F}$; that is,

$$M_{\mathcal{F}} = \sum_{A \in \mathcal{F}} A.$$

We'll call $\mathcal{F}$ *direct* if the sum $M_{\mathcal{F}} = \sum_{A \in \mathcal{F}} A$ is a direct sum; i.e.,

$$\sum_{A \in \mathcal{F}} A = \bigoplus_{A \in \mathcal{F}} A.$$

Now consider the set

$$\Lambda = \{\mathcal{F} \subseteq X_I \mid \mathcal{F} \text{ is direct}\}.$$

Since $I \neq 0$ there exists a prime $p \in \operatorname{Ass}_R I$. By Remark 1.17(a), $E_R(R/p)$ is isomorphic to a submodule T of I. Thus, $\{T\} \in \Lambda$ and consequently $\Lambda \neq \emptyset$. We consider Λ as a poset by inclusion. One can easily check that Zorn's lemma applies, since to check whether a sum of submodules is direct one only needs to consider finitely many submodules at a time. So let $\mathcal{F}$ be a maximal element of Λ and set $L = M_{\mathcal{F}}$. We claim that $L = I$. Since R is Noetherian, L is injective. Thus, $I = L \oplus N$ for some submodule N of I. If $N \neq 0$ then $N = N_1 \oplus N_2$, where $N_1 \cong E_R(R/p)$ for some $p \in \operatorname{Ass}_R N$ (again by Remark 1.17(a)). Thus, $L + N_1 = L \oplus N_1$ and so $\mathcal{F} \cup \{N_1\} \in \Lambda$, contradicting the maximality of $\mathcal{F}$. Thus, $N = 0$ and $I = L$.

□

Lemma 1.19. *Let R be a Noetherian ring and $p \in \operatorname{Spec} R$.*

(a) Every element of $E_R(R/p)$ is annihilated by a power of p.

(b) $E_R(R/p) \cong E_{R_p}(k(p))$, where $k(p) \cong R_p/pR_p$.

Proof. For part (a), let $x \in E_R(R/p)$, $x \neq 0$. Then $\operatorname{Ass}_R Rx \subseteq \operatorname{Ass}_R E_R(R/p) = \operatorname{Ass}_R R/p = \{p\}$. Hence, $\operatorname{Ass}_R Rx = \{p\}$. Since $Rx \cong R/\operatorname{Ann}_R x$, we have $p = \sqrt{\operatorname{Ann}_R x}$ (see Grifo's 905 notes, Theorem 6.50). Hence, x is annihilated by a power of p.

For part (b), let $E = E_R(R/p)$. We will first show the map $\phi : E \to E_p$ given by $\phi(e) = \frac{e}{1}$ is an isomorphism. Let $e \in E$ and suppose $\phi(e) = \frac{e}{1} = 0$. Then there exists $s \notin p$ such that $se = 0$. But as $\operatorname{Ass}_R E = \{p\}$, every zero-divisor on E is contained in p (cf., Theorem 6.27 of

Grifo's 905 notes). Hence, s is a non-zero-divisor on E and thus $e = 0$. This shows ϕ is injective. Now let $\frac{e}{s} \in E_p$, where $s \notin p$. Since $\operatorname{Ass}_R Re = \{p\}$ (see the proof of (a)) we have that s is a non-zero-divisor on Re. Then the map $f : Re \to Re$ given by $f(re) = sre$ is injective. Let $i : Re \to E$ be the inclusion map. As E is injective, there exists a map $h : Re \to E$ such that $i = hf$. Let $e' = h(e)$ Then

$$e = i(e) = hf(e) = h(se) = sh(e) = se'.$$

Thus, $\frac{e}{s} = \frac{se'}{s} = \frac{e'}{1} = \phi(e')$. Hence, ϕ is surjective and $E_R(R/p) \cong E_R(R/p)_p$.

Applying Proposition 1.15(b) we have that $E_R(R/p)_p \cong E_{R_p}((R/p)_p) \cong E_{R_p}(k(p))$ since $(R/p)_p \cong R_p/pR_p = k(p)$. Hence, $E_R(R/p) \cong E_{R_p}(k(p))$.

□

Lemma 1.20. *Let R be a Noetherian ring, I an ideal of R and $p \in \operatorname{Spec} R$. Then*

$$\operatorname{Hom}_R(R/I, E_R(R/p)) = \begin{cases} E_{R/I}(R/p), & \text{if } I \subseteq p \\ 0, & \text{if } I \not\subseteq p. \end{cases}$$

Proof. If $I \subseteq p$, then R/p is an R/I-module. The result then follows from Problem 2 of Homework # 1. If $I \not\subseteq p$ let $s \in I$, $s \notin p$ and let $f : R/I \to E_R(R/p)$ be a homomorphism. Then for all $r \in R$, $sf(\overline{r}) = f(\overline{sr}) = f(\overline{0}) = 0$. Since s is a non-zero-divisor on $E_R(R/p)$, $f = 0$.

□

Proposition 1.21. *Let R be a Noetherian ring and I an injective module. Suppose $I \cong E_R(R/p)^\alpha \oplus I'$, where $E_R(R/p)$ is not a summand of I' (equivalently, $p \notin \operatorname{Ass}_R I'$). Then*

$$\alpha = \operatorname{rank}_{k(p)} \operatorname{Hom}_{R_p}(k(p), I_p).$$

Consequently, the number of copies of $E_R(R/p)$ appearing in any decomposition of I into indecomposables is uniquely determined. We denote α by $\mu(p, I)$.

Proof. Let $E_R(R/q)$ be a summand of I'. By assumption, $p \neq q$. If $p \not\subseteq q$ then $\operatorname{Hom}_R(R/p, E_R(R/q)) = 0$ by Lemma 1.20. If $p \subsetneq q$ then $(R/q)_p = 0$. Since $(R/q)_p \subseteq E_R(R/q)_p$ is essential, we conclude that $E_R(R/q)_p = 0$. Thus, for every summand of $E_R(R/q)$ of I' we have $\operatorname{Hom}_{R_p}(k(p), E_R(R/q)_p) = 0$. Thus, $\operatorname{Hom}_{R_p}(k(p), I'_p) = 0$. Hence,

$$\begin{aligned} \operatorname{Hom}_{R_p}(k(p), I_p) &\cong \operatorname{Hom}_{R_p}(k(p), E_R(R/p)_p^\alpha) \\ &\cong \operatorname{Hom}_R(R/p, E_R(R/p))_p^\alpha \\ &\cong E_{R/p}(R/p)_p^\alpha \\ &\cong E_{k(p)}(k(p))^\alpha \\ &\cong k(p)^\alpha. \end{aligned}$$

The second isomorphism follows Lemma 1.9, the third isomorphism from Problem 2 of HW #1, and the fourth by Proposition 1.15. From these isomorphisms, it follows that

$$\operatorname{rank}_{k(p)} \operatorname{Hom}_{R_p}(k(p), I_p) = \operatorname{rank}_{k(p)} k(p)^\alpha = \alpha.$$

□

Proposition 1.22. *Let R be a Noetherian ring and $L \subseteq M$ R-modules. Then the following are equivalent:*

(a) $L \subseteq M$ is essential;

(b) $L_p \subseteq M_p$ is essential for all primes p;

(c) $(0 :_{L_p} pR_p) = (0 :_{M_p} pR_p)$ for all primes p.

(d) The natural map $\operatorname{Hom}_{R_p}(k(p), L_p) \to \operatorname{Hom}_{R_p}(k(p), M_p)$ is an isomorphism for all primes p.

Proof. (a) $\implies$ (b): This follows by Lemma 1.14.

(b) $\implies$ (c): It suffices to prove that if R is local ring with maximal ideal m and $A \subseteq B$ is essential then $(0 :_A m) = (0 :_B m)$. It is easy to see that $(0 :_A m) \subseteq (0 :_B m)$ is an essential extension of R/m-modules. Since R/m is a field, we must have equality.

(c) $\iff$ (d): This follows from the natural isomorphism $(0 :_{N_p} pR_p) \cong \operatorname{Hom}_{R_p}(k(p), N_p)$ for all R-modules N and prime ideals p.

(c) $\implies$ (a): Let A be a nonzero submodule of M and $p \in \operatorname{Ass}_R A$. Then $p = (0 :_R a)$ for some $a \in A$. Hence, $pR_p = (0 :_{R_p} \frac{a}{1})$, so $\frac{a}{1} \in (0 :_{M_p} pR_p) = (0 :_{L_p} pR_p)$. Thus, $\frac{a}{1} = \frac{u}{s}$ for some $u \in L$ and $s \notin p$. Then $x := tu = tsa$ for some $t \notin p$. Note $x \in L \cap A$ since $u \in L$ and $a \in A$. Also, $x \neq 0$ since $ts \notin p = (0 :_R a)$.

□

2 Minimal injective resolutions and injective dimension

Definition 2.1. Let R be a ring, M an R-module and $I^\bullet$ an injective resolution of M. Let $\partial^\bullet$ denote the differential of $I^\bullet$, where $\partial^i : I^i \to I^{i+1}$. We say that $I^\bullet$ is *minimal* if $I^0 \cong E_R(M)$ and $I^i \cong E_R(\operatorname{im} \partial^{i-1})$ for all $i > 0$.

Lemma 2.2. *Let R be ring and M an R-module. Then M has a minimal injective resolution.*

Proof. Let $I^0 = E_R(M)$ and $\partial^{-1} : M \to I^0$ be the inclusion map. Let $C^0 = \operatorname{coker} \partial^{-1}$ and $I^1 = E_R(C^0)$. Let ∂^0 be the composition $I^0 \to C^0 \to I^1$, where the first map is the canonical projection and the second is the natural inclusion. Then $0 \to M \xrightarrow{\partial^{-1}} I^0 \xrightarrow{\partial^0} I^1$ is exact. And as $C \cong \operatorname{im} \partial^0$ we have $I^1 \cong E_R(\operatorname{im} \partial^0)$. Continuing in this fashion, we obtain a minimal injective resolution of M.

□

Proposition 2.3. *Let R be Noetherian and M an R-module. Let $I^\bullet$ be an injective resolution of M. Then $I^\bullet$ is minimal if and only if for all $p \in \operatorname{Spec} R$ and for all $i \geqslant 0$ the map $\operatorname{Hom}_{R_p}(k(p), I^i_p) \to \operatorname{Hom}_{R_p}(k(p), I^{i+1}_p)$ is zero.*

Proof. Let $\partial^\bullet$ be the differential of $I^\bullet$. Let $C^0 = M$ and $C^i = \operatorname{im} \partial^{i-1}$ for $i > 0$. Applying the left exact functor $\operatorname{Hom}_{R_p}(k(p), R_p \otimes_R -)$ to the exact sequence $0 \to C^i \to I^i \to I^{i+1}$, we obtain for all $p \in \operatorname{Spec} R$

$$0 \to \operatorname{Hom}_{R_p}(k(p), C^i_p) \to \operatorname{Hom}_{R_p}(k(p), I^i_p)) \to \operatorname{Hom}_{R_p}(k(p), I^{i+1}_p)$$

is exact. Now, $I^\bullet$ is minimal if and only if $I^i \cong E_R(C^i)$ for all i. But as I^i is injective, this holds if and only if $C^i \subseteq I^i$ is essential for each i. By Proposition 1.22, $C^i \subseteq I^i$ is essential if and only if the map $\operatorname{Hom}_{R_p}(k(p), C_p^i) \to \operatorname{Hom}_{R_p}(k(p), I_p^i)$ is an isomorphism for all primes p. But from the exact sequence above, this is equivalent to the maps $\operatorname{Hom}_{R_p}(k(p), I_p^i) \to \operatorname{Hom}_{R_p}(k(p), I_p^{i+1})$ being zero for all $p \in \operatorname{Spec} R$.

□

Corollary 2.4. *Let R be a Noetherian ring, M an R-module, and $I^\bullet$ a minimal injective resolution of M. For any mulitiplicatively closed set S of R, $I_S^\bullet$ is a minimal injective resolution of M_S.*

Theorem 2.5. *Let R be a Noetherian ring, M an R-module, and $I^\bullet$ a minimal injective resolution of M. Then for $i \geqslant 0$ and each $p \in \operatorname{Spec} R$ we have*

$$\mu(p, I^i) = \operatorname{rank}_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), M_p).$$

Consequently, $\mu(p, I^i)$ does not depend on M or the choice of minimal injective resolution $I^\bullet$. The number $\mu(p, I^i)$ is called the ith Bass number *of M with respect to p and is denoted $\mu_i(p, M)$.*

Proof. Using Propositions 1.21 and 2.3

$$\begin{aligned}\operatorname{Ext}^i_{R_p}(k(p), M) &\cong \operatorname{H}^i(\operatorname{Hom}_{R_p}(k(p), I_p)) \\ &\cong \operatorname{Hom}_{R_p}(k(p), I_p^i)\end{aligned}$$

Thus, $\mu(p, I^i) = \operatorname{rank}_{k(p)} \operatorname{Hom}_{R_p}(k(p), I_p^i) = \operatorname{rank}_{k(p)} \operatorname{Ext}^i_{R_p}(k(p), M_p)$. □

Corollary 2.6. *Let R be a Noetherian ring and M a finitely generated R-module. Then $\mu_i(p, M) < \infty$ for all i and all $p \in \operatorname{Spec} R$.*

Proof. This follows from the fact that over a Noetherian ring R, $\operatorname{Ext}^i_R(A, B)$ is finitely generated for all i whenever A and B are finitely generated. (Let $F_\bullet$ be a free resolution of A consisting of finitely generated free modules. Hence for each i, $\operatorname{Hom}_R(F_i, B)$ is finitely generated. Then $\operatorname{Ext}^i_R(A, B) = \operatorname{H}^i(\operatorname{Hom}_R(F_\bullet, B)$ is isomorphic to a subquotient of $\operatorname{Hom}_R(F_i, B)$.) □

Remark 2.7. Let R be a Noetherian ring and M an R-module. Then $\mu_0(p, M) \neq 0$ if and only if $p \in \operatorname{Ass}_R M$. This follows from $\mu_0(p, M) = \operatorname{rank}_{k(p)} \operatorname{Hom}_{R_p}(k(p), M_p)$. Thus, if M is finitely generated, there are only finitely many primes p such that $\mu_0(p, M) \neq 0$.

Definition 2.8. Let M be an R-module. Then the *injective dimension* of M, denoted $\operatorname{id}_R M$, is defined to be the infimum of the lengths of all injective resolutions of M. (Recall that the length of a resolution $I^\bullet$ is $\sup\{n \mid I^n \neq 0\}$.)

Proposition 2.9. *Let R be a Noetherian ring and M an R-module.*

(a) $\operatorname{id}_R M = \sup\{n \mid \mu_n(p, M) \neq 0$ *for some* $p \in \operatorname{Spec} R\}$.

(b) The length of any minimal injective resolution of M is equal to $\operatorname{id}_R M$.

Proof. Observe that by Theorem 2.5, given any minimal injective resolution $I^\bullet$, $I^n \neq 0$ if and only if $\mu_n(p, M) \neq 0$ for some p. Now let $r = \operatorname{id}_R M$ and let ℓ denote the right-hand side of the equality. Note that if $\mu_n(p, M) \neq 0$ for some n then $\operatorname{Ext}^n_R(R/p, M) \neq 0$, which implies $r \geqslant n$. Hence, $r \geqslant \ell$. If $\ell = \infty$ there is nothing left to show, so assume $\ell < \infty$. Let $I^\bullet$ be a minimal injective resolution of M. By the observation above, $I^{\ell+1} = 0$, which means that $I^\bullet$ is an injective resolution of M of length at most ℓ. Hence, $r \leqslant \ell$. This proves (a).

For part (b), again by the observation about, the length of any minimal injective resolution is ℓ (the right-hand-side of the equality in (a)).

□

We next recall the notion of length and some of its elementary properties. Proofs of these properties can be found in Atiyah-Macdonald.

Definition 2.10. Let R be a ring. An R-module is called *simple* if $M \neq 0$ and M has no non-trivial submodules.

Remark 2.11. An R-module M is simple if and only if $M \cong R/m$ for some maximal ideal m of R. For, if M is simple and $x \in M$, $x \neq 0$, then $M = Rx \cong R/I$ where $I = (0 :_R x)$. Since M is simple, I must be maximal.

Definition 2.12. Let M be an R-module. A filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ is called a *composition series* for M if M_{i+1}/M_i is simple for $i = 1, \ldots, n$. In this case, we say the length of the composition series is n. If M has a composition series, the *length* of M, denoted $\lambda_R(M)$, is defined to be infimum of the lengths of all composition series for M. If M does not have a composition series, we say M has infinite length; i.e., $\lambda_R(M) = \infty$.

Proposition 2.13. *Let R be a ring and M an R-module.*

(a) If M has a composition series then all composition series for M have the same length.

(b) Every filtration of M can be refined to a composition series.

(c) If $0 \to L \to M \to N \to 0$ is a short exact sequence then $\lambda_R(M) = \lambda_R(L) + \lambda_R(N)$.

Proof. See Atiyah-Macdonald. □

Definition 2.14. An R-module M is called *Artinian* if M satisfies the descending chain condition on submodules. The ring R is said to be Artinian if R is Artinian as an R-module.

Remark 2.15. Let k be a field and V a k-vector space. The following are equivalent:

(a) V is Noetherian;

(b) V is Artinian;

(c) $\operatorname{rank}_k V < \infty$.

Proof. Basic vector space theory. □

Proposition 2.16. *Let R be a ring and M an R-module. Then M has finite length if and only if M is both Artinian and Noetherian.*

Proof. If M has finite length then any chain of submodules with proper containments has length at most $\lambda_R(M)$ (since it can be refined to a composition series). Thus, M satisfies both chain conditions. Conversely, suppose M is both Noetherian and Artinian. Since M is Noetherian there exists a maximal proper submodule, say M_1. Then let M_2 be a maximal proper submodule of M_1, etc. As M is Artinian, this eventually terminates in a composition series. Hence, M has finite length. □

.

Proposition 2.17. *Let R be a Noetherian ring and M a finitely generated R-module. Then M has finite length if and only if $\dim M = 0$.*

Proof. Recall that $\dim M = \dim R/\operatorname{Ann}_R M$. Since the length of M as R-module and as an $R/\operatorname{Ann}_R M$-module are the same, we may assume $\operatorname{Ann}_R M = 0$ and $\dim M = \dim R$. Let p be a minimal prime of R. Then p is minimal over $\operatorname{Ann}_R M$ and thus $p \in \operatorname{Ass}_R M$. (See Theorem 6.40 of Grifo's 905 notes.) Then there is an injection of R/p into M, which implies that R/p has finite length. It suffices to show that every Artinian domain S is a field: let $x \in S$, $x \neq 0$. Then the descending chain $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots$ stabilizes, so there exists an $n > 0$ such that $(x^n) = (x^{n+1})$. Thus, $x^n = x^{n+1}y$ for some $y \in S$. Canceling x^n, we see that x is a unit. Hence, R/p is a field, so p is maximal. Thus, $\dim M = \dim R = 0$.

Conversely, suppose $\dim M = 0$. Again, we may assume $\operatorname{Ann}_R M = 0$ and thus $\dim R = 0$. Then R has finitely many prime ideals, say $\{m_1, \ldots, m_r\}$, all of which are both maximal and minimal. Then $\sqrt{(0)} = m_1 \cap \cdots \cap m_r$. Hence, $(m_1 \cdots m_r)^s = 0$ for some s. Thus, 0 is the product of finitely many maximal ideals. Let $0 = p_1 \cdots p_t$ where p_i are maximal ideals (not necessarily distinct). For $i = 1, \ldots, t$ let $I_i = p_1 \cdots p_i$, and let $I_i = R$ for all $i \leqslant 0$. Note $I_i = 0$ for $i \geqslant t$. We claim that $I_i M$ has finite length for all i. This is trivially true when $i \geqslant t$. Assume $i < t$ and that $I_{i+1}M$ has finite length. Consider the exact sequence

$$0 \to I_{i+1}M \to I_i M \to I_i M/I_{i+1}M \to 0.$$

All three modules are Noetherian since M is. By assumption $I_{i+1}M$ has finite length. Note that $p_{i+1} \cdot I_i M/I_{i+1}M = 0$, since $p_{i+1}I_i = I_{i+1}$. Thus, $I_i M/I_{i+1}M$ is a (Noetherian) R/p_{i+1}-vector space. By Remark 2.15, $I_i M/I_{i+1}M$ is also Artinian as an R/p_{i+1}-vector space, and hence as an R-module as well. Thus, $I_i M/I_{i+1}M$ has finite length by Proposition 2.16. Now by part (c) of Proposition 2.13, we conclude $I_i M$ has finite length. This proves the claim. Hence $M = I_0 M$ has finite length.

□

Theorem 2.18. *Let R be a ring. Then R is Artinian if and only if R is Noetherian and $\dim R = 0$.*

Proof. See Atiyah-Macdonald. □

Remark 2.19. While every Artinian ring is Noetherian, the same cannot be said of modules. Let $R = \mathbb{Z}_{(2)}$. Then $\mathbb{Q}/R$ is an Artinian R-module which is not Noetherian.

Proposition 2.20. *If (R, m) is a local ring containing a field k such that the composition $k \hookrightarrow R \to R/m$ is an isomorphism. Then for any R-module M, $\lambda_R(M) = \operatorname{rank}_k M$.*

Proof. Any chain of R-modules is a chain of k-vector spaces. Thus, if M has infinite length as an R-module, M has infinite rank as a k-vector space. Suppose $\lambda_R(M) = n < \infty$. If $n = 1$ then $M \cong R/m \cong k$, so $\operatorname{rank}_k M = 1$. If $n > 1$, let M_1 be a simple submodule of M and consider the short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$. (This is a s.e.s. as both R-modules and k-vector spaces.) Then $\lambda_R(M/M_1) = n - 1$. By the (unstated) induction hypothesis, $\operatorname{rank}_k M/M_1 = n - 1$. Since $\operatorname{rank}_k M_1 = 1$, we conclude $\operatorname{rank}_k M = n$. □

Example 2.21. Let $R = k[x]/(x^n)$, where k is a field and x a variable. Note R is local with maximal ideal $m = (\overline{x})$. Also, the map $k \to R \to R/m$ is an isomorphism. Hence, $\lambda_R(R) = \dim_k R = n$ as $\{\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}\}$ is a k-basis for R.

Lemma 2.22. *Let (R, m) be a local ring with residue field k and M an R-module. Suppose $\operatorname{Ext}^i_R(k, M) = 0$ for some i. Then $\operatorname{Ext}^i_R(L, M) = 0$ for any finite length module L.*

Proof. Let $\lambda_R(L)$ denote the length of L. If $\lambda_R(L) = 1$ then $L \cong k$ and the result holds. Suppose now that $\lambda_R(L) > 0$ and that the lemma holds for all modules of length less than $\lambda_R(L)$. Then there exists an exact sequence $0 \to k \to L \to C \to 0$ such that $\lambda_R(C) = \lambda_R(L) - 1$. Thus, $\operatorname{Ext}^i_R(k, C) = 0$. From the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(C, M) \to \operatorname{Ext}^i_R(L, M) \to \operatorname{Ext}^i_R(k, M) \to \cdots$$

we conclude that $\operatorname{Ext}^i_R(L, M) = 0$.

□

Theorem 2.23. *Let R be a Noetherian ring and M a finitely generated R-module. Let $p \subseteq q$ be prime ideals and set $h = \operatorname{ht} q/p$. If $\mu_i(p, M) \neq 0$ for some i then $\mu_{i+h}(q, M) \neq 0$.*

Proof. By induction on h, it suffices to prove the case when $h = 1$. Furthermore, since $\mu_i(p, M) = \mu_i(pR_q, M_q)$ (use either Corollary 2.4 or Theorem 2.5), we may assume (R, m) is local, $q = m$ and $\operatorname{ht} m/p = 1$. Choose $x \in m \setminus p$. Note that m is minimal over (p, x), so $\dim R/(p, x) = 0$. Hence, by Proposition 2.17, $\lambda_R(R/(p, x)) < \infty$. Since x is a non-zero-divisor on R/p we have the short exact sequence

$$0 \to R/p \xrightarrow{x} R/p \to R/(p, x) \to 0.$$

Applying $\operatorname{Hom}_R(-, M)$ we have the exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/p, M) \xrightarrow{x} \operatorname{Ext}^i_R(R/p, M) \to \operatorname{Ext}^{i+1}_R(R/(p, x), M) \to \cdots$$

Since $\mu_i(p, M) \neq 0$ we have $\operatorname{Ext}^i_R(R/p, M) \neq 0$. Note also that $\operatorname{Ext}^i_R(R/p, M)$ is finitely generated (see the proof of Corollary 2.6). Now suppose by way of contradiction that $\mu_{i+1}(m, M) = 0$. Then $\operatorname{Ext}^{i+1}_R(k, M) = 0$, which implies by Lemma 2.22 that $\operatorname{Ext}^{i+1}_R(R/(p, x), M) = 0$. Hence, $\operatorname{Ext}^i_R(R/p, M) = x \operatorname{Ext}^i_R(R/p, M)$ which forces $\operatorname{Ext}^i_R(R/p, M) = 0$ by Nakayama's Lemma. This is a contradiction.

□

Corollary 2.24. *Let (R, m) be a local ring with residue field k and M a finitely generated R-module. Then*

$$\operatorname{id}_R M = \sup\{n \mid \operatorname{Ext}^n_R(k, M) \neq 0\}.$$

Proof. Let $r = \operatorname{id}_R M$ and ℓ denote the right-hand-side of the equality. Since $\operatorname{Ext}^i_R(k, M) = 0$ for all $i > r$, certainly $\ell \leqslant r$. Suppose $\ell < r$. Then by Proposition 2.9 $\mu_i(p, M) \neq 0$ for some $i > \ell$ and some $p \in \operatorname{Spec} R$. By Theorem 2.23, $\mu_{i+h}(m, M) \neq 0$ where $h = \dim R/p$. But this means $\operatorname{Ext}^{i+h}_R(k, M) \neq 0$, a contradiction as $i + h > \ell$. □

Corollary 2.25. *Let R be a Noetherian ring and M a nonzero finitely generated R-module. Then $\operatorname{id}_R M \geqslant \dim M$.*

Proof. Since $\dim_R M = \sup\{\dim M_p \mid p \in \operatorname{Spec} R\}$ and $\operatorname{id}_R M \geqslant \operatorname{id}_{R_p} M_p$ for all $p \in \operatorname{Spec} R$, it suffices to prove the statement in the case (R, m) is a local ring. Let $d = \dim M$ and choose p in $\operatorname{Supp}_R M = \operatorname{V}(\operatorname{Ann}_R M)$ such that $\dim R/p = d$. Then $p \in \operatorname{Ass}_R M$ (cf. Theorem 6.40 of Grifo's 905 notes). Hence, $\mu_0(p, M) = \dim_{k(p)} \operatorname{Hom}_{R_p}(k(p), M_p) \neq 0$. By Theorem 2.23, we obtain that $\mu_d(m, M) \neq 0$. Thus, $\operatorname{id}_R M \geqslant d$ by Corollary 2.24.

□

3 Grade and depth

Definition 3.1. Let R be a ring, M an R-module, and $\mathbf{x} = x_1, \ldots, x_n \in R$. We say $\mathbf{x}$ is a *regular sequence on* M, or simply an M*-sequence*, if $(x_1, \ldots, x_n)M \neq M$ and x_i is regular (i.e., a non-zero-divisor) on $M/(x_1, \ldots, x_{i-1})M$ for each $i = 1, \ldots, n$. An M-sequence $\mathbf{x}$ is called *maximal* if $\mathbf{x}$ cannot be extended to a longer M-sequence.

Example 3.2. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over the field k. Then $x_1, \ldots, x_n$ is a maximal R-sequence.

Lemma 3.3. *Let R be Noetherian and M a finitely generated R-module. Then*

(a) The length of every M-sequence is finite.

(b) Every M-sequence can be extended to a maximal M-sequence.

Proof. For part (a), note that if $x_1, \ldots, x_n$ is an M-sequence then

$$0 \subset (x_1)M \subset (x_1, x_2)M \subset \cdots \subset (x_1, \ldots, x_n)M$$

is a strictly ascending chain of submodule of M. For if $(x_1, \ldots, x_i)M = (x_1, \ldots, x_{i+1})M$ for some i, then $x_{i+1}M \subseteq (x_1, \ldots, x_i)M$. As $M \neq (x_1, \ldots, x_i)M$, then x_{i+1} is a zero-divisor on $M/(x_1, \ldots, x_i)M$, a contradiction. Consequently, as M is Noetherian, this chain must terminate. Hence, there are no M-sequences of infinite length. The same argument (using ACC) proves (b) as well.

□

Example 3.4. Let $R = k[x, y, x]$. Then x, y, z is a maximal R-sequence.

Example 3.5. Let $R = k[x, y, z]_{(x,y,z)}$ and $M = R/(xy, yz)$. Then $y - z$ is a maximal M -sequence. To see this, note that $(xy, xz) = (y) \cap (x, z)$ is a irredundant primary decomposition of I. Thus, (y) and (x, z) are the associated primes of M. Since $y - z$ is not in either associated prime, it is a non-zero-divisor on M. Also, $M \neq (y - z)M$, so $y - z$ is M-regular. Note that $M/(y - z)M \cong k[x, y, z]_{(x,y,z)}/(xy, xz, y - z) \cong k[x, y]_{(x,y)}/(xy, y^2)$. Since $(x, y)x = 0$ in $k[x, y]_{(x,y)}/(xy, y^2)$, (x, y) consists of zero-divisors on $M/(y - z)M$. Thus, $y - z$ is a maximal M-sequence.

Here we summarize some essential facts about primary decompositions for modules, and then use them to prove Krull's Intersection Theorem.

Definition 3.6. Let M be an R-module. A submodule Q of M is called *primary* if $Q \subsetneq M$ and for every $r \in R$ multiplication by r on M/Q is either injective or nilpotent; i.e., r is either a non-zero-divisor on M/Q or $r^n M \subseteq Q$ for some n.

Remark 3.7. Let Q be a primary submodule of M. Then $p := \sqrt{\operatorname{Ann}_R M/Q}$ is a prime ideal of R. We say that Q is a p-primary submodule of M.

Theorem 3.8. *Let R be a Noetherian ring, M a finitely generated R-module and $N \subsetneq M$ a submodule. Then there exists primary submodules $Q_1, \ldots, Q_n$ such that*

- *$N = Q_1 \cap \cdots \cap Q_n$;*
- *$N \subsetneq Q_1 \cap \cdots \cap \hat{Q}_i \cap \cdots Q_n$ for $i = 1, \ldots, n$;*

- *$p_1, \ldots, p_n$ are distinct prime ideals, where $p_i = \sqrt{\operatorname{Ann}_R M/Q_i}$.*

The decomposition $N = Q_1 \cap \cdots \cap Q_n$ is called an **irredundant primary decomposition for** *$N \subset M$. The prime ideals $p_1, \ldots, p_n$ are uniquely determined by $N \subset M$ and are called the* **associated primes** *of $N \subset M$ (or more commonly, of M/N). We denote the set of associated primes of M/N by $\operatorname{Ass}_R M/N$. Moreover, a prime ideal $p \in \operatorname{Ass}_R M/N$ if and only if $p = (N :_R x)$ for some $x \in M$. Additionally, if p_i is a minimal associated prime of M/N, the $Q_i = \phi^{-1}(N_{p_i})$ where $\phi : M \to M_{p_i}$ is the natural localization map.*

Proof. See Atiyah-Macdonald. □

Theorem 3.9. (Krull's Intersection Theorem) *Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module. Then there exist an $s \in I$ such that*

$$(1-s)\bigcap_{n=1}^{\infty} I^n M = 0.$$

Proof. Let $N = \bigcap_{n=1}^{\infty} I^n M$. We claim that $IN = N$. If $IN = M$, there is nothing to prove. Suppose $IN \subsetneq M$ and let $IN = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ be a primary decomposition of $IN \subset M$. Then for each i, $IN \subseteq Q_i$. If $N \not\subset Q_i$ then I consists of zero-divisors on M/Q_i. As Q_i is a primary submodule of M, we must have $I^n M \subseteq Q_i$ for some n. But $N \subseteq I^n M$, so $N \subseteq Q_i$, a contradiction. Thus, $N \subseteq Q_1 \cap \cdots \cap Q_r = IN$. Consequently, $IN = N$. By a homework exercise, this implies there exists $s \in I$ such that $(1-s)N = 0$.

□

Definition 3.10. Let R be a ring. The *Jacobson radical of R*, denoted $J(R)$, is the intersection of all maximal ideals of R. It is easily seen that if $r \in J(R)$ then $1 - r$ is a unit.

Corollary 3.11. *Let R be a Noetherian ring, $I \subseteq J(R)$ an ideal, and M a finitely generated R-module. Then*

$$\bigcap_{n=1}^{\infty} I^n M = 0.$$

Proof. Apply Krull's Intersection Theorem and use that $1 - s$ is a unit for every $s \in I$. □

.

Proposition 3.12. *Let R be Noetherian, M a finitely generated R-module, and $\mathbf{x} = x_1, \ldots, x_n \in J(R)$ an M-sequence. Then any permutation of $\mathbf{x}$ is an M-sequence.*

Proof. It suffices to show that if $x, y \in J(R)$ is an M-sequence then so is y, x. First note that $(y,x)M = (x,y)M \neq M$. We next show that y is regular on M: suppose $yu = 0$ for some $u \in M$. Then $yu \in xM$ so $u \in xM$. Write $u = xu_1$ where $u_1 \in M$. Then $0 = yu = xyu_1$, so $yu_1 = 0$. Repeating the argument above, we get $u_1 \in xM$ and hence $u \in (x)^2 M$. Continuing in this way, we obtain that $u \in \bigcap_{n\geqslant 1}(x)^n M$. As $(x) \subseteq J(R)$ we have $\bigcap_{n\geqslant 1}(x)^n M = 0$ and thus $u = 0$. This shows y is regular on M. Now assume $xv \in yM$ for some $v \in M$. Then $xv = yw$ for some $w \in M$. Since y is regular on M/xM, we obtain that $w = xz$ for some $z \in M$. Consequently, $xv = xyz$. As x is regular on M, we then have $v = yz \in yM$. Hence, x is regular on M/yM.

□

Definition 3.13. Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module such that $IM \neq M$. Then the *grade of I on M*, denote $\operatorname{grade}_I M$ or $\operatorname{grade}(I, M)$, is defined to be the supremum of the lengths of all M-sequences contained in I.

Remark 3.14. Note that it is not clear from the definition that $\operatorname{grade}_I M < \infty$. Although every maximal M-sequence is finite, the supremum of the lengths of such sequences might be infinite.

Notation (temporary): Let R be a ring and L and M R-modules. Define

$$g(L, M) := \inf\{n \mid \operatorname{Ext}_R^n(L, M) \neq 0\}.$$

Note that $g(L, M) \geqslant 0$ for all R-modules L and M. (Recall $\inf \emptyset = \infty$.)

Proposition 3.15. *Let R be a ring, L and M R-modules, and $x \in R$ such that $xL = 0$ and x is regular on M. Then $g(L, M/xM) = g(L, M) - 1$. Furthermore, if $g := g(L, M) < \infty$ then $\operatorname{Ext}_R^{g-1}(L, M/xM) \cong \operatorname{Ext}_R^g(L, M)$.*

Proof. Let $f : L \to M$ be a homomorphism. Then $xf(L) = f(xL) = f(0) = 0$. As x is regular on M, we see that $f(L) = 0$. Hence, $\operatorname{Hom}_R(L, M) = 0$ and so $g > 0$. Now applying $\operatorname{Hom}_R(L, -)$ to the s.e.s. $0 \to M \xrightarrow{x} M \to M/xM \to 0$ we get the long exact sequence

$$\cdots \to \operatorname{Ext}_R^i(L, M) \xrightarrow{x} \operatorname{Ext}_R^i(L, M) \to \operatorname{Ext}_R^i(L, M/xM) \to \operatorname{Ext}_R^{i+1}(L, M) \xrightarrow{x} \cdots$$

Since $xL = 0$ we have $x \operatorname{Ext}_R^i(L, M) = 0$ for all i (cf. Grifo's 915 notes, Exercise 73(c).) Thus, for each i we have an exact sequence

$$0 \to \operatorname{Ext}_R^i(L, M) \to \operatorname{Ext}_R^i(L, M/xM) \to \operatorname{Ext}_R^{i+1}(L, M) \to 0.$$

Both conclusions follow easily from this sequence.

□

Theorem 3.16. *Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module such that $IM \neq M$. Let $x_1, \ldots, x_r$ be a maximal M-sequence contained in I. Then $r = \sup\{n \mid \operatorname{Ext}_R^n(R/I, M) \neq 0\}$. Consequently, all maximal M-sequences contained in I have the same length and*

$$\operatorname{grade}_I M = \inf\{n \mid \operatorname{Ext}_R^n(R/I, M) \neq 0\}.$$

In particular, $\operatorname{grade}_I M < \infty$.

Proof. Notice that the right-hand-side is $g := g(R/I, M)$. Suppose $r = 0$. Since $IM \neq M$, it must be that I consists of zero-divisors on M. Thus, I is contained in the union of the associated primes of M (Grifo 905, Theorem 6.27). By the prime avoidance lemma (Grifo 905, Theorem 3.29), $I \subseteq p$ for some associated prime p of M. Then the composition $R/I \to R/p \hookrightarrow M$ is nonzero. Hence, $\operatorname{Hom}_R(R/I, M) = \operatorname{Ext}_R^0(R/I, M) \neq 0$ and $g = 0 = r$. Proceeding by induction on r, we may assume $r > 0$ and that the result holds for all finitely generated R-modules N with $IN \neq N$ and having a maximal N-sequence of length at most $r - 1$. Let $N = M/x_1M$. Then $IN \neq N$ and $x_2, \ldots, x_r$ is a maximal N-sequence contained in I. Hence, $r - 1 = g(R/I, N) = g - 1$ by Proposition 3.15. Thus, $r = g$, which completes the proof. □

Corollary 3.17. *Let R be a Noetherian ring, I an ideal of R, and M a finitely generated R-module such that $IM \neq M$. If $x \in I$ is a regular element on M, then*

$$\operatorname{grade}_{I/(x)} M/xM = \operatorname{grade}_I M/xM = \operatorname{grade}_I M - 1.$$

Proof. For the first equality, observe that any sequence $\mathbf{y}$ in I is an M/xM-sequence if and only if its image $\overline{\mathbf{y}}$ in $I/(x)$ is an M/xM-sequence. For the second equality, we have by Theorem 3.16 and Proposition 3.15,

$$\operatorname{grade}_I M/xM = g(R/I, M/xM) = g(R/I, M) - 1 = \operatorname{grade}_I M - 1.$$

□

Definition 3.18. Let (R, m) be a local ring and M a finitely generated R-module. Then the *depth of* M, denoted $\operatorname{depth} M$, is defined to be $\operatorname{grade}_m M$; i.e., the length of the longest M-sequence from R.

Remark 3.19. Let R be a local ring and M a finitely generated R-module. By Theorem 3.16, we have

$$\operatorname{depth} M = \inf\{n \mid \operatorname{Ext}_R^n(R/m, M) \neq 0\}.$$

Note: By convention, the depth of the zero module is infinity, since $\inf \emptyset = \infty$.

Theorem 3.20. (Ischebeck's Theorem) *Let (R, m) be a local ring and M and N finitely generated R-modules. Then $\operatorname{Ext}_R^i(M, N) = 0$ for $i < \operatorname{depth} N - \dim M$.*

Proof. First, if either module is zero the result trivially holds. So assume M and N are nonzero and let $d = \dim M$. If $d = 0$ then M has finite length. Since $\operatorname{Ext}_R^i(R/m, N) = 0$ for all $i < \operatorname{depth} N$ by Remark 3.19, we have $\operatorname{Ext}_R^i(M, N) = 0$ for all $i < \operatorname{depth} N$ by Lemma 2.22. Hence, the result holds for the case $d = 0$.

Suppose $d > 0$ and assume the result holds for all finitely generated modules of dimension less than d. Consider a filtration of M:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_j/M_{j-1} \cong R/p_j$ for some primes $p_1, \ldots, p_n$ of R. (See Grifo 905 :Theorem 6.33.) Note that each p_j contains $\operatorname{Ann}_R M$, so $\dim R/p_j \leqslant d$ for all j. Hence, if we show $\operatorname{Ext}_R^i(R/p_j, N) = 0$ for all $i < \operatorname{depth} N - \dim R/p_j$ and all j, then we'll have $\operatorname{Ext}_R^i(M_j/M_{j-1}, N) = 0$ for all $i < \operatorname{depth} N - d$ and all j. Using the long exact sequences on Ext arising from the short exact sequences

$$0 \to M_{j-1} \to M_j \to M_j/M_{j-1} \to 0,$$

we can conclude $\operatorname{Ext}_R^i(M_j, N) = 0$ for all $i < \operatorname{depth} N - d$. Since $M_n = M$, we'll be done.

Thus, assume $M \cong R/p$ for some prime p with $\dim R/p = d > 0$. Since $p \subsetneq m$, let $x \in m \setminus p$ and consider the short exact sequence

$$0 \to R/p \xrightarrow{x} R/p \to R/(p, x) \to 0.$$

Since $\dim R/(p, x) \leqslant d - 1$, we have $\operatorname{Ext}_R^i(R/(p, x), N) = 0$ for all $i < \operatorname{depth} N - d + 1$ by the induction hypothesis. Rewriting this, we have $\operatorname{Ext}_R^{i+1}(R/(p, x), N) = 0$ for all $i < \operatorname{depth} N - d$. From the long exact sequence on Ext, we have

$$\operatorname{Ext}_R^i(R/p, N) \xrightarrow{x} \operatorname{Ext}_R^i(R/p, N) \to \operatorname{Ext}_R^{i+1}(R/(p, x), N).$$

By Nakayama's Lemma, we see that $\operatorname{Ext}_R^i(R/p, N) = 0$ for all $i < \operatorname{depth} N - d$. This completes the proof.

□

Corollary 3.21. *Let (R, m) be a local ring and M a nonzero finitely generated R-module. Then* $\operatorname{depth} M \leqslant \dim R/p$ *for all* $p \in \operatorname{Ass}_R M$. *In particular,* $\operatorname{depth} M \leqslant \dim M$.

Proof. Let $p \in \operatorname{Ass}_R M$. Then $\operatorname{Ext}^0_R(R/p, M) = \operatorname{Hom}_R(R/p, M) \neq 0$. By Theorem 3.20, $\operatorname{depth} M \leqslant \dim R/p$. □

Example 3.22. Let k be a field and $R = k[x_1, \dots, x_{d+1}]/(x_1, \dots, x_{d+1})(x_{d+1})$. Let $m = (x_1, \dots, x_{d+1})R$. Then R_m has depth zero and dimension d. To see $\operatorname{depth} R_m = 0$, note that $mx_{d+1} = 0$ in R_m, so mR_m consists of zero-divisors. For $\dim R_m$, note that $\dim k[x_1, \dots, x_{d+1}]_m$ is a $(d+1)$-dimensional domain, so $\dim k[x_1, \dots, x_d]_m/I \leqslant d$ for any nonzero ideal I. Hence, $\dim R_m \leqslant d$. On the other hand, $\dim R_m \geqslant \dim R_m/(x_{d+1})R_m = \dim k[x_1, \dots, x_d]_{(x_1,\dots,x_d)} = d$. Thus, $\dim R_m = d$.

4 The Koszul complex

Let C and D be chain complexes with differentials ∂^C and ∂^D, respectively. Recall the definition of the tensor product complex $C \otimes_R D$ (see Grifo 915: Remark 6.14): For all n,

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

and

$$\partial_n^{C \otimes_R D}(c \otimes d) = \partial_p^C(c) \otimes d + (-1)^p c \otimes \partial_q^D(d)$$

for $c \in C_p$ and $d \in D_q$ and $p + q = n$.

For ease of notation, we often suppress the superscripts and subscripts on the differentials so long as they are clear from the context. For an element c of a complex C, we define the *degree* of c, denoted $|c|$, to be p if $c \in C_p$. Thus, the differential of $C \otimes_R D$ can be expressed as

$$\partial(c \otimes d) = \partial(c) \otimes d + (-1)^{|c|} c \otimes \partial(d).$$

Definition 4.1. Let R be a ring and $\mathbf{x} = x_1, \dots, x_n$ a sequence of elements in R. We define the *Koszul complex* $K(\mathbf{x})$ of $\mathbf{x}$ (on R) inductively as follows: When $n = 1$, we define $K(x_1)$ to be the complex

$$0 \to R \xrightarrow{x} R \to 0,$$

where the R on the right is in homological degree 0. Suppose $n > 1$ and let $\mathbf{x}' = x_1, \dots, x_{n-1}$. Assume that $K(\mathbf{x}')$ has been defined. Then we set $K(\mathbf{x}) := K(\mathbf{x}') \otimes_R K(x_n)$.

Example 4.2. Let R be a ring and x, y elements of R. Let's find $K(x, y) = K(x) \otimes_R K(y)$. So $K(x, y)$ is the complex

$$(0 \to R \xrightarrow{x} R \to 0) \otimes (0 \to R \xrightarrow{y} R \to 0).$$

To track degrees, let's write $K(x)$ as $0 \to K_1 \xrightarrow{x} K_0 \to 0$ and $K(y)$ as $0 \to L_1 \xrightarrow{y} L_0 \to 0$, where $K_i = L_i = R$ for all $i = 0, 1$. Then $K(x) \otimes K(y)$ has the form

$$0 \to K_1 \otimes_R L_1 \xrightarrow{\partial_2} (K_0 \otimes L_1) \oplus (K_1 \otimes_R L_0) \xrightarrow{\partial_1} K_0 \otimes_R L_0 \to 0.$$

Using the rule for the differential of a tensor product of complexes, we obtain

$$\begin{aligned}\partial_2(1\otimes 1) &= x\otimes 1 - 1\otimes y\\ \partial_1(1_{K_0}\otimes 1_{L_1}) &= 0\otimes 1 + 1\otimes y = 1\otimes y\\ \partial_1(1_{K_1}\otimes 1_{L_0}) &= x\otimes 1 - 1\otimes 0 = x\otimes 1\\ \partial_0(1\otimes 1) &= 0\otimes 1 + 1\otimes 0 = 0\otimes 0.\end{aligned}$$

Now, each $K_i \otimes L_j \cong R$ by virtue of the map sending $1\otimes 1$ to 1. Under this identification, the resulting complex is

$$0 \to R \xrightarrow{\begin{pmatrix} x \\ -y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & x \end{pmatrix}} R \to 0,$$

where again the rightmost R is in degree 0.

Definition 4.3. Let $\mathbf{x}$ be a (finite) sequence of elements of R and M an R-module. Then the Koszul complex of $\mathbf{x}$ on M is defined to be $K(\mathbf{x}; M) := K(\mathbf{x}) \otimes_R M$.

Example 4.4. Let $R = k[x, y]$ be a polynomial ring over a field k and let $M = R/(y)$. Then using Example 4.2 and that $M \cong k[x]$, we have $K(x, y; M) = K(x, y) \otimes_R R/(y)$ is isomorphic to

$$0 \to k[x] \xrightarrow{\begin{pmatrix} x \\ 0 \end{pmatrix}} k[x]^2 \xrightarrow{\begin{pmatrix} 0 & x \end{pmatrix}} k[x] \to 0.$$

Definition 4.5. Let R be a ring and C a complex of R-module. Then the *shift* or *suspension* ΣC of C is the complex defined by $(\Sigma C)_i = C_{i-1}$ and $\partial^{\Sigma C} = -\partial^C$. Note that $\mathrm{H}_i(\Sigma C) = \mathrm{H}_{i-1}(C)$.

Construction 4.6. Let C be a complex of R-modules and $x \in R$. Let $C(x)$ denote the complex $C \otimes_R K(x)$. Note that for all i

$$C(x)_i = (C_{i-1} \otimes_R R) \oplus (C_i \otimes_R R) \cong C_{i-1} \oplus C_i.$$

Let $(u, v) \in C(x)_{i-1}$, where $u \in C_{i-1}$ and $v \in C_i$. Then

$$\begin{aligned}\partial^{C(x)}(u, v) &= \partial^{C(x)}(u\otimes 1 + v\otimes 1)\\ &= \partial(u)\otimes 1 + (-1)^{|u|} u\otimes x + \partial(v)\otimes 1 + (-1)^{|v|} v\otimes 0\\ &= (\partial(u), \partial(v) + (-1)^{|u|} xu) \in C(x)_{i-1}.\end{aligned}$$

There exists chain maps $\alpha : C \to C(x)$ and $\beta : C(x) \to \Sigma C$ given by

$$\begin{aligned}\alpha(v) &:= (0, v) \in C(x)_i\\ \beta(u, v) &:= (-1)^{|u|} u \in (\Sigma C)_i\end{aligned}$$

for $u \in C_{i-1}$ and $v \in C_i$. One can easily check that these maps commute with differentials, and so are indeed chain maps. (Keep in mind that the differential of ΣC is $-\partial^C$.) Hence, we have a short exact sequence of complexes

$$0 \to C \xrightarrow{\alpha} C(x) \xrightarrow{\beta} \Sigma C \to 0.$$

This leads to the long exact sequence on homology

$$\cdots \to \mathrm{H}_i(C) \xrightarrow{\alpha_*} \mathrm{H}_i(C(x)) \xrightarrow{\beta_*} \mathrm{H}_{i-1}(C) \xrightarrow{\delta_{i-1}} \mathrm{H}_{i-1}(C) \to \cdots,$$

where here we have used the identification $\mathrm{H}_i(\Sigma C)$ with $\mathrm{H}_{i-1}(C)$.

Lemma 4.7. *The connecting homomorphism $\delta_{i-1} : \mathrm{H}_{i-1}(C) \to \mathrm{H}_{i-1}(C)$ in the long exact sequence above is multiplication by x for all i.*

Proof. This is a classic diagram chase argument. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\alpha} & C_{i-1} \oplus C_i & \xrightarrow{\beta} & C_{i-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^C} & & \downarrow{\scriptstyle \partial^{C(x)}} & & \downarrow{\scriptstyle -\partial^C} & & \\ 0 & \longrightarrow & C_{i-1} & \xrightarrow{\alpha} & C_{i-2} \oplus C_{i-1} & \xrightarrow{\beta} & C_{i-2} & \longrightarrow & 0 \end{array}$$

Let $\overline{z} \in \mathrm{H}_i(\Sigma C) = \mathrm{H}_{i-1}(C)$, where $z \in C_{i-1}$ is a cycle. Lift z via β to $((-1)^{|z|}z, 0) \in C_{i-1} \oplus C_i$. Then $\partial^{C(x)}((-1)^{|z|}z, 0) = (0, xz)$ and $\alpha^{-1}(0, xz) = xz$. (Note: here we are using z is a cycle in C.) Hence, $\delta_{i-1}(\overline{z}) = \overline{xz} = x\overline{z}$, where $\overline{*}$ denotes image in homology.

□

Definition 4.8. Let R be a ring, $\mathbf{x} = x_1, \ldots, x_n$ elements of R and M an R-module. Then $\mathrm{H}_i(\mathbf{x}; M) := \mathrm{H}_i(K(\mathbf{x}; M)$ is called the ith *Koszul homology* of M with respect to $\mathbf{x}$.

Proposition 4.9. *Let R be a ring, M an R-module, and $\mathbf{x} = x_1, \ldots, x_n \in R$ be a sequence of elements. Let $K^M = K(\mathbf{x}; M)$.*

(a) $K_i^M \cong M^{\binom{n}{i}}$ *for all i; in particular, $K_i^M = 0$ for $i < 0$ and $i > n$.*

(b) For all i, $\partial(K_i^M) \subseteq (\mathbf{x})K_{i-1}^M$;

(c) $\mathrm{H}_0(\mathbf{x}; M) = M/(\mathbf{x})M$.

(d) $\mathrm{H}_n(\mathbf{x}; M) = (0 :_M (\mathbf{x}))$.

(e) If $\mathbf{x}$ is an M-sequence then $\mathrm{H}_i(\mathbf{x}; M) = 0$ *for $i \geqslant 1$.*

Proof. We proceed by induction on n. In the case $n = 1$, K^M is the complex

$$0 \to M \xrightarrow{x_1} M \to 0.$$

It is clear part (a) holds and that $\mathrm{H}_0(x_1; M) \cong M/x_1M$, $\mathrm{H}_1(x_1; M) = (0 :_M x_1)$ and $\partial(K_1^M) = x_1M$. If x_1 is regular on M then $\mathrm{H}_1(x_1; M) = 0$.

Suppose $n > 1$ and all parts hold for smaller values. Let $\mathbf{x}' = x_1, \ldots, x_{n-1}$ and $C = K(\mathbf{x}'; M)$. Note that $K^M = C(x_n)$. From Construction 4.6 we have that $K_i^M = C(x_n)_i \cong C_{i-1} \oplus C_i$. By the $n-1$ case, we have $C_{i-1} \cong M^{\binom{n-1}{i-1}}$ and $C_i \cong M^{\binom{n-1}{i}}$ for all i. Part (a) now follows. We also have by the $n-1$ case that $\partial^C(C_i) \subseteq (\mathbf{x}')C_i$ for all i. From Construction 4.6, we have

$$\partial(K_i^M) = \partial(C(x)_i) \subseteq ((\mathbf{x}')C_{i-2}, (\mathbf{x}')C_{i-1} + x_nC_{i-1}) \subseteq (\mathbf{x})C(x)_i = (\mathbf{x})K_{i-1}^M.$$

For the remaining parts, we have by Construction 4.6 and Lemma 4.7 the long exact sequence

$$\cdots \to \mathrm{H}_i(\mathbf{x}'; M) \to \mathrm{H}_i(\mathbf{x}; M) \to \mathrm{H}_{i-1}(\mathbf{x}'; M) \xrightarrow{x_n} \mathrm{H}_{i-1}(\mathbf{x}'; M) \to \cdots$$

By the induction hypothesis, $\mathrm{H}_{n-1}(\mathbf{x}'; M) = (0 :_M (\mathbf{x}'))$ and $\mathrm{H}_0(\mathbf{x}'; M) \cong M/(\mathbf{x}')M$. From the l.e.s. above, we have the exact sequences

$$0 \to \mathrm{H}_n(\mathbf{x}; M) \to (0 :_M (\mathbf{x}')) \xrightarrow{x_n} (0 :_M (\mathbf{x}'))$$

and

$$M/(\mathbf{x}')M \xrightarrow{x_n} M/(\mathbf{x}')M \to \mathrm{H}_0(\mathbf{x}; M) \to 0.$$

This proves (b) and (c).

Suppose $\mathbf{x}$ is an M-sequence. By induction, we have that $\mathrm{H}_i(\mathbf{x}'; M) = 0$ for all $i \geqslant 1$. From the long exact sequence above we obtain $\mathrm{H}_i(\mathbf{x}; M) = 0$ for $i \geqslant 2$. It also yields the exact sequence

$$0 \to \mathrm{H}_1(K^M) \to M/(\mathbf{x}')M \xrightarrow{x_n} M/(\mathbf{x}')M.$$

Since x_n is a regular element on $M/(\mathbf{x}')M$ we conclude $\mathrm{H}_1(\mathbf{x}; M) = 0$. □

Remark 4.10. Let R be a Noetherian ring, $\mathbf{x} = x_1, \ldots, x_n \in R$, and M a finitely generated R-module. Then $\mathrm{H}_i(\mathbf{x}; M)$ is a finitely generated R-module for all i. This is because $\mathrm{H}_i(\mathbf{x}; M)$ is a subquotient of $K_i^M = M^{\binom{n}{i}}$, which is finite direct sum of copies of M. As M is Noetherian, so is K_i^M.

The converse of part (d) of Proposition 4.9 holds in strengthened form under certain conditions:

Theorem 4.11. *Let R be a Noetherian ring, M a nonzero finitely generated R-module, and $\mathbf{x} = x_1, \ldots, x_n \in J(R)$. The following are equivalent:*

(a) $\mathbf{x}$ *is an M-sequence;*

(b) $\mathrm{H}_i(\mathbf{x}; M) = 0$ *for all $i \geqslant 1$;*

(c) $\mathrm{H}_1(\mathbf{x}; M) = 0$.

Proof. From Proposition 4.9, we have (a) implies (b), and the implication (b) to (c) is obvious. We prove (c) implies (a) by induction on n. Suppose $n = 1$. Since $(0 :_M x_1) = \mathrm{H}_1(x_1; M) = 0$ we have that x_1 is a non-zero-divisor on M. And as $x_1 M \neq M$ by Nakayama's lemma, we see that x_1 is M-regular.

Suppose that $n > 1$. As in the proof of Proposition 4.9, we have the long exact sequence

$$\cdots \to \mathrm{H}_i(\mathbf{x}'; M) \to \mathrm{H}_i(\mathbf{x}; M) \to \mathrm{H}_{i-1}(\mathbf{x}'; M) \xrightarrow{x_n} \mathrm{H}_{i-1}(\mathbf{x}'; M) \to \cdots,$$

where $\mathbf{x}' = x_1, \ldots, x_{n-1}$. From the assumption $\mathrm{H}_1(\mathbf{x}; M) = 0$ we obtain the exact sequence

$$\mathrm{H}_1(\mathbf{x}'; M) \xrightarrow{x_n} \mathrm{H}_1(\mathbf{x}'; M) \to 0.$$

Thus, $x_n \mathrm{H}_1(\mathbf{x}'; M) = \mathrm{H}_1(\mathbf{x}'; M)$. Since $x_n \in J(R)$ and $\mathrm{H}_1(\mathbf{x}'; M))$ is finitely generated by Remark 4.10, we obtain that $\mathrm{H}_1(\mathbf{x}'; M) = 0$. By the induction hypothesis, $\mathbf{x}'$ is an M-sequence. From the exact sequence

$$0 \to \mathrm{H}_0(\mathbf{x}'; M) \xrightarrow{x_n} \mathrm{H}_0(\mathbf{x}'; M)$$

we conclude that x_n is a non-zero-divisor on $M/(\mathbf{x}')M$. Since $(\mathbf{x})M \neq M$ by Nakayama, we conclude that $\mathbf{x}$ is an M-sequence. □

Corollary 4.12. *Let R be a ring and $\mathbf{x} = x_1, \ldots, x_n$ an R-sequence. Then $K(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$ of length n. If R is local then $K(\mathbf{x})$ is a minimal free resolution of $R/(\mathbf{x})$.*

Proof. The first statement is clear from parts (a) and(e) of Proposition 4.9. For the second statement, note that from part (b) of Proposition 4.9, $\partial(K(\mathbf{x})) \subseteq (\mathbf{x})K(\mathbf{x}) \subseteq mK(\mathbf{x})$. Hence, $K(\mathbf{x})$ is minimal (cf. Definition 5.7 and Lemma 5.9 of Grifo's 915 Notes). □

Corollary 4.13. *Let R be a ring, $\mathbf{x} = x_1, \ldots, x_n$ an R-sequence, and M an R-module. Then for all i,*

$$\mathrm{H}_i(\mathbf{x}; M) \cong \mathrm{Tor}_i^R(R/(\mathbf{x}), M).$$

Proof. By Corollary 4.12, $K(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$. Hence,

$$\mathrm{Tor}_i^R(R/(\mathbf{x}), M) \cong \mathrm{H}_i(K(\mathbf{x}) \otimes_R M) = \mathrm{H}_i(K(\mathbf{x}; M) = \mathrm{H}_i(\mathbf{x}; M).$$

□

Proposition 4.14. *Let R be a ring, $\mathbf{x} = x_1, \ldots, x_n \in$, and M an R-module. Let $T_1, \ldots, T_n$ be indeterminates and $S = R[T_1, \ldots, T_n]$. Let $\phi : S \to R$ be the ring homomorphism given by $\phi(T_i) = x_i$. Consider M as an S-module via restriction of scalars; i.e., $T_i u = x_i u$ for all i and all $u \in M$. Then for all i,*

$$\mathrm{H}_i(\mathbf{x}; M) \cong \mathrm{Tor}_i^S(S/(\mathbf{T}), M).$$

Proof. First note that $\mathrm{H}_i(\mathbf{x}; M) \cong \mathrm{H}_i(\mathbf{T}; M)$ for all i. This is because the variables T_i act as the x_i's on M. Since $\mathbf{T}$ forms an S-sequence, we have by Corollary 4.13 that $\mathrm{H}_i(\mathbf{T}; M) \cong \mathrm{Tor}_i^R(S/(\mathbf{T}), M)$ for all i. □

Corollary 4.15. *Let R be a ring, $\mathbf{x} = x_1, \ldots, x_n \in R$, and M an R-module. Then $(\mathbf{x})\,\mathrm{H}_i(\mathbf{x}; M) = 0$ for all i.*

Proof. By Proposition 4.14, $\mathrm{H}_i(\mathbf{x}; M) \cong \mathrm{Tor}_i^S(S/(\mathbf{T}), M)$, where $S = R[T_1, \ldots, T_n]$. By a standard fact about Tor, we have $T_j \cdot \mathrm{Tor}_i^S(S/(\mathbf{T}), M) = 0$ for all j. Hence, $x_j\,\mathrm{H}_i(\mathbf{x}; M) = T_j\,\mathrm{H}_i(\mathbf{x}, M) = 0$ for all j. □

Proposition 4.16. *Let R be a ring and $\mathbf{x} = x_1, \ldots, x_n \in R$. Then for any short exact sequence of R-modules*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

there is a corresponding long exact sequence on Koszul homology

$$\cdots \to \mathrm{H}_i(\mathbf{x}; L) \to \mathrm{H}_i(\mathbf{x}; M) \to \mathrm{H}_i(\mathbf{x}; N) \to \mathrm{H}_{i-1}(\mathbf{x}; L) \to \cdots.$$

Proof. Let $K = K(\mathbf{x}; R)$. For each i we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i \otimes_R L & \xrightarrow{1\otimes f} & K_i \otimes_R M & \xrightarrow{1\otimes g} & K_i \otimes_R N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial\otimes 1} & & \downarrow{\scriptstyle \partial\otimes 1} & & \downarrow{\scriptstyle \partial\otimes 1} & & \\ 0 & \longrightarrow & K_{i-1} \otimes_R L & \xrightarrow{1\otimes f} & K_{i-1} \otimes_R M & \xrightarrow{1\otimes g} & K_{i-1} \otimes_R N & \longrightarrow & 0 \end{array}$$

where the two rows are exact as K_i is free (hence flat) for all i. Since $K(\mathbf{x}; A) = K \otimes_R A$ for all R-modules A, we have a short exact sequence of complexes

$$0 \to K(\mathbf{x}; L) \xrightarrow{1\otimes f} K(\mathbf{x}; M) \xrightarrow{1\otimes g} K(\mathbf{x}; N) \to 0.$$

Applying Theorem 2.28 of Grifo's 915 Notes, we obtain the desired long exact sequence.

□

Lemma 4.17. *Let (R, m) be a local ring and M a finitely generated R-module with $\mathrm{pd}_R M = t$. Then $\mathrm{Ext}_R^t(M, N) \neq 0$ for every nonzero finitely generated R-module N.*

Proof. Let $F_\bullet$ be a minimal free resolution of M. Let $F_\bullet$ be

$$0 \to F_t \xrightarrow{\phi_t} F_{t-1} \to \cdots \to F_0 \to 0,$$

where $\phi_t(F_t) \subseteq mF_{t-1}$. Applying $\operatorname{Hom}_R(-, N)$, we have the complex

$$\operatorname{Hom}_R(F_{t-1}, N) \xrightarrow{\phi_t^*} \operatorname{Hom}_R(F_t, N) \to 0.$$

It is easy to see that $\phi_t^*(\operatorname{Hom}_R(F_{t-1}, N)) \subseteq m \operatorname{Hom}_R(F_t, N)$. Since $\operatorname{Hom}_R(F_t, N) \cong N^{\operatorname{rank} F_t} \neq 0$, we see that $\operatorname{Ext}_R^t(M, N) = \operatorname{coker} \phi_t^* \neq 0$ by Nakayama.

□

Theorem 4.18. *Let (R, m) be a local ring and M a nonzero finitely generated R-module of finite injective dimension. Then $\operatorname{id}_R M = \operatorname{depth} R$.*

Proof. Let $p = \operatorname{depth} R$ and $r = \operatorname{id}_R M$. Let $\mathbf{x} = x_1, \ldots, x_p \in m$ be a maximal R-sequence. Then $\operatorname{pd}_R R/(\mathbf{x}) = p$ by Corollary 4.12. Hence, by Lemma 4.17, we obtain that $\operatorname{Ext}_R^p(R/(\mathbf{x}), M) \neq 0$. Thus, $r = \operatorname{id}_R M \geqslant p$. Also note $\operatorname{depth} R/(\mathbf{x}) = 0$ since $\mathbf{x}$ is a maximal R-sequence. Thus, $m \in \operatorname{Ass}_R R/(\mathbf{x})$, so there exists an injection $R/m \hookrightarrow R/(\mathbf{x})$. Applying $\operatorname{Hom}_R(-, M)$ to the resulting short exact sequence,

$$\cdots \to \operatorname{Ext}_R^r(R/(\mathbf{x}), M) \to \operatorname{Ext}_R^r(R/m, M) \to 0,$$

where we have used that $\operatorname{Ext}^i(A, M) = 0$ for all $i > r = \operatorname{id}_R M$ for all R-modules A. Since $\operatorname{id}_R M = r$, we have $\operatorname{Ext}_R^r(R/m, M) \neq 0$ by Proposition 2.24. Thus, $\operatorname{Ext}_R^r(R/(\mathbf{x}), M) \neq 0$, which implies $r \leqslant p$.

□

5 Cohen-Macaulay rings and modules

We first do a quick review of dimension theory in Noetherian rings:

Theorem 5.1. (Krull's Principal Ideal Theorem) *Let R be a Noetherian ring and $p \in \operatorname{Spec} R$ such that p is minimal over an ideal generated by n elements. Then $\operatorname{ht} p \leqslant n$.*

Proof. See Grifo's 905 notes, Theorem 8.5. □

Proposition 5.2. *Let (R, m) be a local ring. Then*

$$\dim R = \min\{n \mid \textit{there exists } x_1, \ldots, x_n \in m \textit{ such that } m = \sqrt{(x_1, \ldots, x_n)}\}.$$

Proof. See Grifo's 905 notes, Corollary 8.14. □

Definition 5.3. Let (R, m) be a local ring of dimension d. Then any d elements $x_1, \ldots, x_d \in m$ such that $m = \sqrt{(x_1, \ldots, x_d)}$ is called a *system of parameters* for R.

Proposition 5.4. *Let (R, m) be a d-dimensional local ring, M a nonzero finitely generated R-module, and $x \in m$. Then $\dim M/xM \geqslant \dim M - 1$ with equality if and only if $x \notin p$ for all $p \in \operatorname{Min}_R M$ such that $\dim R/p = \dim M$.*

Proof. Note that $\sqrt{\operatorname{Ann}_R M/xM} = \sqrt{(x) + \operatorname{Ann}_R M}$. (This is left as an exercise.) Thus, $\dim M/xM = \dim R/((x) + \operatorname{Ann}_R M) = \dim \overline{R}/(\overline{x})$, where $\overline{R} = R/\operatorname{Ann}_R M$. Since $\dim M = \dim \overline{R}$ and the minimal primes of M correspond to the minimal primes of $\overline{R}$, it suffices to prove the result in the case $M = R$. Suppose first that $x \in m$ and that $\dim R/(x) \leqslant \dim R - 2 = d - 2$. Then by Proposition 5.2, there exist $x_1, \ldots, x_{d-2} \in m$ such that $m/(x) = \overline{m} = \sqrt{(\overline{x}_1, \ldots, \overline{x}_{d-2})}$, where $^-$ here means image in $R/(x)$. Lifting to R, we get $m = \sqrt{(x, x_1, \ldots, x_{d-1})}$. But this means m is minimal over an ideal generated by $d-1$ elements, contradicting that $\dim R = \operatorname{ht} m = d$. Hence, $\dim R/(x) \geqslant d - 1$. Thus, $\dim R/(x)$ is either d or $d-1$. Suppose that $x \in p$ where $\dim R/p = d$. Then $\overline{p} = p/(x) \in \operatorname{Spec} \overline{R}$ and $\dim \overline{R}/\overline{p} = \dim R/p = d$. So $\dim R/(x) = d$. Conversely, suppose $\dim R/(x) = d$. Then there exists a prime ideal $q \in \operatorname{Spec} \overline{R}$ such that $\dim \overline{R}/q = d$. Lifting q to a prime p of R containing x, we have $\dim R/p = \dim \overline{R}/q = d$. Thus, $\dim R/(x) = d$ if and only if $x \in p$ for some prime p such that $\dim R/p = d = \dim R$.

□

Definition 5.5. Let (R, m) be a Noetherian local ring. A finitely generated R-module M is called *Cohen-Macaulay* (or *CM* for short) if $M = 0$ or $\operatorname{depth} M = \dim M$.

Examples 5.6.

- Zero-dimensional local rings are CM. E.g., $R = k[x, y]/(x^2, xy, y^2)$.
- One-dimensional local domains are CM. E.g., $R = \mathbb{Z}_{(2)}$ or $R = k[x, y]_{(x,y)}/(x^3 - y^2)$
- Polynomials rings over a field (localized) are CM; E.g. $R = k[x_1, \ldots, x_d]_{(\mathbf{x})}$.
- Two-dimensional local UFDs are CM. (Exercise)
- $R = k[x, y]/(x^2, xy)$ (localized at (x, y)) is not CM, since $\operatorname{depth} R = 0 < 1 = \dim R$. However, $M = R/(x)$ is a CM R-module.

Proposition 5.7. *Let (R, m) be a local ring and M a finitely generated R-module.*

(a) If M is CM then $\dim R/p = \dim M$ for all $p \in \operatorname{Ass}_R M$.

(b) Suppose $\mathbf{x}$ is an M-sequence. Then M is CM if and only if $M/(\mathbf{x})M$ is CM.

Proof. Part (a) follows immediately from Corollary 3.21 and the definition of CM. To prove (b), it suffices to consider the case $\mathbf{x} = x$, a single element. Suppose M is CM and x is M-regular. We know $\operatorname{depth} M/xM = \operatorname{depth} M - 1$ by Corollary 3.17. Since x is not in any associated prime of M, we have $\dim M/xM = \dim M - 1$ by Proposition 5.4. Thus, $\operatorname{depth} M/xM = \dim M/xM$ and M/xM is Cohen-Macaulay. Conversely, assume M/xM is Cohen-Macaulay, where x is M-regular. As above, we have $\operatorname{depth} M/xM = \operatorname{depth} M - 1$ and $\dim M/xM = \dim M - 1$. Hence, $\dim M = \operatorname{depth} M$ and M is Cohen-Macaulay.

□

Examples 5.8.

- Let $R = k[x_1, \ldots, x_d]_{(\mathbf{x})}$ and f a nonzero element of R (and a nonunit). Then $R/(f)$ is CM. Such rings are called *local hypersurface rings.*
- Let $R = k[x, y, z]/(x) \cap (y, z)$ (localized at (x, y, z)) is not CM. (Why?)

Remark 5.9. Let R be a ring, M an R-module, and $\mathbf{x}$ an M-sequence. Let S be a multiplicatively closed set of R. If $(\mathbf{x})M_S \neq M_S$, then $\frac{\mathbf{x}}{1} \in R_S$ is an M_S-sequence. To see this, it suffices to consider the case $\mathbf{x} = x$ is a single element. But if $0 \to M \xrightarrow{x} M$ is exact, then so is $0 \to M_S \xrightarrow{\frac{x}{1}} M_S$, as localization is exact.

An important consequence of this remark is:

Lemma 5.10. *Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module such that $IM \neq M$. Let S be a multiplicatively closed set of R. If $IM_S \neq M_S$ then* $\operatorname{grade}(I_S, M_S) \geqslant \operatorname{grade}(I, M)$*.*

Proposition 5.11. *Let (R, m) be a local ring and M a finitely generated CM R-module. Then for every $p \in \operatorname{Supp}_R M$, $\dim M_p = \operatorname{grade}(p, M)$. In particular, M_p is CM for all $p \in \operatorname{Spec} R$.*

Proof. Let $p \in \operatorname{Supp}_R M$. By Lemma 5.10 we have

$$\dim M_p \geqslant \operatorname{depth} M_p = \operatorname{grade}(pR_p, M_p) \geqslant \operatorname{grade}(p, M).$$

Hence, if we show $\dim M_p = \operatorname{grade}(p, M)$ then M_p is CM. We'll prove this by induction of $\operatorname{grade}(p, M)$. If $\operatorname{grade}(p, M) = 0$ then $p \subseteq q$ for some $q \in \operatorname{Ass}_R M$. By Proposition 5.7, $\dim R/q = \dim M$. Hence q, and therefore p, is minimal in $\operatorname{Supp}_R M$. Consequently, $\dim M_p = 0$. Now suppose $\operatorname{grade}(p, M) > 0$. Let $x \in p$ be an M-regular element. By Proposition 5.7, M/xM is a CM R-module and $\operatorname{grade}(p, M/xM) = \operatorname{grade}(p, M) - 1$. By the induction hypothesis, $\dim(M/xM)_p = \operatorname{grade}(p, M/xM)$. Since $\dim(M/xM)_p = \dim M_p/xM_p = \dim M_p - 1$ by Proposition 5.4, we obtain that $\dim M_p = \operatorname{grade}(p, M)$ as desired. □

Definition 5.12. Let R be a Noetherian ring. A finitely generated R-module M is called *CM* if M_m is CM for all maximal ideals m of R. Equivalently, by the previous proposition, M is CM if M_p is CM for all $p \in \operatorname{Spec} R$.

Examples 5.13.

- Artinian rings are CM.
- Polynomial rings over a field are CM (see Theorem 5.21 below).
- One-dimensional domains (e.g., $\mathbb{Z}$) are CM.

Definition 5.14. Let R be a Noetherian ring. For an ideal I of R, the *height* of I, denoted $\operatorname{ht} I$, is defined to the minimum of $\operatorname{ht} p$ for all primes p containing I.

Lemma 5.15. *Let R be a Noetherian ring and I a proper ideal. Then* $\operatorname{grade} I \leqslant \operatorname{ht} I$*.*

Proof. Let p be a prime containing I. Then using Corollary 3.21 and Lemma 5.10 we have

$$\operatorname{grade} I \leqslant \operatorname{grade} p \leqslant \operatorname{grade}(pR_p, R_p) = \operatorname{depth} R_p \leqslant \dim R_p = \operatorname{ht} p.$$

Since this holds for all primes containing I, we conclude that $\operatorname{grade} I \leqslant \operatorname{ht} I$. □

Theorem 5.16. *Let R be a Noetherian ring. Then R is CM if and only if* $\operatorname{grade} I = \operatorname{ht} I$ *for all proper ideals I of R.*

Proof. Suppose grade $I = \operatorname{ht} I$ for all proper ideals I. Let m be a maximal ideal of R. Then

$$\operatorname{depth} R_m = \operatorname{grade}(mR_m, R_m) \geqslant \operatorname{grade} m = \operatorname{ht} m = \dim R_m.$$

Thus, R_m is CM, and since m was arbitrary, R is CM.

Now suppose R is CM. Let $I \neq R$ be an ideal and $g = \operatorname{grade} I$. Let $\mathbf{x} = x_1, \ldots, x_g$ be a maximal R-sequence contained in I. Then I consists of zero-divisors on $R/(\mathbf{x})$, so I is contained in some $p \in \operatorname{Ass}_R R/(\mathbf{x})$. Since $R/(\mathbf{x})R$ is CM, $\operatorname{Ass}_R R/(\mathbf{x}) = \operatorname{Min}_R R/(\mathbf{x})$ and so p is minimal over $(\mathbf{x})$. Thus, $\operatorname{ht} I \leqslant \operatorname{ht} p \leqslant g = \operatorname{grade} I$.

□

Proposition 5.17. *Let (R, m) be a local ring. The following are equivalent:*

(a) R is CM;

(b) Every system of parameters for R forms a regular sequence;

(c) Some system of parameters for R forms a regular sequence.

Proof. $(c) \implies (a)$: let $x_1 \ldots, x_d$ be an s.o.p. for R which forms an R-sequence. Then $d = \dim R$ and depth $R \geqslant d$. Hence, R is CM.

$(b) \implies (c)$: *a fortiori.*

$(a) \implies (b)$: We use induction on $d = \dim R$. If $d = 0$ there is nothing to show. Suppose $d > 0$ and the implication holds for all CM local rings of dimension less than d. Let $\mathbf{x} = x_1, \ldots, x_d$ be an s.o.p. for R. Thus, m is minimal over $(x_1, \ldots, x_d)$. Suppose x_1 is a zero-divisor on R. Then, as R is CM, x_1 is in some minimal prime of R and $\dim R/(x_1) = d$. But then $m/(x_1)$ is minimal over $(\overline{x_2}, \ldots, \overline{x_d})$) (where $^-$ denotes the image of a in $R/(x_1)$). But this contradicts KPIT, as $\operatorname{ht} m/(x_1) = \dim R/(x_1) = d$. Thus, x_1 is regular on R and $R/(x_1)$ is a $d-1$-dimensional CM local ring by Proposition 5.7(a). By the induction hypothesis, $\overline{x_2}, \ldots, \overline{x_d}$ forms a regular sequence on $R/(x_1)$. Thus, $x_1, \ldots, x_d$ is an R-sequence.

□

Theorem 5.18. *Let (R, m) be a CM local ring. Then* $\dim R/I + \operatorname{ht} I = \dim R$ *for any ideal I of R.*

Proof. We first do the case $I = p$ is a prime ideal. Let $\mathbf{x} = x_1, \ldots, x_g$ be a maximal R-sequence contained in p. Note that $g = \operatorname{grade} p = \operatorname{ht} p$. Since p consists of zero-divisors on $R/(\mathbf{x})$, and $R/(\mathbf{x})$ is CM, $\dim R/p = \dim R/(\mathbf{x}) = \dim R - g$, where we have used Proposition 5.4 for the last equality. Hence, the formula holds when I is prime.

Let I be an arbitrary ideal. Let $p \supseteq I$ be a prime ideal such that $\dim R/p = \dim R/I$. Then

$$\dim R/I + \operatorname{ht} I = \dim R/p + \operatorname{ht} I \leqslant \dim R/p + \operatorname{ht} p = \dim R.$$

Now let $p \supseteq I$ such that $\operatorname{ht} p = \operatorname{ht} I$. Then

$$\dim R/I + \operatorname{ht} I = \dim R/I + \operatorname{ht} p \geqslant \dim R/p + \operatorname{ht} p = \dim R.$$

□

Definition 5.19. A Noetherian ring R is called *catenary* if for any primes $p \subset q$ of R, every saturated chain of primes from p to q has the same length, namely $\operatorname{ht} q/p$ in R/p.

Corollary 5.20. *Let R be a Noetherian ring which is the quotient of a CM ring. Then for all primes $p \subset q$ of R, $\operatorname{ht} q/p = \operatorname{ht} q - \operatorname{ht} p$. In particular, R is catenary.*

Proof. We first establish the equality. Let $R = S/J$ where S is a CM ring. Since the primes of R are in bijective inclusion-preserving correspondence with the primes of S which contain J, it suffices to show the equality holds for primes in S. So let $p \subset q$ be primes of S. Since S_q is CM, we have by Theorem 5.18 that

$$\operatorname{ht} q = \dim S_q = \dim S_q/pS_q + \operatorname{ht} pS_q = \operatorname{ht} q/p + \operatorname{ht} p.$$

To show R is catenary, let $p \subset q$ be primes and $p = p_0 \subset p_1 \subset \cdots \subset p_n = q$ be a saturated chain; i.e. $\operatorname{ht} p_i/p_{i-1} = 1$ for all $i = 1, \dots, n$. Thus $\operatorname{ht} p_i - \operatorname{ht} p_{i-1} = 1$ for $i = 1, \dots n$. Summing these up, we obtain $n = \operatorname{ht} q - \operatorname{ht} p$.

□

Theorem 5.21. *Let R be a CM ring and $t_1, \dots, t_n$ indeterminates. Then $R[t_1, \dots, t_n]$ is CM.*

Proof. It suffices to prove the case $n = 1$. Let n be a maximal ideal of $S = R[t]$ and $m = n \cap R$. Then m is a prime ideal of R and let $W = R \setminus m$. Then $S_n \cong (S_W)_{n_W} \cong R_W[t]_{n_W}$. Note that R_W is a CM local ring with maximal ideal m_W, and $n_W \cap R_W = m_W$. Thus, to prove S_n is CM, we may assume $S = R[t]$ where (R, m) is a CM local ring and $n \cap R = m$. Note that n/mS is a maximal ideal of $S/mS \cong (R/m)[t]$, which is a PID. Thus, $n = (m, f(t))S$, where $f \in R[t]$ such that the leading coefficient of $f = f(t)$ is a unit in R. Let $x_1, \dots, x_d$ be an s.o.p. for R. Then $x_1, \dots, x_d, f$ is an s.o.p. for S_n. (It is easy to see that the height of nS is $d + 1$.) Now, as S_n is a faithfully flat R-algebra, $\mathbf{x}$ is an S-sequence by a homework exercise. And as the leading coefficient of f is a unit in R, f is a non-zero-divisor on $R/(\mathbf{x})[t]$ and hence also on $S_n/(\mathbf{x})S_n$. Thus, $x_1, \dots, x_d, f$ is an S_n-sequence and S_n is CM.

□

Definition 5.22. Let R be a ring. An R-algebra S is said to be *finite type* over R if $S = R[u_1, \dots, u_n]$ for some $u_1, \dots, u_n \in S$. Additionally, S is said to be *essentially* of finite type over R if S is the localization of an R-algebra of finite type.

Corollary 5.23. *Any algebra essentially of finite type over a field or the integers is catenary.*

Proof. Let S be such an algebra. Then S is a localization of a quotient of the polynomial ring $k[t_1, \dots, t_n]$ for some n, where k is a field or $\mathbb{Z}$. As k is CM, so is $k[t_1, \dots, t_n]$ by Theorem 5.21. Thus, any quotient of $k[t_1, \dots, t_n]$ is catenary by Corollary 5.20. Noting that localizations of catenary rings are catenary completes the proof. □

6 Gorenstein rings

Definition 6.1. A Noetherian local ring R is said to be *Gorenstein* if R has finite injective dimension as an R-module.

Proposition 6.2. *Gorenstein local rings are CM.*

Proof. By Theorem 4.18 and Corollary 2.25, we have $\operatorname{depth} R = \operatorname{id}_R R \geqslant \dim R$. Hence, R is CM. □

Lemma 6.3. *Let (R, m) be a local ring and M an R-module of finite length. Let $E = E_R(R/m)$. Then $\lambda_R(M) = \lambda_R(\operatorname{Hom}_R(M, E))$.*

Proof. We proceed by induction on $\lambda_R(M)$. Suppose $\lambda_R(M) = 1$. Then $M \cong R/m$. Then $\operatorname{Hom}_R(M, E) \cong \operatorname{Hom}_R(R/m, E) \cong E_{R/m}(R/m) \cong R/m$. Hence, $\lambda_R(\operatorname{Hom}_R(M, E)) = 1$. Now assume M is a module of finite length $n > 0$ and that the result holds for all R-modules of length less than n. Then there exists a short exact sequence

$$0 \to L \to M \to N \to 0$$

where $\lambda_R(L) = 1$ and $\lambda_R(N) = n - 1$. Applying the exact functor $\operatorname{Hom}_R(-, E)$, we obtain the exact sequence

$$0 \to \operatorname{Hom}_R(N, E) \to \operatorname{Hom}_R(M, E) \to \operatorname{Hom}_R(L, E) \to 0.$$

We have by induction that $\lambda_R(N) = \lambda_R(\operatorname{Hom}_R(N, E))$ and $\lambda_R(L) = \lambda_R(\operatorname{Hom}_R(L, E))$. By the additivity of length on short exact sequences, we obtain $\lambda_R(M) = \lambda_R(\operatorname{Hom}_R(M, E))$.

□

Definition 6.4. Let (R, m) be a local ring and M an R-module. The *socle* of M, denoted $\operatorname{Soc}_R M$, is defined by

$$\operatorname{Soc}_R M = (0 :_M m) \cong \operatorname{Hom}_R(R/m, M).$$

Note that if M is finitely generated then $\operatorname{Soc}_R M$ is a finite dimensional R/m-vector space.

Proposition 6.5. *Let (R, m) be a zero-dimensional local ring. Then R is Gorenstein if and only if $\dim_{R/m} \operatorname{Soc}_R R = 1$.*

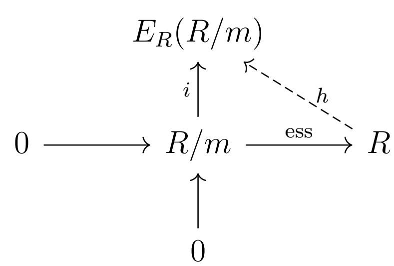
Proof. Suppose R is a zero-dimensional local Gorenstein ring. Then $\operatorname{id}_R R < \infty$, so $\operatorname{id}_R R = \operatorname{depth} R \leqslant \dim R = 0$ by Theorem 4.18. Thus, R is injective. As R is indecomposable, we must have $R \cong E_R(R/m)$. Then

$$\operatorname{Soc}_R R = \operatorname{Soc}_R E_R(R/m) \cong \operatorname{Hom}_R(R/m, E_R(R/m)) \cong R/m.$$

Hence, $\dim_{R/m} \operatorname{Soc} R = 1$.

We claim quite generally that for zero-dimensional local rings R is essential over $\operatorname{Soc}_R R$: since $m^n = 0$ for some n, we have that for every nonzero element $y \in R$, there exists an ℓ such that $m^\ell y \neq 0$ but $m^{\ell+1} y = 0$. Thus, $m^\ell y \cap \operatorname{Soc}_R R \neq 0$, proving the claim.

Given that $\dim_R \operatorname{Soc}_R R = 1$, we have $\operatorname{Soc}_R R \cong R/m$. Then we have the diagram



where h restricted to $\operatorname{Soc}_R R \cong R/m$ is i (inclusion). Since i is injective and the inclusion $R/m \cong \operatorname{Soc}_R R \subseteq R$ is essential, we must have h is injective. But since $\lambda_R(R) = \lambda_R(E_R(R/m))$ by Lemma 6.3, we obtain that h is an isomorphism. Hence, R is an injective R-module and R is Gorenstein.

□

Examples 6.6. The examples below follow by computing the socle dimension. (Here, k is an arbitrary field.)

- k is Gorenstein.
- $k[x]/(x^3)$ is Gorenstein.
- $k[x,y]/(x^2,y^3)$ is Gorenstein.
- $k[x,y]/(x^2,xy,y^2)$ is not Gorenstein.
- $\mathbb{Z}/(4)$ is Gorenstein.
- $k[x,y,z]/(x^2-y^2,x^2-z^2,xy,xz,yz)$ is Gorenstein.

Proposition 6.7. *If R is a local Gorenstein ring and $p \in \operatorname{Spec} R$ then R_p is Gorenstein.*

Proof. By either Proposition 1.15 or using Corollary 2.4 together with Proposition 2.9, we have $\operatorname{id}_{R_p} R_p \leqslant \operatorname{id}_R R$. Hence, R_p is Gorenstein. □

Definition 6.8. A Noetherian ring R is called *Gorenstein* if R_m is a Gorenstein local ring for all maximal ideals m. By the previous proposition, this is equivalent to R_p being Gorenstein for all primes p of R.

Lemma 6.9. *Let S be a ring and $\{F_i\}_{i\geqslant 0}$ a set of additive functors on the category of S-modules such that the following hold:*

- *$F^0(-)$ is naturally equivalent to the functor $\operatorname{Hom}_S(-,N)$ for some S-module N;*
- *$F^i(P)=0$ for all $i>0$ and projective S-modules P;*
- *For any short exact sequence of S-modules $0 \to A \to B \to C \to 0$ there is a long exact sequence (which is natural on the category of short exact sequences of S-modules)*

$$\cdots \to F^i(C) \to F^i(B) \to F^i(A) \to F^{i+1}(C) \to \cdots .$$

Then for each $i \geqslant 0$, $F^i(-)$ is naturally equivalent to $\operatorname{Ext}^i_S(-,N)$.

Proof. This is a straightforward exercise, using the fact the $\operatorname{Ext}^i_S(-,N)$ also satisfies the three properties above. □

Theorem 6.10. *Let R be a ring, M and N R-modules, and $x \in R$ such that x is regular on both R and N, and $xM=0$. Let $S=R/(x)$. Then*

(a) $\operatorname{Hom}_R(M,N)=0$, *and*

(b) $\operatorname{Ext}^i_R(M,N)$ *is naturally isomorphic to* $\operatorname{Ext}^{i-1}_S(M,N/xN)$ *for all* $i \geqslant 1$.

Proof. For part (a), let $f: M \to N$ be an R-homomorphism. Then $0=f(0)=f(xM)=xf(M) \subseteq N$. As x is regular on N, we see that $f=0$.

For part (b), for $i \geqslant 0$ let $F^i(-) := \operatorname{Ext}^{i+1}_R(-,N)$. Consider the exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0.$$

Then for any S-module M we have (using part (a)):

$$0 \to \operatorname{Hom}_R(M, N/xN) \to \operatorname{Ext}^1_R(M,N) \xrightarrow{x} \operatorname{Ext}^1_R(M,N) \to \cdots .$$

Since $xM = 0$, we have $x \operatorname{Ext}^1_R(M,N) = 0$. Thus,

$$F^0(M) = \operatorname{Ext}^1_R(M,N) \cong \operatorname{Hom}_R(M, N/xN) \cong \operatorname{Hom}_S(M, N/xN)$$

and the isomorphisms are natural. Now let P be a projective S-module. We wish to show that $F^i(P) = \operatorname{Ext}^{i+1}_R(P,N) = 0$ for all $i \geqslant 1$. Since P is direct summand of a free S-module, it suffices to prove this when $P = \oplus_{i\in I} S$ is a free S-module. But since $S = R/(x)$ and x is R-regular,

$$0 \to \bigoplus_{i\in I} R \xrightarrow{x} \bigoplus_{i\in I} R \to 0$$

is a free resolution of P as an R-module. Hence, $\operatorname{Ext}^j_R(P,N) = 0$ for all $j \geqslant 2$.

Finally, let $0 \to A \to B \to C \to 0$ be a short exact sequence of S-modules. As $\operatorname{Hom}_R(A,N) = \operatorname{Hom}_R(B,N) = \operatorname{Hom}_R(C,N) = 0$ by part (a), we have a natural long exact sequence

$$0 \to \operatorname{Ext}^1_R(C,N) \to \operatorname{Ext}^1_R(B,N) \to \operatorname{Ext}^1_R(A,N) \to \operatorname{Ext}^2_R(C,N) \to \cdots .$$

Thus, we have a natural long exact sequence

$$0 \to F^0(C) \to F^0(B) \to F^0(A) \to F^1(C) \to \cdots .$$

Hence, by Lemma 6.9, we have for $i \geqslant 1$ that

$$\operatorname{Ext}^i_R(M,N) = F^{i-1}(M) \cong \operatorname{Ext}^{i-1}_S(M, N/xN),$$

and these isomorphisms are natural.

□

Theorem 6.11. *Let (R, m) be a Noetherian local ring, M a finitely generated R-module, and $x \in m$ a regular element on R and M. Then*

$$\operatorname{id}_{R/(x)} M/xM = \operatorname{id}_R M - 1.$$

Proof. Recall from Corollary 2.24 that

$$\operatorname{id}_R M = \sup\{n \mid \operatorname{Ext}^n_R(k,M) \neq 0\}.$$

But by Theorem 6.10, for all $i \geqslant 1$, $\operatorname{Ext}^i_R(k,M) \cong \operatorname{Ext}^{i-1}_{R/xR}(k, M/xM)$. The result now follows. □

Corollary 6.12. *Let (R, m) be a Noetherian local ring and $\mathbf{x}$ a regular sequence on R. Then R is Gorenstein if and only if $R/(\mathbf{x})$ is Gorenstein.*

Proof. It suffices to prove the result in the case $\mathbf{x} = x_1$, a regular sequence of length 1. But by Theorem 6.11, $\operatorname{id}_R R < \infty$ if and only if $\operatorname{id}_{R/xR} R/xR < \infty$. Thus, the result follows by the definition of Gorenstein. □

Definition 6.13. Let (R, m, k) be a CM local ring of dimension d. The *CM type* of R, denoted $r(R)$, is defined to be $\dim_k \operatorname{Ext}^d_R(k,R)$.

Proposition 6.14. *Let (R, m, k) be a CM local ring. Then:*

(a) If $x \in R$ is a regular element then $r(R) = r(R/(x))$.

(b) For any s.o.p. $\mathbf{x}$ of R, $r(R) = \dim_k \operatorname{Soc} R/(\mathbf{x})$.

(c) R is Gorenstein if and only $r(R) = 1$.

Proof. Note that by Theorem 6.10, $\operatorname{Ext}^{d-1}_{R/(x)}(k, R/(x)) \cong \operatorname{Ext}^d_R(k, R)$. Part (a) follows readily. For (b), recall that any s.o.p. in a CM local ring generates an R-sequence. Hence, using part (a) and induction, we obtain $r(R) = r(R/(\mathbf{x}))$. Since $\dim R/(\mathbf{x}) = 0, r(R/(\mathbf{x})) = \dim_k \operatorname{Hom}_R(k, R/(\mathbf{x})) = \dim_k \operatorname{Soc} R/(\mathbf{x})$.

Finally, let $\mathbf{x}$ be an s.o.p. for R. By Corollary 6.12, R is Gorenstein and only if $R/(\mathbf{x})$ is Gorenstein. As $R/(\mathbf{x})$ is zero-dimensional, by Proposition 6.5, $R/(\mathbf{x})$ is Gorenstein if and only if $\dim_k \operatorname{Soc} R/(\mathbf{x}) = 1$, which is if and only if $r(R) = 1$ by part (b).

□

Corollary 6.15. *Let (R, m, k) be a local ring. Then the following are equivalent:*

(a) R is Gorenstein.

(b) $\mu_i(p, R) = \delta_{i \operatorname{ht} p}$ for all i and $p \in \operatorname{Spec} R$. (Here, δ_{ij} is the Kronecker delta function.)

Proof. Let $d = \dim R$. Assume (b) holds. Then clearly $\mu_i(p, R) = 0$ for $i > d$ for all $p \in \operatorname{Spec} R$. Hence, $\operatorname{id}_R R \leqslant d$ by Proposition 2.9(b). Hence R is Gorenstein.

Now suppose R is Gorenstein. As R is CM, $\operatorname{depth} R = d$, so $\operatorname{Ext}^i_R(k, R) = 0$ for all $i < d$ by Remark 3.19. Since $\operatorname{id}_R R = \operatorname{depth} R = d$ by Theorem 4.18, we have $\operatorname{Ext}^i_R(k, R) = 0$ for all $i > d$. Thus, $\mu_i(m, R) = 0$ for all $i \neq d$. Since $r(R) = 1$, $\mu_d(m, R) = 1$. This establishes (b) in the case $p = m$. But for any $p \in \operatorname{Spec} R$, $\mu_i(p, R) = \mu_i(pR_p, R_p)$ (by Theorem 2.5) and R_p is Gorenstein. Thus, as pR_p is the maximal ideal of R_p, we have by the maximal ideal case that $\mu_i(p, R) = \delta_{i \operatorname{ht} p}$.

□

Corollary 6.16. *Let R be a local Gorenstein ring. Then a minimal injective resolution of R has the form*

$$0 \to \bigoplus_{\operatorname{ht} p = 0} E_R(R/p) \to \bigoplus_{\operatorname{ht} p = 1} E_R(R/p) \to \cdots \to \bigoplus_{\operatorname{ht} p = d-1} E_R(R/p) \to E_R(R/m) \to 0.$$

Proof. This follows from Corollary 6.15 and Theorem 2.5. □

Proposition 6.17. *Let R be a Gorenstein ring and x an indeterminate. Then $R[x]$ is Gorenstein.*

Proof. Homework exercise. □

7 Regular local rings and modules of finite projective dimension

Definition 7.1. Let (R, m, k) be a local ring. Then the *embedding dimension* of R, denoted $\operatorname{edim} R$, is defined to be the least number of elements needed to generate m; i.e., $\operatorname{edim} R = \operatorname{rank}_k m/m^2$ (by Nakayama).

Remark 7.2. For any local ring R we have $\operatorname{edim} R \geqslant \dim R$ by Krull's Principal Ideal Theorem.

Lemma 7.3. *Let (R, m, k) be a local ring and $I \subseteq m$ an ideal. Then*

$$\operatorname{edim} R/I = \operatorname{edim} R - \dim_k I/I \cap m^2.$$

In particular, for $x \in m$, $\operatorname{edim} R/(x) \geqslant \operatorname{edim} R - 1$ with equality if and only if $x \notin m^2$.

Proof. Let $n = m/I$ be the maximal ideal of R/I. Then $n/n^2 \cong m/(m^2 + I)$. Thus,

$$\begin{aligned}
\operatorname{edim} R/I &= \dim_k n/n^2 \\
&= \dim_k m/(m^2 + I) \\
&= \dim_k m/m^2 - \dim_k(m^2 + I)/m^2 \\
&= \operatorname{edim} R - \dim_k I/I \cap m^2.
\end{aligned}$$

If $I = (x)$, then $I/I \cap m^2 = k\overline{x}$ where $\overline{x}$ is the image of x in $I/I \cap m^2$. Hence, $\dim_k I/I \cap m^2 = \dim_k k\overline{x} \leqslant 1$, with equality if and only if $\overline{x} \neq 0$, i.e., $x \notin m^2$. □

Example 7.4. Let $R = k[x, y, z]_{(x,y,z)}$. Since $\operatorname{edim} R \geqslant \dim R = 3$, we see that $\operatorname{edim} R = 3$ as the maximal ideal is 3-generated. Now let $S = R/(xy - z^2, x^3z + y^4, x^2 - yz^3)$. Then $\operatorname{edim} S = 3$ by the above lemma. What about $R/(xy - z^2, y + xz^3)$?

Definition 7.5. Let (R, m, k) be a local ring. Then R is said to be a *regular local ring* (or RLR, short) if $\operatorname{edim} R = \dim R$. If $m = (x_1, \ldots, x_d)$ where $d = \dim R$, then $x_1, \ldots, x_d$ is called a *regular system of parameters*.

Examples 7.6. The following are examples of regular local rings:

- Any field
- $\mathbb{Z}_{(2)}$
- $\mathbb{Z}[x]_{(2.x)}$
- $k[x_1, \ldots, x_d]_{(\mathbf{x})}$, where k is any field
- $k[[x_1, \ldots, x_d]]$ (formal power series over the field k)

Proposition 7.7. *Regular local rings are domains.*

Proof. Let (R, m) be a regular local ring. We use induction on $d = \dim R$ to show R is a domain. Suppose $d = 0$. Then $m = 0$; i.e., R is a field. Next, assume $d = 1$. Then $m = (x)$ for some $x \in m$. Let p be a minimal prime of R. Let $a \in p$. Then $a = rx$ for some $r \in R$. As $rx \in p$ and $x \notin p$, we must have $r \in p$. Hence, $a \in xp$. As $a \in p$ is arbitrary, we have $p = xp$. By Nakayama's lemma, $p = 0$ and thus R is a domain.

Now suppose $d > 1$. By prime avoidance, choose $x \in m \setminus m^2$ and such that $x \notin p$ for any $p \in \operatorname{Min} R$. Then $\operatorname{edim} R/(x) = \operatorname{edim} R - 1$ and $\dim R/(x) = \dim R - 1$ by Lemma 7.3 and Proposition 5.4. Thus $R/(x)$ is a regular local ring of dimension $d - 1$. By the induction hypothesis, we have $R/(x)$ is a domain and hence (x) is a (non-minimal) prime ideal of R. Let $p \subset\neq (x)$ be a minimal prime of R. Repeating the argument from the $d = 1$ case, we get $p = xp$, so $p = 0$. Thus, R is a domain. □

Corollary 7.8. *Let (R, m) be a regular local ring and $x \in m \setminus \{0\}$. Then $R/(x)$ is regular if and only if $x \notin m^2$.*

Proof. Let $d = \dim R = \operatorname{edim} R$ and suppose $x \in m \setminus m^2$. Then, as R is a domain and $x \neq 0$, $\dim R/(x) = d - 1$. Also, $\operatorname{edim} R/(x) = d - 1$ by Lemma 7.3. Hence, $R/(x)$ is regular. Now suppose $x \in m^2$ and $x \neq 0$. Then $\operatorname{edim} R/(x) = \operatorname{edim} R = \dim R$ by Lemma 7.3. However, as R is a domain and $x \neq 0$, $\dim R/(x) = d - 1$. Thus, $R/(x)$ is not regular. □

Example 7.9. Let $R = k[x, y]/(x^2, xy)$ localized at (x, y). Let $m = (x, y)R$. Then $x \in m \setminus m^2$ and $R/(x) \cong k[y]_{(y)}$ is a RLR. However, R is not a RLR.

Corollary 7.10. *Let R be an RLR and $x_1, \ldots, x_d$ a regular system of parameters. Then $(x_1, \ldots, x_i)$ is a prime ideal of height i for each $i = 1, \ldots, d$.*

Proof. We use induction on i to show $(x_1, \ldots, x_i)$ is a prime ideal. As in the proof of the previous proposition, we have $\operatorname{edim} R/(x_1) \leqslant d - 1 \leqslant \dim R/(x_1)$, so $R/(x_1)$ is a RLR, hence a domain. Thus, (x_1) is a prime ideal. Suppose $i > 1$. Then $R/(x_1)$ is a RLR and $\overline{x_2}, \ldots, \overline{x_d}$ a regular system of parameters for $R/(x_1)$. Hence, $(\overline{x_2}, \ldots, \overline{x_i})$ is a prime ideal of $R/(x_1)$ (by the induction hypothesis on i). Lifting to R, we see that $(x_1, \ldots, x_i)$ is a prime ideal of R. The height of $(x_1, \ldots, x_i)$ is at most i by KPIT. On the other hand, the chain of primes

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_i)$$

shows that the height of $(x_1, \ldots, x_i)$ is at least i. □

Proposition 7.11. *Regular local rings are Gorenstein.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_d$ be a regular system of parameters. As R is a domain, x_1 is a regular element on R. As (x_1) is a prime ideal by the previous corollary, we have x_2 is regular on $R/(x_1)$. Continuing in this fashion, we conclude that $\mathbf{x}$ is a regular sequence on R. Thus, $\operatorname{depth} R = d$ and R is CM. Since $\mathbf{x} = x_1, \ldots, x_d$ is a regular sequence, we have $r(R) = r(R/(\mathbf{x}))$ by Proposition 6.14. Since $m = (\mathbf{x})$, $\operatorname{Soc}_R R/(\mathbf{x}) = R/m$, which is a one-dimensional R/m-vector space. Thus, $r(R) = 1$ and R is Gorenstein by Proposition 6.14. □

We now arrive at a key question that proved perplexing for ring theorists in the 1950s:

Localization Question: Suppose R is a RLR and $p \in \operatorname{Spec} R$. Must R_p be a RLR?

The solution to this question resulted in the "homological revolution" in commutative algebra. We first need several results on modules of finite projective dimension.

Definition 7.12. Let R be a ring and M an R-module. Then the *projective dimension* of M, denoted $\operatorname{pd}_R M$, is defined to be the supremum of the lengths of all projective resolutions of M. (Recall that the length of a resolution $P_\bullet$ is $\sup\{n \mid P_n \neq 0\}$.)

Lemma 7.13. *Let R be a ring N an R-module and $n \geqslant 0$ an integer. Suppose $\operatorname{Ext}^i_R(R/I, N) = 0$ for all ideals I of R and $i > n$. Then $\operatorname{id}_R N \leqslant n$.*

Proof. We use induction on n. If $n = 0$, the result follows from Remark 1.1(b). Suppose $n > 0$ and the result holds for all integers less than n. If $\operatorname{id}_R N = 0$ there is nothing to prove. Suppose $\operatorname{id}_R N > 0$. Let $E = E_R(N)$ and $C = E/N$. Consider the exact sequence

$$0 \to N \to E \to C \to 0.$$

Let I be an ideal and apply $\operatorname{Hom}_R(R/I, -)$ to the above sequence:

$$\cdots \to \operatorname{Ext}^{i-1}_R(R/I, E) \to \operatorname{Ext}^{i-1}_R(R/I, C) \to \operatorname{Ext}^i_R(R/I, N) \to \operatorname{Ext}^i_R(R/I, E) \to \cdots .$$

We have $\operatorname{Ext}^i_R(R/I, N) = 0$ for all $i > n$, where $n > 0$. As E is injective, $\operatorname{Ext}^{i-1}_R(R/I, E) = 0$ for all $i - 1 > 0$, so certainly for all $i - 1 > n - 1 \geqslant 0$. Thus, $\operatorname{Ext}^{i-1}_R(R/I, C) \cong \operatorname{Ext}^i_R(R/I, N) = 0$ for all $i - 1 > n - 1$. By the induction hypothesis, we obtain $\operatorname{id}_R C \leqslant n - 1$. Hence, $\operatorname{id}_R N \leqslant n$. □

Lemma 7.14. *Let R be a ring M an R-module and $n \geqslant 0$ an integer. Suppose $\operatorname{Ext}^i_R(M, N) = 0$ for all R-modules N and $i > n$. Then $\operatorname{pd}_R M \leqslant n$.*

Proof. The proof is similar to the proof of Lemma 7.13, where here we use that M is projective if and only if $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > 0$ and R-modules N. □

Lemma 7.15. *Let R be a ring and $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then*

(a) If any two of L, M and N have finite projective dimension, so does the third.

(b) If $\operatorname{pd}_R N > \operatorname{pd}_R M$, then $\operatorname{pd}_R L = \operatorname{pd}_R N - 1$.

Proof. Part (a) follows readily from Lemma 7.14 and the long exact sequence on $\operatorname{Ext}^*_R(-, B)$ for an arbitrary R-module B. For part (b), let $\ell = \operatorname{pd}_R L$, $m = \operatorname{pd}_R M$, and $n = \operatorname{pd}_R N$. As we are assuming, $n > m$, we must have $m < \infty$. If $n = \infty$ then $\ell = \infty$ by part (a), and so (b) holds. Assume $n < \infty$. Then $\operatorname{Ext}^{i+1}_R(N, B) = \operatorname{Ext}^i_R(M, B) = 0$ for all $i > n - 1$ and all R-modules B (here we are using $n - 1 \geqslant m$). From the l.e.s. on $\operatorname{Ext}^*_R(-, B)$, we obtain that $\operatorname{Ext}^i_R(L, B) = 0$ for all $i > n - 1$ and all B, so $\operatorname{pd}_R L \leqslant n - 1$. Since $\operatorname{pd}_R N = n$, there exists an R-module C such that $\operatorname{Ext}^n_R(N, C) \neq 0$. Since $\operatorname{Ext}^n_R(M, C) = 0$, we have from the l.e.s.

$$\operatorname{Ext}^{n-1}_R(L, C) \to \operatorname{Ext}^n_R(N, C) \to 0$$

is exact. Hence, $\operatorname{Ext}^{n-1}_R(L, C) \neq 0$ and $\operatorname{pd}_R L = n - 1$.

□

An important consequence of this lemma is the following:

Proposition 7.16. *Let R be a ring, M an R-module and $n \geqslant 0$ an integer. The following conditions are equivalent:*

(a) $\operatorname{pd}_R M \leqslant n$;

(b) Given any exact sequence

$$P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$$

such that P_i is projective for each i, we have $\ker \partial_{n-1}$ is projective.

Proof. (*b*) $\Longrightarrow$ (*a*): By the existence of projective resolutions one can construct an exact sequence as in (b) for any $n \geqslant 0$. Let $P_n = \ker \partial_{n-1}$. Then

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$$

is a projective resolution of M of length at most n.

(*a*) $\Longrightarrow$ (*b*): Let $K_i = \ker \partial_{i-1}$ for $i = 1, \ldots, n$ and set $K_0 = M$. We wish to show that K_n is projective. Observe we have short exact sequences

$$0 \to K_i \to P_{i-1} \to K_{i-1} \to 0$$

for $i = 1, \ldots, n$. Suppose K_{i-1} is projective for some $1 \leqslant i \leqslant n$. Then the map $P_{i-1} \to K_{i-1}$ splits and $P_{i-1} \cong K_i \oplus K_{i-1}$. Hence, K_i is projective. Repeating this argument, we obtain that K_j is projective for all $j \geqslant i-1$, and in particular K_n is projective. Assume now that K_{i-1} is not projective for all $1 \leqslant i \leqslant n$. Then $\operatorname{pd}_R K_{i-1} > \operatorname{pd}_R P_i = 0$ for all $1 \leqslant i \leqslant n$. By part (b) of Lemma 7.15, we have $\operatorname{pd}_R K_i = \operatorname{pd}_R K_{i-1} - 1$ for $i = 1, \ldots, n$. From this we obtain that $\operatorname{pd}_R K_i = \operatorname{pd}_R K_0 - i$ for all $i = 1, \ldots, n$. In particular, $\operatorname{pd}_R K_n = \operatorname{pd}_R K_0 - n \leqslant 0$, and so K_n is projective.

□

Theorem 7.17. *Let R be a ring and $n \geqslant 0$ an integer. The following conditions are equivalent:*

(a) $\operatorname{pd}_R R/I \leqslant n$ *for every ideal I of R;*

(b) $\operatorname{pd}_R M \leqslant n$ *for every R-module M;*

(c) $\operatorname{id}_R N \leqslant n$ *for all R-modules N;*

(d) $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i > n$ and all R-modules M and N.*

Proof. (*a*) $\Longrightarrow$ (*c*): Let N be an R-module. By assumption (a), $\operatorname{pd}_R R/I \leqslant n$ for all ideals I of R. Thus, $\operatorname{Ext}^i_R(R/I, N) = 0$ for all $i > n$. Hence, by Lemma 7.13, $\operatorname{id}_R N \leqslant n$.

(*c*) $\Longrightarrow$ (*d*): Trivial.

(*d*) $\Longrightarrow$ (*b*): This follows from Lemma 7.14.

(*b*) $\Longrightarrow$ (*a*): Trivial.

□

Definition 7.18. Let R be a ring. Then the *global dimension* of R, denoted $\operatorname{gl-dim} R$, is defined to be the least integer n (if it exists) such that R satisfies any of the equivalent conditions for n in Theorem 7.17. If such an n does exist, R is said to have *finite global dimension.*

Theorem 7.19. *Let (R, m) be a local ring. The following conditions are equivalent for an integer n:*

(a) R has global dimension n;

(b) $\operatorname{pd}_R k = n$;

(c) $\operatorname{id}_R k = n$.

Proof. $(a) \implies (b)$: By (a), we have $\operatorname{pd}_R k \leqslant n$. Suppose $\operatorname{pd}_R k < n$. Then $\operatorname{Tor}_i^R(k, R/I) = 0$ for all $i > n-1$ and all ideals I, and hence $\operatorname{pd}_R R/I \leqslant n-1$. This implies $\operatorname{gl-dim} R \leqslant n-1$, a contradiction.

$(b) \implies (c)$: Since $\operatorname{pd}_R k = n$, $\operatorname{Ext}_R^i(k,k) = 0$ for all $i > n$ and $\operatorname{Ext}_R^n(k,k) \neq 0$ by Proposition 7.22. Then $\operatorname{id}_R k = n$ by Corollary 2.24.

$(c) \implies (a)$: First, since $\operatorname{id}_R k = n$, we have $\operatorname{Ext}_R^n(k,k) \neq 0$ by Corollary 2.24. Thus, $\operatorname{gl-dim} R \geqslant n$. On the other hand, for all ideals I of R we have $\operatorname{Ext}_R^i(R/I, k) = 0$ for all $i > n$, which implies $\operatorname{pd}_R R/I \leqslant n$ by Proposition 7.22. Thus, $\operatorname{gl-dim} R \leqslant n$ by Theorem 7.17. $\square$

For local rings, we have the concept of a *minimal* projective resolutions for finitely generated modules:

Definition 7.20. Let (R, m) be a local ring and M a finitely generated R-module. A projective resolution $P_\bullet$ of M is said to be *minimal* if $\partial_i(P_i) \subseteq mP_{i-1}$ for all $i \geqslant 1$.

Remark 7.21. Let (R, m) be a local ring. By Grifo's 915 notes, we know that every finitely generated module has a minimal projective (in fact, free) resolution (Lemma 5.9) and that the length of any minimal projective resolution is the projective dimension of the module (Theorem 5.18).

Proposition 7.22. *Let (R, m, k) be a local ring and M a finitely generated R-module. Then*

$$\begin{aligned}\operatorname{pd}_R M &= \sup\{n \mid \operatorname{Tor}_n^R(k, M) \neq 0\}\\ &= \sup\{n \mid \operatorname{Ext}_R^n(M, k) \neq 0\}.\end{aligned}$$

Proof. Let $F_\bullet$ be a minimal free resolution of M. Then $k \otimes_R F_\bullet$ is a complex with zero differentials. Hence, for all i

$$\operatorname{Tor}_i^R(k, M) = \operatorname{H}_i(k \otimes_R F_\bullet) = k \otimes_R F_i \cong k^{\operatorname{rank} F_i}.$$

Hence, $\sup\{n \mid \operatorname{Tor}_n^R(k, M) \neq 0\} = \sup\{n \mid F_n \neq 0\}$, which is $\operatorname{pd}_R M$. Similarly, $\operatorname{Hom}_R(F_\bullet, k)$ is a complex with zero differentials, so

$$\operatorname{Ext}_R^i(M, k) = \operatorname{H}^i(\operatorname{Hom}_R(F_\bullet, k)) = \operatorname{Hom}_R(F_i, k) \cong k^{\operatorname{rank} F_i}.$$

$\square$

Proposition 7.23. *Let R be a ring, M and N finitely generated R-modules, and $x \in R$ such that $xN = 0$ and x is R- and M-regular. Then for all $i \geqslant 0$,*

$$\operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_{R/(x)}^i(M/xM, N).$$

Proof. We first claim that $\operatorname{Tor}_i^R(M, R/(x)) = 0$ for all $i \geqslant 1$. Note that $0 \to R \xrightarrow{x} R \to 0$ is a free resolution of $R/(x)$. Hence, $\operatorname{Tor}_i^R(M, R/(x)) = 0$ for all $i \geqslant 2$. To compute $\operatorname{Tor}_1^R(M, R/(x))$, apply $M \otimes_R -$ to the resolution of $R/(x)$ above:

$$0 \to M \xrightarrow{x} M \to 0.$$

Thus, $\operatorname{Tor}_1^R(M, R/(x)) \cong (0 :_M x) = 0$ since x is M-regular.

Now, let $F_\bullet$ be a free resolution of M. Then $\mathrm{H}_i(F_\bullet \otimes_R R/(x)) \cong \operatorname{Tor}_i^R(M, R/(x)) = 0$ for $i \geqslant 1$ by the claim above. Also, $\mathrm{H}_0(F_\bullet \otimes_R R/(x)) \cong M \otimes_R R/(x) \cong M/xM$. Since $F_i \otimes_R R/(x)$ is a free $R/(x)$-module for all i, we obtain that $F_\bullet \otimes_R R/(x)$ is a free $R/(x)$-resolution of M/xM. Then using Hom-tensor adjointness along with $xN = 0$, we have

$$\begin{aligned}
\operatorname{Ext}_R^i(M, N) &\cong \mathrm{H}^i(\operatorname{Hom}_R(F_\bullet, N)) \\
&\cong \mathrm{H}^i(\operatorname{Hom}_R(F_\bullet, \operatorname{Hom}_{R/(x)}(R/(x), N)) \\
&\cong \mathrm{H}^i(\operatorname{Hom}_{R/(x)}(F_\bullet \otimes_R R/(x), N) \\
&\cong \operatorname{Ext}_{R/(x)}^i(M/xM, N).
\end{aligned}$$

□

Proposition 7.24. *Let (R, m) be a local ring and M a nonzero finitely generated R-module. Let $x \in m$ be R-regular and M-regular. Then*

$$\operatorname{pd}_{R/(x)} M/xM = \operatorname{pd}_R M.$$

Proof. Using Propositions 7.22 and 7.23, we have

$$\begin{aligned}
\operatorname{pd}_R M &= \sup\{n \mid \operatorname{Ext}_R^n(M, k) \neq 0\} \\
&= \sup\{n \mid \operatorname{Ext}_{R/(x)}^n(M/xM, k) \neq 0\} \\
&= \operatorname{pd}_{R/(x)} M/xM.
\end{aligned}$$

□

Theorem 7.25. (Auslander-Buchsbaum formula) *Let (R, m) be a local ring and M a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R.$$

Proof. We proceed by induction on $\operatorname{depth} R$. Suppose $\operatorname{depth} R = 0$. Suppose $n = \operatorname{pd}_R M > 0$ and let $F_\bullet$ be a minimal free resolution of M. Consider the tail end of the resolution:

$$0 \to F_n \xrightarrow{\partial} F_{n-1} \to \cdots .$$

Since $\operatorname{depth} R = 0$, $\operatorname{Soc}_R R = (0 :_R m) \neq 0$ and hence $\operatorname{Soc}_R F_n \neq 0$. Since $\partial(F_n) \subseteq mF_{n-1}$ one easily checks that $\partial(\operatorname{Soc}_R F_n) \subseteq m(\operatorname{Soc}_R F_{n-1}) = 0$, contradicting the injectivity of ∂.

Suppose $\operatorname{depth} R > 0$. If $\operatorname{pd}_R M = 0$ then M is free and $\operatorname{depth} M = \operatorname{depth} R$. Therefore, the formula holds. Assume by way of (double) induction that $\operatorname{pd}_R M > 0$. If $\operatorname{depth} M > 0$ one can choose (by prime avoidance) $x \in R$ which is both R-regular and M-regular. Then $\operatorname{depth} M/xM = \operatorname{depth} M - 1$, $\operatorname{depth} R/(x) = \operatorname{depth} R - 1$ and $\operatorname{pd}_{R/(x)} M/xM = \operatorname{pd}_R M$ by Proposition 7.24. Thus, using the induction hypothesis, we have

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{pd}_{R/(x)} M/xM + \operatorname{depth} M/xM + 1 = \operatorname{depth} R/(x) + 1 = \operatorname{depth} R,$$

which is what we wanted to show.

Now assume $\operatorname{depth} M = 0$ (but still in the case $\operatorname{depth} R > 0$ and $\operatorname{pd}_R M > 0$). Consider the short exact sequence $0 \to K \to F \to M \to 0$, where F is a finitely generated free R-module. Then $\operatorname{pd}_R K = \operatorname{pd}_R M - 1$ by Lemma 7.15. Let $x \in R$ be an R-regular element.

Then x is F-regular and hence also K-regular as $K \subseteq F$. Thus, $\operatorname{pd}_R K = \operatorname{pd}_{R/(x)} K/xK$, $\operatorname{depth} R/(x) = \operatorname{depth} R - 1$, and $\operatorname{depth} K/xK = \operatorname{depth} K - 1$. Applying $\operatorname{Hom}_R(R/m, -)$ to the s.e.s. above and using that $\operatorname{Hom}_R(k, F) = 0$, we have

$$0 \to \operatorname{Hom}_R(k, M) \to \operatorname{Ext}^1_R(k, K)$$

is exact. Since $\operatorname{Hom}_R(k, M) \neq 0$ (as $\operatorname{depth} M = 0$) we have $\operatorname{Ext}^1_R(k, M) \neq 0$, so $\operatorname{depth} K = 1$. Thus, since $\operatorname{depth} K > 0$, we have

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{pd}_R M = \operatorname{pd}_R K + 1 = \operatorname{pd}_R K + \operatorname{depth} K = \operatorname{depth} R.$$

□

Theorem 7.26. *(Auslander-Buchsbaum-Serre Theorem) Let (R, m, k) be a local ring of dimension d. The following conditions are equivalent:*

(a) R is a regular local ring;

(b) $\operatorname{pd}_R k < \infty$;

(c) $\text{gl-dim}\, R = d$.

Proof. $(a) \implies (c)$: Let $\mathbf{x} = x_1, \ldots, x_d$ be a regular system of parameters. Since R is Gorenstein (and hence CM), $x_1, \ldots, x_d$ is an R-sequence. Thus, the Koszul complex $K(\mathbf{x})$ is a minimal free resolution of $R/(\mathbf{x}) = R/m$ by Corollary 4.12, which implies $\operatorname{pd}_R k = d$. By Theorem 7.19, $\text{gl-dim}\, R = d$.

$(c) \implies (b)$: Immediate from the definition of global dimension.

$(b) \implies (a)$: We proceed by induction on d. If $d = 0$ then $\operatorname{depth} R = 0$. Thus, by the Auslander-Buchsbaum formula, $\operatorname{pd} R/m = 0$, i.e., R/m is a free R-module. But since $m \cdot R/m = 0$ and the annihilator of any nonzero free module is 0, we must have $m = 0$. Thus, R is a field, which is a regular local ring.

Suppose $d > 0$. Note we must have $\operatorname{depth} R > 0$. Otherwise, by the argument in the preceding paragraph, $\operatorname{pd}_R R/m = 0$ and R is a field, contradicting that $d > 0$. Thus, $m \notin \operatorname{Ass}_R R$. By prime avoidance, we can choose $x \in m \setminus m^2$ such that x is not in any associated prime of R. Hence, x is a minimal generator for m and a regular element on R. Therefore, $\dim R/(x) = d - 1$ and $\operatorname{edim} R = \operatorname{edim} R - 1$. If we show that $\operatorname{pd}_{R/(x)} R/m < \infty$, then $R/(x)$ is regular and thus R is regular. From the exact sequence $0 \to m/(x) \to R/(x) \to R/m \to 0$, it suffices to show that $\operatorname{pd}_{R/(x)} m/(x) < \infty$. Since x is regular on both R and m, we have by Proposition 7.24 that $\operatorname{pd}_{R/(x)} m/xm = \operatorname{pd}_R m = \operatorname{pd}_R R/m - 1 < \infty$ (see Lemma 7.15 for the last equality). Since $(x) \supset xm$, we have a natural surjection $f : m/xm \to m/(x)$. We claim this map splits. Let $x_1, \ldots, x_s$ be a minimal generating set for m, where $x_1 = x$. Then $m/(x)$ is generated over $R/(x)$ by $x_2, \ldots, x_s$. Define $g : m/(x) \to m/xm$ by $g(\sum_{i=2}^s r_i x_i + (x)) = \sum_{i=2}^s r_i x_i + xm$ (using coset notation). To show g is well-defined, suppose $\sum_{i=2}^s r_i x_i + (x) = 0 + (x)$. Then $\sum_{i=2}^s r_i x_i = r_1 x$. Since $x_1, \ldots, x_s$ is a minimal generating set for m, we must have $r_1 \in m$. Hence, $\sum_{i=2}^s r_i x_i \in xm$, so $g(\sum_{i=2}^s r_i x_i + (x)) = 0 + xm$. Thus, g is well-defined. It is easily seen that $fg = \operatorname{id}_{m/(x)}$. As f splits, $m/xm \cong m/(x) \oplus T$ for some $R/(x)$-module T. Since $\operatorname{Tor}_i^{R/(x)}(k, m/xm) \cong \operatorname{Tor}_i^{R/(x)}(k, m/(x)) \oplus \operatorname{Tor}_i^{R/(x)}(k, T)$ for all i, we see that $\operatorname{pd}_{R/(x)} m/(x) \leqslant \operatorname{pd}_{R/(x)} m/xm < \infty$ by Proposition 7.22. This proves the claim and finishes the proof.

□

We can now finally answer the localization question:

Corollary 7.27. *Let (R, m) be a regular local ring and $p \in \operatorname{Spec} R$. Then R_p is a RLR.*

Proof. Let $d = \dim R$ and $p \in \operatorname{Spec} R$. By Theorem 7.26, gl-dim $R = d$ and so $\operatorname{pd}_R R/p \leqslant d$. As localization is exact, if $P_\bullet$ is a projective resolution of R/p, then $(P_\bullet)_p$ is a projective R_p-resolution of $(R/p)_p = k(p)$. Thus, $\operatorname{pd}_{R_p} k(p) \leqslant d < \infty$ and R_p is a RLR by Theorem 7.26. □

Definition 7.28. A Noetherian ring R is said to be *regular* if R_m is a regular local ring for all maximal ideals m of R (equivalently, for all prime ideals p of R.

Example 7.29. The following are examples of regular rings:

- $\mathbb{Z}$
- $\mathbb{Q} \times \mathbb{Q}$ (exercise). Hence, regular rings are not necessarily domains.
- Any ring of algebraic integers; i.e., the integral closure of $\mathbb{Z}$ in a finite field extension of $\mathbb{Q}$ (proof later).
- $k[x_1, \ldots, x_n]$ and $\mathbb{Z}[x_1, \ldots, x_n]$, where k is a field and $x_1, \ldots, x_n$ are indeterminates (see below).
- There exist regular rings of infinite Krull dimension (Nagata).

Proposition 7.30. *Let R be a regular ring and x an indeterminate. Then $R[x]$ is regular.*

Proof. Homework exercise. □

8 Serre's conditions and normal rings

Definition 8.1. Let R be a Noetherian ring and $n \geqslant 0$ an integer. The ring R is said to satisfy S_n if $\operatorname{depth} R_p \geqslant \min\{n, \dim R_p\}$ for all $p \in \operatorname{Spec} R$. R is said to satisfy R_n if R_p is a regular local ring for all primes p with $\dim R_p \leqslant n$. The conditions S_n and R_n are called *Serre's conditions.*

Remark 8.2. Let R be a Noetherian ring.

(a) R satisfies S_n (respectively, R_n) for all n if and only R is Cohen-Macaulay (respectively, regular).

(b) R satisfies S_1 if and only if $\operatorname{Ass}_R R = \operatorname{Min}_R R$.

(c) R satisfies R_0 if and only if R_p is a field for all $p \in \operatorname{Min} R$.

Proposition 8.3. *A Noetherian ring R is reduced if and only if R satisfies S_1 and R_0.*

Proof. Suppose R is reduced. Then (0) is the intersection of all primes of R, and hence the intersection of all the minimal primes: $p_1 \cap \cdots \cap p_n = 0$ where $\operatorname{Min}_R R = \{p_1, \ldots, p_n\}$. This is an irredundant primary decomposition for 0 (or more properly, $R = R/(0)$). Thus, $\operatorname{Min}_R R = \operatorname{Ass}_R R$ and R satisfies S_1. Since $p_i R_{p_j} = R_{p_i}$ for all $i \neq j$ and using that localization commutes with finite intersections, we obtain

$$\begin{aligned} p_i R_{p_i} &= p_1 R_{p_i} \cap \cdots \cap p_i R_{p_i} \cap \cdots \cap p_n R_{p_i} \\ &= (p_1 \cap \cdots \cap p_n) R_{p_i} \\ &= (0) R_{p_i}. \end{aligned}$$

Thus, R_{p_i} is a field and R satisfies R_0.

Conversely, suppose R satisfies S_1 and R_0. Thus, $\operatorname{Min}_R R = \operatorname{Ass}_R R$ and R_p is a field for all $p \in \operatorname{Min}_R R$. Suppose $r \in R$ is nilpotent, say $r^n = 0$. Let $p \in \operatorname{Min}_R R$. As R_p is a field, $\frac{r}{1} = 0$ in R_p. Thus, there exists $s \in R \setminus p$ such that $sr = 0$. Hence, $(0 :_R r) \not\subset p$ for all $p \in \operatorname{Min}_R R = \operatorname{Ass}_R R$. By prime avoidance, this implies there exists $z \in (0 :_R r)$ which is a non-zero-divisor on R. Since $zr = 0$ we conclude $r = 0$. Hence, R is reduced.

□

Definition 8.4. Let R be a ring and $W \subset R$ the set of all non-zero-divisors of R. The ring R_W is called the *total quotient ring* or *total ring of fractions* of R and is denoted $\operatorname{TQ}(R)$.

The total quotient ring is a natural generalization of the ring of fractions of a domain. Note that the map $R \to R_W$ is injective, so one can consider R as a subring of $\operatorname{TQ}(R)$. Note that if R is Noetherian, $W = R \setminus \bigcup_{p \in \operatorname{Ass}_R R} p$.

Remark 8.5. If $R = R_1 \times \cdots \times R_n$ then $\operatorname{TQ}(R) = \operatorname{TQ}(R_1) \times \cdots \times \operatorname{TQ}(R_n)$.

Proof. Exercise. □

Let $R \subseteq S$ be rings. Recall that the set of elements of S which are integral over R forms a subring of S, called the *integral closure* of R in S. It is straightforward to show that if T is the integral closure of R in S, and W is any multiplicatively closed subset of R, then T_W is the integral closure of R_W in S_W.

Recall that a domain R is called *normal* if it is integrally closed in its field of fractions (cf. Grifo 905 notes, Definition 1.28). By the preceeding paragraph, if R is a normal domain, so is R_W for any multiplicatively closed set W of R. The following definition generalizes this notion to (Noetherian) reduced rings:

Definition 8.6. A Noetherian ring R is called *normal* if R is reduced and integrally closed in $\operatorname{TQ}(R)$.

Theorem 8.7. *Let R be a Noetherian ring. The following are equivalent:*

(a) R is normal.

(b) R is isomorphic to a direct product of finitely many normal domains.

(c) R_p is a normal domain for all $p \in \operatorname{Spec} R$.

(d) R_m is normal for all maximal ideals m of R.

Moreover, if any of these conditions hold then R/p is normal for all $p \in \operatorname{Min}_R R$ and

$$R \cong \prod_{p \in \operatorname{Min} R} R/p.$$

Proof. (*c*) $\implies$ (*d*) is *a fortiori*.

(*d*) $\implies$ (*a*): Since R_m is reduced for all maximal ideals m, we know that R_m satisfies S_1 and R_0 for all m. But since these conditions are defined locally at every prime, and as every prime is contained in a maximal ideal, we have that R satisfies S_1 and R_0. Hence, R is reduced. Now let $\frac{r}{w} \in \operatorname{TQ}(R)$ be integral over R and let m be a maximal ideal of R. Since $\frac{w}{1} \in R_m$ is a non-zero-divisor on R_m, we have that $\frac{r}{w} = \frac{r}{1}/\frac{w}{1} \in \mathrm{TQ}(R_m)$. Further, $\frac{r}{w}$ and is integral over R_m using the same equation demonstrating the integrality of $\frac{r}{w}$ over R. Since R_m is normal, we have $\frac{r}{w} = \frac{a}{s}$ for some $a \in R$ and $s \in R \setminus m$. Then there exists $t \in R \setminus m$ such that $trs = twa$. Hence, $ts \in ((w) :_R r)$. As $ts \notin m$, $((w) :_R r) \not\subset m$. Since m was arbitrary, this implies $((w) :_R r) = R$. Hence, $r = bw$ and $\frac{r}{w} = \frac{b}{1}$ for some $b \in R$. Thus, R is integrally closed in $\mathrm{TQ}(R)$.

$(a) \implies (b)$: Since R is reduced, we have $\mathrm{Ass}_R R = \mathrm{Min}_R R$ by Proposition 8.3. Let $\mathrm{Min}_R R = \{p_1, \dots, p_n\}$. Then $\mathrm{TQ}(R) = R_W$ where $W = R \setminus \bigcup_{i=1}^{n} p_i$. Note that $\operatorname{Spec} R_W = \{(p_1)_W, \dots, (p_n)_W\}$ and each $(p_i)_W$ is both minimal and maximal. Also, as R is reduced, $(p_1)_W \cap \dots \cap (p_n)_W = (p_1 \cap \dots \cap p_n)_W = 0$. Finally, note that as $(p_i)_W$ is maximal in R_W, we have $R_W/(p_i)_W \cong (R/p_i)_W \cong \mathrm{TQ}(R/p_i)$ for all i. By the Chinese Remainder Theorem, we have

$$\begin{aligned} R_W &\cong (R/(p_1 \cap \dots \cap p_n))_W \\ &\cong R_W/((p_1)_W \cap \dots \cap (p_n)_W) \\ &\cong R_W/(p_1)_W \cap \dots \cap R_W/(p_n)_W \\ &\cong \mathrm{TQ}(R/p_1) \times \dots \cap \mathrm{TQ}(R/p_n). \end{aligned}$$

The image of R in $\mathrm{TQ}(R) = R_W \cong \mathrm{TQ}(R/p_1) \times \dots \times \mathrm{TQ}(R/p_n)$ under this isomorphism is $R(\overline{1}, \dots, \overline{1}) = \{(\overline{r}, \dots, \overline{r}) \mid r \in R\}$. Let $e_i = (\overline{0}, \dots, \overline{1}, \dots, \overline{0})$ where the $\overline{1}$ sits in the ith component. Since $e_i^2 - e_i = 0$, we see that $e_i \in \mathrm{TQ}(R)$ is integral over $R(\overline{1}, \dots, \overline{1})$ for each i. As $R \cong R(\overline{1}, \dots, \overline{1})$ is integrally closed in R_W, we conclude that each $e_i \in R = R(\overline{1}, \dots, \overline{1})$. Thus,

$$\begin{aligned} R &\cong R(\overline{1}, \dots, \overline{1}) \\ &= Re_1 \times \dots \times Re_n \\ &= R/p_1 \times \dots \times R/p_n. \end{aligned}$$

Since R is integrally closed in R_W, each R/p_i is integrally closed in $\mathrm{TQ}(R/p_i)$. Thus, R is isomorphic to a direct product of finitely many normal domains. (We've also proved that (a) implies the final statement.)

$(b) \implies (c)$: Suppose $R \cong R_1 \times \dots \times R_n$ where each R_i is a normal domain. It is clear that R is reduced. For each i, let $p_i = \{(r_1, \dots, r_n) \in R \mid r_i = 0\}$. Then $\mathrm{Min}_R R = \{p_1, \dots, p_n\}$ and $R/p_i \cong R_i$, which is normal for each i.

□

Example 8.8. Let $R = k[x,y]/(xy)$. Note that R is reduced and $\operatorname{Min} R = \{xR, yR\}$. Also observe that $R/xR \cong k[y]$ and $R/yR \cong k[x]$ are both normal domains. However, R is not normal since $R_{(x,y)}$ is not a domain.

Proposition 8.9. *Let (R, m) be a one-dimensional local domain. The following are conditions are equivalent:*

(a) R is normal.

(b) $m = (x)$ for some $x \in m$; i.e., R is a RLR.

(c) There exists an $x \in m$ such that every ideal is equal to (x^n) for some $n \geqslant 0$.

Proof. $(a) \implies (b)$: Let $x \in m \setminus m^2$. We claim $m = (x)$. Since R is one-dimensional and $x \neq 0$, we must have $\dim R/(x) = 0$. Hence, $m \in \mathrm{Ass}_R R/(x)$, so $m = ((x) : y)$ for some

$y \in R$. If $y \notin m$ then y is a unit and $m = (x)$. So we may assume $y \in m$. Hence, $ym \subseteq xm$. Thus, in $\mathrm{TQ}(R)$, $\frac{y}{x}m \subseteq m$. As m is a finitely generated R-submodule of $\mathrm{TQ}(R)$, we see (by the determinant trick) that $\frac{y}{x}$ is integral over R. As R is normal, this implies $\frac{y}{x} \in R$, i.e., $y \in (x)$. But then $m = ((x) : y) = R$, a contradiction.

$(b) \implies (c)$: Let I be a nonzero ideal of R. Since R is local and Noetherian, we have $\cap_n m^n = 0$ by Krull's Intersection Theorem. Thus, there exists an n such that $I \subseteq m^n$ but $I \not\subseteq m^{n+1}$. Since $m = (x)$, then $I \subseteq m^n = (x^n)$. Choose $y \in I \notin m^{n+1} = (x^{n+1})$. Then $y = rx^n$. If $r \in m = (x)$, then $y \in m^{n+1}$, a contradiction. Hence, $r \notin m$, so r is a unit. Thus, $x^n = r^{-1}y \in I$. Hence, $I = (x^n)$.

$(c) \implies (b)$: Condition (c) implies that R is a PID, and PIDs are integrally closed in their fraction fields.

□

Definition 8.10. A domain R is called a *Dedekind domain* if R is Noetherian, one-dimensional, and normal.

Examples 8.11. The following are examples of Dedekind domains:

- Any PID which is not a field
- Any ring of algebraic integers; e.g., $\mathbb{Z}[\sqrt{d}]$ if $d \equiv 3 \pmod 4$ or $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1 \pmod 4$. In particular, $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain but not a PID. (See Dummit-Foote.)

Remark 8.12. Any Dedekind domain is a regular ring

Proof. Let R be a Dedekind domain and p a prime ideal of R. If $\operatorname{ht} p = 0$ then $p = 0$ and R_p is a field, hence regular. If $\operatorname{ht} p = 1$ then R_p is one-dimensional, Noetherian and normal, so R_p is a RLR by Proposition 8.9. □

Theorem 8.13. *Let R be a Noetherian ring. Then R is normal if and only if R satisfies S_2 and R_1.*

Proof. Suppose R is normal. Then R is reduced and so satisfies S_1 and R_0. Let $p \in \operatorname{Spec} R$ be a height one prime. Then R_p is a normal, Noetherian, one-dimensional domain, so R_p is a RLR by Proposition 8.9. Thus R satisfies R_1. To prove R satisfies S_2, it suffices to prove it locally at every prime. Let $p \in \operatorname{Spec} R$. If $\operatorname{ht} p \leqslant 1$ then R_p is a RLR (which is CM) and so R_p satisfies S_n for all n. Suppose $\operatorname{ht} p \geqslant 2$. Then R_p is a normal domain of dimension at least two. Reset notation by replacing R_p with R and pR_p with m. We wish to show that $\operatorname{depth} R \geqslant 2$. Suppose not. Then, as R is a domain, we must have $\operatorname{depth} R = 1$. Let $u \in m$ be a nonzero element. Then $\operatorname{depth} R/(u) = \operatorname{depth} R - 1 = 0$. Thus $m \in \operatorname{Ass}_R R/(u)$. Hence, $m = ((u) :_R y)$ for some $y \in R$. Consequently, $ym \subseteq (u)$ and so $ym = u(ym :_R u)$. If $(ym :_R u) = R$ then $u = yt$ for some $t \in m$. Then $m = ((yt) :_R y) = (t)$, contradicting that $\operatorname{ht} m \geqslant 2$ (by KPIT). Thus, $((u) :_R y) \subseteq m$. This implies $ym \subseteq um$. In $\mathrm{TQ}(R)$, we have $\frac{y}{u}m \subseteq m$. By the determinant trick, we conclude that $\frac{y}{u}$ is integral over R, and so $\frac{y}{u} \in R$ as R is normal. Thus $y = ru$ for some $r \in R$. But then $m = ((u) :_R y) = ((u) :_R ru) = R$, a contradiction. Thus, $\operatorname{depth} R \geqslant 2$.

Conversely, suppose R satisfies S_2 and R_1. Then R is reduced by Proposition 8.3. Let $\frac{r}{w} \in \mathrm{TQ}(R)$ be integral over R. Let p be a prime and $\phi_p : \mathrm{TQ}(R) \to \mathrm{TQ}(R_p)$ the composition

$$\mathrm{TQ}(R) = R_W \to (R_p)_{\frac{W}{1}} \hookrightarrow \mathrm{TQ}(R_p).$$

Then $\phi_p(\frac{r}{w}) = \frac{r/1}{w/1}$ is integral over R_p for any prime p (by applying ϕ_p to the equation of integral dependence for $\frac{r}{w}$.) If p has height one, then R_p is normal and so $\phi_p(\frac{r}{w}) \in R_p$; i.e., $\frac{r}{w} = \frac{a}{s}$ in R_p for some $s \in R \setminus p$. Thus, there exists $s' \in R \setminus p$ such that $s'sr = s'wa$. Thus, $((w) :_R r) \not\subseteq p$ for all height one primes p. We claim that $((w) :_R r) = R$. Suppose $((w) :_R r) \subseteq m$ for some maximal ideal m of R. Then $w \in m$, w is a regular element of R_m, and $(wR_m :_{R_m} r) \neq R_m$. Note that $(wR_m :_{R_m} r)$ consists of zero-divisors on R_m/wR_m (since $r \notin wR_m$), and so $(wR_m :_{R_m} r)$ must be contained in an associated prime pR_m of R_m/wR_m. Then $0 = \operatorname{depth} R_p/wR_p = \operatorname{depth} R_p - 1$, so $\operatorname{depth} R_p = 1$. As R satisfies S_2, we must have $\operatorname{ht} p = 1$. But then $((w) :_R r) \subseteq p$, which is a contradiction. Hence, $((w) :_R r) = R$, which implies $\frac{r}{w} \in R$. Thus, R is integrally closed in $\operatorname{TQ}(R)$ and hence normal. □

Theorem 8.14. (Jacobian criterion for hypersurfaces) *Let k be a perfect field and $f \in k[x_1, \ldots, x_n]$ a nonzero polynomial. Let $J_f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_x}, \ldots, \frac{\partial f}{\partial x_n})$, called the Jacobian ideal of f. Let $R = k[x_1, \ldots, x_n]/(f)$ and $p \in \operatorname{Spec} R$. Then R_p is a RLR if and only if $p \not\supseteq J_f R$.*

Proof. See Matsumura's *Commutative Ring Theory*, Theorem 30.10. □

Example 8.15. Let k be a perfect field and $R = k[x,y,z]/(x^2 + yz)$. Then R is normal.

Proof. First note that $k[x,y,z]$ is CM and x^2+yz is a non-zero-divisor. Thus, $R = k[x,y,z]/(x^2+yz)$ is CM. Hence, R satisfies S_n for all n. Let $f = x^2 - yz$. Then $J_f = (2x, y, z)$. Let p be a prime of $\operatorname{Spec} R$. If $p \supseteq J_f R$ then $p \supseteq (y,z)R$ and since in R, $x^2 = -yz \in p$, we see that p also contains x. (This argument allows for k to have characteristic 2, in which case $2x = 0$.) Thus, by the Jacobian criterion for hypersurfaces, R_p is a RLR unless $p = (x,y,z)R$. Note that $(x,y,z)R$ has height two. Thus, for all primes p of height at most one, R_p is a RLR. Thus, R satisfies R_1 and so R is normal by Theorem 8.13. □

We recall the following characterization of unique factorization domains (UFDs) from Dummit-Foote:

Proposition 8.16. *Let D be a domain. Then D is a UFD if and only if the following conditions are satisfied:*

- *D satisfies the ascending chain condition on principal ideals.*
- *Every irreducible element of D is prime (i.e., generates a prime ideal).*

Of course, Noetherian domains satisfy the first condition automatically. We can restate the second condition also in the Noetherian context:

Proposition 8.17. *Let R be a Noetherian domain. Then R is a UFD if and only if every height one prime is principal.*

Proof. Suppose R is a UFD and let p be a height one prime. As $p \neq 0$, p contains a nonzero (and non-unit) element f. Since f is the product of irreducibles and p is prime, p must contain some irreducible element π. As R is a UFD, (π) is a prime ideal. As R is a domain, $\operatorname{ht}(\pi) \geqslant 1$. But since $(\pi) \subseteq p$ and $\operatorname{ht} p = 1$, we must have $p = (\pi)$.

Conversely, suppose every height one prime of R is principal. Let $\pi \in R$ be an irreducible element and let p be a prime minimal over (π). By assumption, $p = (d)$ for some $d \in R$. Then $\pi = cd$ for some $c \in R$. As π is irrreducible and d is a nonunit, we must have c is a unit. Thus, $p = (\pi)$ and hence π is a prime element. By Proposition 8.16, we see that R is a UFD.

□

Remark 8.18. Let R be a UFD and F it's field of fractions and let $f(x), g(x) \in R[x]$. Suppose the gcd of the coefficient of f is a unit (i.e., f is primitive). If f divides g in $F[x]$ then f divides g in $R[x]$.

Proof. This follows easily from Gauss' Lemma (Dummit-Foote, Proposition 5 of Section 9.3). □

The following two results are true without the Noetherian hypothesis. We prove them in the Noetherian case to illustrate the utility of Proposition 8.17.

Corollary 8.19. *Let R be a Noetherian UFD and $x_1, \ldots, x_n$ indeterminates. Then $R[x_1, \ldots, x_n]$ is a UFD.*

Proof. It suffices to prove the case $n = 1$. Let P be a height one prime of $R[x]$. By Proposition 8.17, it suffices to prove P is principal. Let $P \cap R = q$. Suppose $q \neq 0$. Then $\operatorname{ht} q \geqslant 1$. Note that $qR[x]$ is a prime ideal contained in P. Then

$$1 \leqslant \operatorname{ht} q \leqslant \operatorname{ht} qR[x] \leqslant \operatorname{ht} P = 1.$$

Thus, $\operatorname{ht} q = 1$ and $P = qR[x]$. Since R is a Noetherian UFD, $q = (a)$ for some $a \in R$. Then $P = aR[x]$ is principal.

Suppose $q = 0$. Let $W = R \setminus \{0\}$. Then $R_W = F$, the fraction field of R. Then P_W is a height one prime ideal of $R_W[x] = F[x]$. Since $F[x]$ is a PID, $P_W = (f)$ for some polynomial f in $F[x]$. We multiplying by a nonzero constant in F, we can assume $f \in P$ and the gcd of the coefficients of f is a unit. We claim that $P = fR[x]$. One containment is clear. Suppose $g \in P$. Then f divides g in $F[x]$. By the remark above, f divides g in $R[x]$. Thus, $g \in fR[x]$. □

Corollary 8.20. *Let R be a Noetherian UFD. Then R is normal.*

Proof. It suffices to show that R satisfies S_2 and R_1. Since all height one primes are principal, it follows immediately that R satisfies R_1. Since R is a domain, it is clear that R satisfies S_1. So let $p \in \operatorname{Spec} R$ be a prime of height at least two. Then p necessarily contains a height one prime q, which is principal, say $q = (f)$. Clearly, $f \in pR_p$ is a regular element and $R_p/fR_p = R_p/qR_p$ is a domain, but not a field (since $pR_p \supsetneq qR_p$). Thus, $\operatorname{depth} R_p/fR_p \geqslant 1$, hence $\operatorname{depth} R_p \geqslant 2$. Thus, R satisfies S_2. □

Definition 8.21. A finitely generated R-module M is called *stably free* if $M \oplus F \cong G$ for some finitely generated free R-modules F and G.

Clearly, every f.g. free module is stably free and every stably free module is projective. In the local case, every f.g. projective is free, so all three concepts coincide. In general, however, there exists stably free modules which are not free, and projective modules which are not stably free. We'll give an example of the latter after we prove a few results. Here is an example of the former:

Example 8.22. Let $S = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$. By the Jacobian Criteria (Theorem 8.14), R is a regular ring (in fact, a regular domain). Consider the homomorphism $f : S \to S^3$ given by $f(s) = (sx, sy, sz)$, where I am using x, y and z to denote the images of X, Y, and Z in S. Note that this map splits: Let $g : S^3 \to S$ be given by $g(u, v, w) = ux + vy + wz$. It is easily checked that $gf = \operatorname{id}_S$, since $x^2 + y^2 + z^2 = 1$ in S. Let P denote the cokernel of f. Then by the splitting lemma, $P \oplus S \cong S^3$, so P is stably free. However, by a result from differential geometry (about combing the hair on a sphere), P is not free.

Definition 8.23. Let R be a ring and M an R-module. A *finite free resolution* (FFR) for M is a finite complex $F_\bullet$ of finitely generated free R-modules

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

which is exact except in degree zero and $\mathrm{H}_0(F_\bullet) \cong M$. In other words, an FFR for M is a finite projective resolution of M in which all the projective modules in the resolution are finitely generated free R-modules.

Of course, if a f.g. module has an FFR, then it is of finite projective dimension. The converse true over a local ring. However, not all f.g. projectives have FFRs, as we'll see.

Proposition 8.24. *Let R be a Noetherian ring. A finitely generated projective module is stably free if and only if it has an FFR.*

Proof. Suppose P is stably free. Then $P \oplus F \cong G$ for some f.g. free R-modules F and G. This isomorphism yields the exact sequence $0 \to F \to G \to P \to 0$, so P has an FFR. Conversely, let P be a f.g. projective which has an FFR. We'll use induction on the length n of an FFR for P. The case $n = 0$ is trivial (P is free), so suppose $n = 1$. Then there exists an exact sequence $0 \to F_1 \to F_0 \to P \to 0$ for some f.g. free R-modules F_0 and F_1. As P is projective, this sequence splits, so $F_0 \cong F_1 \oplus P$, which shows that P is stably free. Suppose $n > 1$ and the result holds for projectives with FFRs of length less than n. Then there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \xrightarrow{\partial_2} F_1 \to F_0 \to P \to 0,$$

where F_i are f.g. free modules. Let $C = \operatorname{im} \partial_2$. As $0 \to C \to F_0 \to P \to 0$ is exact and P is projective, we have that C is projective and $P \oplus C \cong F_0$. We also see from the exact sequence above that C has an FFR of length $n - 1$. By the induction hypothesis, C is stably free, say $C \oplus G \cong F$ for some f.g. free R-modules F and G. Let $F' = F_0 \oplus G$, which is a f.g. free R-module. Then $F' = F_0 \oplus G \cong P \oplus C \oplus G \cong P \oplus F$, demonstrating that P is stably free.

□

Let A be an $n \times n$ matrix with entries from a commutative ring R. For $i, j \in \{1, \dots n\}$, the ijth *cofactor* of A is defined to be $b_{ij} := (-1)^{i+j} \det(A_{ij})$, where A_{ij} is the matrix obtained from A by deleting the ith row and jth column of A. The *adjoint* of A, denoted $\operatorname{adj} A$, is defined to be the matrix $(b_{ij})^{\mathrm{T}}$. The adjoint theorem states that

$$(\operatorname{adj} A)A = A(\operatorname{adj} A) = \det(A)\,\mathrm{I_n}\,.$$

Lemma 8.25. *Let R be a commutative ring and A an $n \times n$ matrix with entries from R. Then multiplication by A induces an injective map on R^n if and only if* $\det(A)$ *is a non-zero-divisor on R.*

Proof. Suppose first that $d = \det(A)$ is a non-zero-divisor on R. Then certainly multiplication by $d\,\mathrm{I_n}$ on R^n is injective. Hence, multiplication by $(\operatorname{adj} A)A$, and therefore by A, is injective.

Conversely, by way of contradiction suppose that ϕ_A is injective but $d = \det(A)$ is a zero-divisor. Let $w \in R \setminus \{0\}$ such that $dw = 0$. Let T be the prime subring of R (so either $\mathbb{Z}$ or $\mathbb{Z}_m$ for some m) and S the subring of R formed by adjoining all the entries of A and w to T. Then S is Noetherian, multiplication by A is injective on S^n, and d is a zero-divisor in S. Thus, we may assume R is Noetherian. We'll use induction on the number of rows n of A. When $n = 1$ the assumptions clearly result in a contradiction. Suppose $n > 1$. As d is a zero-divisor,

$d \in p$ for some $p \in \operatorname{Ass}_R R$. Localizing at p, we may assume R is local with maximal ideal m and depth $R = 0$. Let u be a nonzero element of $(0 :_R m)$. If every entry of A is in m, then $A(uI_n) = 0$, contradicting that multiplication by A is injective. Hence, some entry of A is a unit. Then using elementary row and column operations there exists invertible matrices P and Q such that

$$PAQ = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

where B is an $(n-1) \times (n-1)$ matrix. Note that as P and Q are invertible, multiplication by PAQ, and hence B, is injective. Also, $\det(B) = \det(P)\det(A)\det(Q)$. Since $\det(P)$ and $\det(Q)$ are units and $\det(A)$ is a zero-divisor, $\det(B)$ is a zero-divisor. But this contradicts the $n-1$ case.

□

Definition 8.26. Let R be a ring and P a finitely generated projective R-module. We say that P has *rank* r (or sometimes *constant rank* r) if $P_m \cong R_m^r$ for all maximal ideals m of R.

Remark 8.27. We note that not all finitely generated projective modules have a rank. For example, let $R = \mathbb{Q} \times \mathbb{Q}$. There are two maximal ideals, $I_1 = \mathbb{Q} \times 0$ and $I_2 = 0 \times \mathbb{Q}$. Both are projective, as $I_1 \oplus I_2 = R$. Then $I_1 R_{I_1} \cong R_{I_1}$ but $I_1 R_{I_2} = 0$. However, it is easy to see that stably free projectives have a rank: if $P \oplus R^r \cong R^s$ then $P_m \cong R_m^{s-r}$ for all maximal ideals m.

Theorem 8.28. *Let R be a ring and I a stably free ideal of R. Then I is free.*

Proof. The result is trivially true if $I = 0$, so assume $I \neq 0$. Then for at least one maximal ideal m of R, $I_m \neq 0$. As I is projective, I_m is free and nonzero, hence $I_m \cong R_m$. And since stably free modules have constant rank, we have that $I_p \cong R_p$ for all prime ideals p. Hence, our assumption that I is nonzero and stably free implies $I \oplus R^{n-1} \cong R^n$ for some $n \geqslant 1$. As $I \subseteq R$, we can consider $I \oplus R^{n-1}$ as a submodule of $R \oplus R^{n-1} = R^n$. Let $e_1, \ldots, e_n$ be a basis for R^n where e_1 is the basis for the first copy of R (the one I sits in) and $e_2, \ldots, e_n$ a basis for R^{n-1}. Let $\psi : R^n \to R^n$ be the composition of

$$R^n \xrightarrow{\cong} I \oplus R^{n-1} \hookrightarrow R^n.$$

Let A be the matrix representing ψ with respect to the basis $\{e_1, \ldots, e_n\}$. Since ψ is injective, $d = \det(A)$ is a non-zero-divisor. Let b_1 be the first column of $\operatorname{adj}(A)$. Then $Ab_1 = de_1$, where we are identifying e_1 with the first column of $\mathrm{I_n}$. Since $e_2, \ldots, e_n$ are in the image of ψ, let $b_2, \ldots, b_n$ be column vectors of R^n such that $Ab_j = e_j$ for $j \geqslant 2$. Let B be the $n \times n$ matrix whose ith column is b_i. Then $AB = \begin{pmatrix} d & 0 \\ 0 & \mathrm{I_{n-1}} \end{pmatrix}$. Note $\det(A)\det(B) = d = \det(A)$. Since $\det(A)$ is a non-zero-divisor, we see that $\det(B) = 1$. Thus, B is invertible and multiplication by B on R^n is an isomorphism. Hence, the image of AB equals the image of A, which is $I \oplus R^{n-1}$. However, the image of AB is $(d) \oplus R^{n-1}$, which means $I \oplus R^{n-1} = (d) \oplus R^{n-1}$ as submodules of $R \oplus R^{n-1}$. Hence, $I = (d)$. As d is a non-zero-divisor, $I = (d) \cong R$.

□

Example 8.29. Let R be a Dedekind domain which is not a PID (e.g., $R = \mathbb{Z}[\sqrt{-5}]$). Let I be a non-principal ideal. Then I is projective but not stably free. (Recall that every ideal in a Dedekind domain is projective as they are locally principal.) For, if I is stably free, then by the above theorem I is free and hence principal. In particular, such an I is an example of a finitely generated module of finite projective dimension that does not have an FFR, by Proposition 8.24.

Lemma 8.30. *Let R be a semilocal ring (not necessarily Noetherian) and M a finitely generated projective R-module of constant rank. Then M is free.*

Proof. Let $m_1, \ldots, m_s$ be the maximal ideals of R. As M is projective of constant rank, there exists an r such that $M_{m_i} \cong R^r_{m_i}$ for all i. Let $J = J(R) = m_1 \cap \cdots \cap m_s$. Then R/J is a zero-dimensional semilocal ring. By the Chinese Remainder Theorem,

$$R/J \cong R/m_1 \times \cdots \times R/m_s, \text{ and}$$
$$M/JM \cong M/m_1M \times \cdots \times M/m_sM.$$

As $M_{m_i} \cong R^r_{m_i}$ for each i, we have

$$M/m_iM \cong M_{m_i}/m_iM_{m_i} \cong R^r_{m_i}/m_iR^r_{m_i} \cong (R/m_i)^r.$$

Hence, $M/JM \cong (R/m_1)^r \times \cdots \times (R/m_s)^r \cong (R/J)^r$. In particular, M/JM is generated by r elements. Hence, by Nakayama's Lemma, M is generated by r elements, say $u_1, \ldots, u_s$. Define $f : R^r \to M$ by $f(e_i) = u_i$, where $\{e_1, \ldots, e_r\}$ is a basis for R^r. Clearly, f is surjective. Let $K = \ker f$. As $M_{m_i} \cong R^r_{m_i}$, f localized at m_i is an isomorphism (again, by NAK), so $K_{m_i} = 0$. Since K is locally zero at every maximal ideal, we see that $K = 0$. Hence, f is an isomorphism and M is free.

□

Lemma 8.31. *Let R be a Noetherian ring and P a rank one stably free R-module. Then P is isomorphic to an ideal of R.*

Proof. Let W be the set of non-zero-divisors of R and R_W, the total quotient ring of R. Let $q_1, \ldots, q_s$ be the maximal associated primes of R, i.e., the maximal elements of $\operatorname{Ass}_R R$. (Here is where the Noetherian hypothesis is used.) Then $W = R \setminus p_1 \cup \cdots \cup p_s$. Thus, $p_1R_W, \ldots, p_sR_W$ are the maximal ideals of R_W. Hence, R_W is semilocal. For ease of notation, let $m_i = p_iR_W$. Since P is rank one stably free R-module, $P_q \cong R_q$ for all primes q of R. Hence, for each i we have $(P_W)_{m_i} \cong P_{p_i} \cong R_{p_i} \cong (R_W)_{m_i}$. Thus, P_W is a rank one stably free module over R_W, which is semilocal. By Proposition 8.30, we have that $P_W \cong R_W$. Consider the composition of R-module homomorphisms

$$P \xrightarrow{i} P_W \xrightarrow{\cong} R_W.$$

Note that as P is isomorphic to a submodule of a free module (every projective is), the elements of W are non-zero-divisors on P. Hence, the map i above is injective. Thus, P is isomorphic to an R-submodule M of R_W. As P is finitely generated, so is M; say $M = R\frac{r_1}{w_1} + \cdots + R\frac{r_t}{w_t}$ for some $r_1, \ldots, r_t \in R$ and $w_1, \ldots, w_t \in W$. Let $w = w_1w_2 \cdots w_t$. As w is a nonzerodivisor, $P \cong M \cong wM$, where wM is an R-submodule of R, i.e., an ideal of R. □

We can now strengthen Theorem 8.28 to all rank one stably free modules:

Theorem 8.32. *Let R be a ring and P a rank one stably free R-module. Then P is free.*

Proof. Suppose $P \oplus R^{n-1} \cong R^n$. Then there exists a split exact sequence

$$0 \to R^{n-1} \xrightarrow{\phi} R^n \to P \to 0.$$

As the sequence splits, there exist $\rho : R^n \to R^{n-1}$ such that $\rho\phi = \mathrm{id}_{R^{n-1}}$. Choose bases for R^{n-1} and R^n and let A and B be the matrices representing ϕ and ρ with respect to these bases. Let

T be the prime subring of R and S be the subring of R obtained by adjoining all the entries of A and B to T. Then S is Noetherian. Let $\pi : S^{n-1} \to S^n$ be given by multiplication by A and let $Q = \operatorname{coker} \pi$. Thus, we have an exact sequence

$$0 \to S^{n-1} \xrightarrow{\pi} S^n \to Q \to 0.$$

Now let $\tau : S^n \to S^{n-1}$ be given by multiplication by B. Then $\tau\pi = \mathrm{id}_{S^{n-1}}$ since $BA = \mathrm{I_{n\text{-}1}}$. Thus, the exact sequence above splits. Hence, $Q \oplus S^{n-1} \cong S^n$, which means Q is a rank one stably free S-module. Since S is Noetherian, we have by Lemma 8.31 that Q is isomorphic to an ideal. By Theorem 8.28, we conclude that $Q \cong S$. If we apply the functor $(-) \otimes_S R$ to the split exact sequence $0 \to S^{n-1} \xrightarrow{\pi} S^n \to Q \to 0$ it stays exact. As both π and ϕ are represented by the matrix A, we have $\pi \otimes \mathrm{id}_R = \phi$. Hence,

$$0 \to R^{n-1} \xrightarrow{\phi} R^n \to Q \otimes_S R \to 0$$

is exact. By the Five Lemma, we see that $Q \otimes_S R \cong P$. As $Q \cong S$, we obtain that $P \cong R$.

□

Here are a couple additional significant results (among many) about projective modules:

- (Quillen-Suslin Theorem, mid-1970s) Let $R = k[x_1, \ldots, x_n]$ where k is field. Then every projective R-module is free.
- (Bass, early 1960s) Suppose R is a Noetherian ring with no nontrivial idempotents (e.g., R is local or a domain). Then every non-finitely generated projective R-module is free.

Back to UFDs: To prove regular local rings are UFDs, we'll need the following lemma:

Lemma 8.33. *Let R be a Noetherian domain and π a prime element of R. If R_π is a UFD, then so is R.*

Proof. It suffices to prove that every height one prime of R is principal. Let p be such a prime. If $\pi \in p$, then since (π) is a height one prime, we have $p = (\pi)$. Assume $\pi \notin p$. Then, as R_π is a UFD, $pR_\pi = aR_\pi$ for some $a \in R$. Consider the nonempty set of ideals $\Lambda = \{aR \mid a \in R \text{ and } pR_\pi = aR_\pi\}$ and choose a maximal element $bR \in \Lambda$. Note that $b \notin (\pi)$, else $bR \subsetneq \frac{b}{\pi}R \in \Lambda$. As $pR_\pi \cap R = p$ we have $b \in p$. Let $c \in p$. Then in R_π, $c = \frac{r}{\pi^n}b$ for some $r \in R$ and $n \geqslant 0$. Thus in R we have $\pi^n c = rb$. Hence, $rb \in (\pi)$. As (π) is prime and $b \notin (\pi)$, $r \in (\pi)$. This yields $\pi^{n-1}c = r'b$ where $r' = \frac{r}{\pi} \in R$. Continuing in this fashion, we obtain that $c \in (b)$. Thus, $p = (b)$.

□

Theorem 8.34. (Auslander-Buchsbaum, 1959) *Any regular local ring is a UFD.*

Proof. Let (R, m) be a regular local ring. We'll proceed by induction on $d = \dim R$. If $d \leqslant 1$ then R is a field or a PID and the result holds. So assume $d > 1$. Let $x \in m \setminus m^2$. Then $R/(x)$ is a regular local ring, and thus a domain. Hence x is a prime element of R. By Lemma 8.33, it suffices to prove R_x is a UFD. Let p_x be a height one prime of R_x, where $p \in \operatorname{Spec} R$ with $x \notin p$. If $p_x \not\subset q_x$, then $(p_x)_{q_x} = R_x$. If $p_x \subseteq q_x$ then $(p_x)_{q_x} \cong pR_q$. As R_q is a RLR of dimension smaller than d, R_q is a UFD. Hence, pR_q is principal and therefore isomorphic to R_q. Thus, p_x is locally free, which implies p_x is a projective R_x-module. Now, as R is a regular local ring, p has an FFR over R. Localizing this FFR at x gives an FFR for p_x over R_x. By Proposition 8.24, we obtain that p_x is stably free. By Theorem 8.28, we have that p_x is free, and hence principal. Thus, every height one prime of R_x is principal and so R_x is a UFD. □

9 Canonical modules

To introduce canonical modules, we first prove the following special case of Matlis duality:

Proposition 9.1. *Let (R, m, k) be an Artinian local ring and $E := E_R(k)$. Let $(-)^{\vee}$ denote the contravariant (and exact) functor $\operatorname{Hom}_R(-, E)$. Then for any finitely generated R-module M the natural evaluation homomorphism $\phi_M : M \to M^{\vee\vee}$ is an isomorphism.*

Proof. We first establish the isomorphism in the case M has length one; i.e. $M \cong R/m = k$. Consider $\phi_k : k \to k^{\vee\vee}$ and let $a \in \ker \phi_k$. Let $f : k \hookrightarrow E$ be an embedding of $k = R/m$ into its injective hull (over R). Then $0 = \phi_k(a)(f) = f(a)$. As f is injective, $a = 0$. Hence, ϕ_k is injective. By Lemma 6.3, $\lambda_R(k) = \lambda_R(k^{\vee}) = \lambda_R(k^{\vee\vee})$. Thus, $\lambda_R(\operatorname{coker} \phi_k) = 0$ and ϕ_k is an isomorphism.

Suppose now that $\lambda_R(M) > 1$. Then there exists a s.e.s. $0 \to L \to M \to N \to 0$ such that $\lambda_R(L)$ and $\lambda_R(N)$ are less than $\lambda_R(M)$. Since evaluation homomorphisms are natural and $(-)^{\vee}$ is exact, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_N} & & \\ 0 & \longrightarrow & L^{\vee\vee} & \longrightarrow & M^{\vee\vee} & \longrightarrow & N^{\vee\vee} & \longrightarrow & 0 \end{array}$$

By induction, ϕ_L and ϕ_N are isomorphisms. Thus ϕ_M is an isomorphism by the Five Lemma. □

We'd like to generalize this duality to local rings of higher dimension. One direction (Matlis duality) utilizes the same "dualizing module" (E) over complete local rings. This is an extremely useful duality, but note that $R^{\vee} = E$ is not a finitely generated module if the dimension of R is positive. There is another generalization of this duality over Cohen-Macaulay local rings (satisfying a mild condition) which uses a finitely generated module to do the dualizing, called the canonical module. This type of duality is best seen at the level of complexes, but we can prove a very pleasing duality on the module level if we restrict to CM modules.

Definition 9.2. A finitely generated module M over a local ring R is called *maximal* Cohen-Macaulay (MCM) if $\operatorname{depth} M = \dim R$; i.e., M is a CM module of maximal possible dimension. If M is an MCM module, its *type*, denoted $r(M)$, is $\mu_d(m, M)$ where $d = \dim R$.

Example 9.3. Let $R = k[[x, y]]/(x^2, xy)$. Then $R/(x)$ is an MCM for R as $\operatorname{depth} R/(x) = 1 = \dim R$. By a direct calculation of $\operatorname{Ext}^1_R(k, R/(x))$ one can show the type of $R/(x)$ (as an R-module) is 1. Note in this example R is not CM.

Definition 9.4. Let (R, m) be a d-dimensional CM local ring. A finitely generated R-module C is called a *canonical module* for R if $\mu_i(m, C) = \delta_{i\,d}$, where δ_{ij} is the Kronecker delta. Equivalently, C is an MCM of type 1 and finite injective dimension.

Remark 9.5. It is a consequence of the New Intersection Theorem that if R has a nonzero finitely generated module of finite injective dimension, then R is CM. Thus, for a local ring to possess a canonical module of the type defined above, R must be CM.

Examples 9.6. Let (R, m, k) be a local ring.

(a) R is Gorenstein if and only if R is a canonical module for R (Proposition 6.14).

(b) If R is Artinian then $E_R(k)$ is a canonical module for R.

We'll need a few results on maximal Cohen-Macaulay modules. An important one is this: if $x \in R$ is regular and M is an MCM, then x is M-regular. This follows since $\dim R/p = \dim R$ for all $p \in \operatorname{Ass}_R M$ (Proposition 5.7). Since M/xM is an MCM $R/(x)$-module, one can strengthen this statement to say that any R-sequence is an M-sequence.

Proposition 9.7. *Let (R, m, k) be a CM local ring. Suppose M and N are MCMs such that $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > 0$. Then*

(a) $\operatorname{Hom}_R(M, N)$ is MCM;

(b) For any R-sequence $\mathbf{x} = x_1, \ldots, x_s$, we have

(i) $\operatorname{Hom}_R(M, N) \otimes_R R/(\mathbf{x}) \cong \operatorname{Hom}_{R/(\mathbf{x})}(M/(\mathbf{x})M, N/(\mathbf{x})N)$, and

(ii) $\operatorname{Ext}^i_{R/(\mathbf{x})}(M/(\mathbf{x})M, N/(\mathbf{x})N) = 0$ for all $i > 0$.

Proof. We'll use induction on $d = \dim R$. In the case $d = 0$ there is nothing to prove. (Every module is MCM and there are no R-sequences.) Suppose $d > 0$. Let x be any regular element on R. By the comments above, x is both M-regular and N-regular, and hence M/xM and N/xN are MCM $R/(x)$-modules. Note $\operatorname{Hom}_R(M/xM, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(R/(x), N)) = 0$. Applying $\operatorname{Hom}_R(-, N)$ to the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ and using that $\operatorname{Ext}^i_R(M, N) = 0$ for $i > 0$, we obtain that

$$0 \to \operatorname{Hom}_R(M, N) \xrightarrow{x} \operatorname{Hom}_R(M, N) \to \operatorname{Ext}^1_R(M/xM, N) \to 0,$$

and $\operatorname{Ext}^i_R(M/xM, N) = 0$ for all $i \geqslant 2$. This gives us that x is a regular element on $\operatorname{Hom}_R(M, N)$ and that $\operatorname{Hom}_R(M, N) \otimes_R R/(x) \cong \operatorname{Ext}^1_R(M/xM, N)$. By Theorem 6.10, we have

$$\operatorname{Hom}_{R/(x)}(M/xM, N/xN) \cong \operatorname{Ext}^1_R(M/xM, N) \cong \operatorname{Hom}_R(M, N) \otimes_R R/(x),$$

and for all $i \geqslant 1$

$$\operatorname{Ext}^i_{R/(x)}(M/xM, N/xN) \cong \operatorname{Ext}^{i+1}_R(M/xM, N) = 0.$$

Thus, $\overline{M} = M/xM$ and $\overline{N} = N/xN$ satisfy the hypotheses of the theorem over $\overline{R} = R/(x)$, which is of dimension $d - 1 < d$. Hence (a) and (b) must hold for $\overline{M}$ and $\overline{N}$. In particular, $\operatorname{Hom}_{\overline{R}}(\overline{M}, \overline{N}) \cong \operatorname{Hom}_R(M, N)/x \operatorname{Hom}_R(M, N)$ is maximal CM over $\overline{R}$. As x is regular on $\operatorname{Hom}_R(M, N)$ we obtain that $\operatorname{Hom}_R(M, N)$ is MCM over R, which proves (a). For (b), let $\mathbf{x} = x_1, \ldots, x_s$ be any R-sequence. We've proved (i) and (ii) hold when $s = 1$. By induction, we have (i) and (ii) hold for the $\overline{R}$-sequence $\overline{x_2}, \ldots, \overline{x_s}$, $\overline{M}$ and $\overline{N}$. But this then shows that (i) and (ii) hold for $\mathbf{x}$, M, and N.

□

A maximal CM module of finite injective dimension has even nicer properties:

Proposition 9.8. *Let (R, m, k) be a CM local ring. Let C be an MCM of finite injective dimension. Let M be a (nonzero) finitely generated CM module of dimension t. Then*

1. *$\operatorname{Ext}^i_R(M, C) = 0$ for $i \neq d - t$.*

2. *$\operatorname{Ext}^{d-t}_R(M, C)$ is nonzero and CM of dimension t.*

Proof. By Theorem 3.20, we have $\operatorname{Ext}_R^i(M,C) = 0$ for $i < d-t$. We use induction on t to prove $\operatorname{Ext}_R^i(M,C) = 0$ for $i > d-t$. If $t = 0$, then M has finite length. By Corollary 2.24 and Theorem 4.18, we have $\operatorname{Ext}_R^i(k,M) = 0$ for $i > d$. By Lemma 2.22, $\operatorname{Ext}_R^i(M,C) = 0$ for all $i > d$. Suppose $t > 0$. Let $x \in m$ be M-regular. Then M/xM is a CM module of dimension $t-1$. Applying $\operatorname{Hom}_R(-,C)$ to the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we have for all j an exact sequence

$$\operatorname{Ext}_R^j(M,C) \xrightarrow{x} \operatorname{Ext}_R^j(M,C) \to \operatorname{Ext}_R^{j+1}(M/xM,C).$$

By induction, we have $\operatorname{Ext}_R^{j+1}(M/xM,C) = 0$ for all $j+1 > d-(t-1)$, or $j > d-t$. By Nakayama, we obtain $\operatorname{Ext}_R^j(M,C) = 0$ for all $j > d-t$. This proves (i).

To prove (ii), we again use induction on t. Assume $t = 0$. Then M has finite length. Thus, $\operatorname{Ext}_R^d(M,C)$ has finite length since it is finitely generated and is locally zero at all primes $p \neq m$. Hence $\operatorname{Ext}_R^d(M,C)$ is CM of dimension zero. It remains to show $\operatorname{Ext}_R^d(M,C) \neq 0$. Suppose $\lambda_R(M) = 1$. Then $M \cong R/m = k$. But $\operatorname{Ext}_R^d(k,C) \neq 0$ since $\operatorname{depth} C = d$. Suppose $\lambda_R(M) > 1$. Then there exists an exact sequence $0 \to k \to M \to N \to 0$. We then obtain an exact sequence

$$\operatorname{Ext}_R^d(M,C) \to \operatorname{Ext}_R^d(k,C) \to \operatorname{Ext}_R^{d+1}(N,C).$$

But $\operatorname{Ext}_R^{d+1}(N,C) = 0$ as $\operatorname{id}_R C = d$, so $\operatorname{Ext}_R^d(M,C) \neq 0$. This proves (ii) in the case $t = 0$. Now suppose $t > 0$. Let $x \in m$ be an M-regular element. Then M/xM is a CM module of dimension $t-1$. Applying $\operatorname{Hom}_R(-,C)$ to the usual exact sequence, we obtain the s.e.s.

$$0 \to \operatorname{Ext}_R^{d-t}(M,C) \xrightarrow{x} \operatorname{Ext}_R^{d-t}(M,C) \to \operatorname{Ext}_R^{d-(t-1)}(M/xM,C) \to 0,$$

where here we are using part (i) to get $\operatorname{Ext}_R^j(M/xM,C) = 0$ for $j \neq d-(t-1)$. Thus, x is a regular element on $A = \operatorname{Ext}_R^{d-t}(M,C)$ and $A/xA \cong \operatorname{Ext}_R^{d-(t-1)}(M/xM,C)$. By the induction hypothesis, we know A/xA is a nonzero CM module of dimension $t-1$. Hence, $A = \operatorname{Ext}_R^{d-t}(M,C)$ is nonzero CM of dimension $d-t$. This completes the proof of (ii). □

Lemma 9.9. *Let (R,m,k) be a local ring, M and N finitely generated R-modules, and $\mathbf{x} = x_1,\ldots,x_s$ an N-sequence. Let $\phi : M \to N$ be a homomorphism. If the induced map $\overline{\phi} : M/(\mathbf{x})M \to N/(\mathbf{x})N$ is an isomorphism, then so is ϕ.*

Proof. It suffices to prove this in the case $s = 1$. Surjectivity of ϕ follows easily from Nakayama's lemma. Let $m \in \ker\phi$. Then $\phi(m) \in xN$, thus $m \in xM$. Write $m = xm_1$ for some $m_1 \in M$. Then $0 = \phi(m) = \phi(xm_1) = x\phi(m_1)$. As x is N-regular, we conclude $\phi(m_1) = 0$. Repeating the same argument for m_1, we obtain $m_1 = xm_2$ for some $m_2 \in M$. Hence, $m = x^2m_2 \in x^2M$. By induction, we see that $m \in x^nM$ for all n, which implies $m = 0$ by Krull's intersection theorem. Thus, ϕ is injective. □

Lemma 9.10. *Let (R,m) be a d-dimensional CM local ring and C a finitely generated R-module. Let $\mathbf{x} = x_1,\ldots,x_s$ be an R-sequence and a C-sequence. Then C is a canonical module for R if and only if $C/(\mathbf{x})C$ is a canonical module for $R/(\mathbf{x})$.*

Proof. It suffices to prove the statement in the case $s = 1$. Clearly, C is an MCM R-module if and only if M/xM is an MCM $R/(x)$-module. By Theorem 6.11, $\operatorname{id}_{R/(x)} C/xC = \operatorname{id}_R C - 1$, so $\operatorname{id}_R C$ is finite if and only if $\operatorname{id}_{R/(x)} C/xC$ is finite. And by Theorem 6.10, $\dim_k \operatorname{Ext}_{R/(x)}^{d-1}(k,C/xC) = \dim_k \operatorname{Ext}_R^d(k,C)$, and so C has type 1 if and only if C/xC has type 1. □

Theorem 9.11. *Let (R, m, k) be a CM local ring and suppose C and D are canonical modules for R. Then $C \cong D$ and the map $\pi_C : R \to \operatorname{Hom}_R(C, C)$ given by $\pi_C(r) = \mu_r$ is an isomorphism, where μ_r is multiplication by r.*

Proof. We proceed by induction on $d = \dim R$. Suppose $d = 0$ and C a canonical module for R. As C is a finitely generated injective module and $\operatorname{Spec} R = \{m\}$, $C \cong E_R(k)^t$ for some t. As C has type 1, we have that $t = 1$. Thus, $C \cong E_R(k)$ and all canonical modules for R are isomorphic. By Proposition 9.1, $R \cong R^{\vee\vee} \cong \operatorname{Hom}_R(E, E)$. Thus, any element which annihilates E also annihilates R, and hence is zero. Consequently, $\pi_E : R \to \operatorname{Hom}_R(E, E)$ is injective. Since $\lambda(R) = \lambda(R^{\vee\vee}) = \lambda(\operatorname{Hom}_R(E, E))$, we conclude that π_E is an isomorphism.

Now assume $d = \dim R > 0$ and let $x \in m$ be a regular element on R. By Lemma 9.10, C/xC and D/xD are canonical modules for $R/(x)$. By induction, $C/xC \cong D/xD$. By part (a) of Proposition 9.8 we have $\operatorname{Ext}^i_R(C, D) = 0$ for all $i > 0$. Thus, by Proposition 9.7 and the induction hypothesis, we obtain

$$\operatorname{Hom}_R(C, D)/x \operatorname{Hom}_R(C, D) \xrightarrow{\cong} \operatorname{Hom}_{R/(x)}(C/xC, D/xD) \cong \operatorname{Hom}_{R/(x)}(C/xC, C/xC) \cong R/(x).$$

By Nakayama's lemma, we see that $\operatorname{Hom}_R(C, D)$ is cyclic. Let $\phi : C \to D$ be a cyclic generator. Then $\phi \otimes_R R/(x)$ corresponds (under the second isomorphism above) to a generator for $\operatorname{Hom}_R(C/xC, C/xC)$. But by the third isomorphism (and the induction hypothesis), this generator has the form μ_r for some generator r of $R/(x)$. But any generator for R must be a unit, which means μ_r is an isomorphism. Consequently, $\phi \otimes_R R/(x)$ is also an isomorphism (this is an elementary exercise, as the second isomorphism is induced by an isomorphism $D/xD \to C/xC$). By Lemma 9.9, we conclude that ϕ is an isomorphism. Finally, consider $\pi_C : R \to \operatorname{Hom}_R(C, C)$. Then $\pi_{C/xC}$ is the composition

$$R/(x) \xrightarrow{\pi_C \otimes R/(x)} \operatorname{Hom}_R(C, C) \otimes_R R/(x) \xrightarrow{\cong} \operatorname{Hom}_{R/(x)}(C/xC, C/xC).$$

Thus, $\pi_C \otimes_R R/(x)$ is an isomorphism, and hence π_C is an isomorphism by Lemma 9.9.

□

As a consequence of Theorem 9.11, we can speak of *the* canonical module for R (assuming one exists), as it is unique up to isomorphism. We'll denote the canonical module by ω_R.

Corollary 9.12. *Let (R, m, k) be a CM local ring. The following are equivalent:*

(a) R is Gorenstein.

(b) R has a canonical module and $\omega_R \cong R$.

Proof. We've already noted that if R is Gorenstein then R is a canonical module. Conversely, if R is a canonical module then R has finite injective dimension, hence Gorenstein. □

Proposition 9.13. *Let (R, m, k) be a CM local ring which possesses a canoncal module ω_R. Then*

(a) $\operatorname{Ann}_R \omega_R = (0)$.

(b) $\operatorname{Supp}_R \omega_R = \operatorname{Spec} R$.

(c) For any R-sequence $\mathbf{x}$, $\omega_{R/(\mathbf{x})} \cong \omega_R/(\mathbf{x})\omega_R$.

(d) For any $p \in \operatorname{Spec} R$, $\omega_{R_p} \cong (\omega_R)_p$.

Proof. Since $R \cong \operatorname{Hom}_R(\omega_R, \omega_R)$, we see that $\operatorname{Ann}_R \omega_R \subseteq \operatorname{Ann}_R R = (0)$, which proves (a). Part (b) is an immediate consequence of (a). Part (c) is a restatement of Lemma 9.10.

For part (d), let $p \in \operatorname{Spec} R$. Since $\operatorname{id}_R \omega_R < \infty$ we have $\operatorname{id}_{R_p} (\omega_R)_p < \infty$ since localization of injective modules are injective (cf. Proposition 1.15). By Proposition 5.11, we have that $(\omega_R)_p$ is a CM R_p-module. As $\operatorname{Ann}_{R_p}(\omega_R)_p = (\operatorname{Ann}_R \omega_R)_p = (0)$ by part (a), we have $\dim(\omega_R)_p = \dim R_p$. Hence, $(\omega_R)_p$ is a MCM for R_p. It remains to show that the type of $(\omega_R)_p$ as R_p-module is 1. Let r be the type of $(\omega_R)_p$, i.e. $r = \dim_{k(p)} \operatorname{Ext}^t_{R_p}(k(p), (\omega_R)_p)$, where $t = \dim R_p$. Since R is CM, we have $\operatorname{grade} p = \operatorname{ht} p$. Let $\mathbf{x}$ be a maximal R-sequence in p. Then the image of $\mathbf{x}$ in R_p is a system of parameters (and still a regular sequence). For ease of notation, let $S = R_p/(\mathbf{x})R_p$, $M = (\omega_R)_p/(\mathbf{x})(\omega_R)_p$, and $E_S = E_S(k(p))$. As $\mathbf{x}$ is a regular sequence on $(\omega_R)_p$ we have by Theorem 6.10 that $r = \dim_{k(p)} \operatorname{Hom}_{R_p}(k(p), M)$. Note also that $\operatorname{id}_S M < \infty$ by Theorem 6.11. Since $\dim S = 0$ we have $\operatorname{id}_S M = 0$, and so $M \cong E_S^r$. On the other hand, by part (c) we know that $\omega_R/(\mathbf{x})\omega_R$ is a canonical module for $R/(\mathbf{x})$. Hence $\operatorname{Hom}_{R/(\mathbf{x})}(\omega_R)/(\mathbf{x})\omega_R, \omega_R/(\mathbf{x})\omega_R) \cong R/(\mathbf{x})$. As $\dim S = 0$ we have by Matlis duality (Proposition 9.1) that

$$\begin{aligned} S^{r^2} &\cong \operatorname{Hom}_S(E_S, E_S)^{r^2} \\ &\cong \operatorname{Hom}_S(E_S^r, E_S^r) \\ &\cong \operatorname{Hom}_S(M, M) \\ &\cong \operatorname{Hom}_{R/(\mathbf{x})}(\omega_R/(\mathbf{x})\omega_R, \omega_R/(\mathbf{x})\omega_R)_p \\ &\cong (R/(\mathbf{x}))_p \\ &\cong S. \end{aligned}$$

Comparing ranks, we conclude that $r = 1$. This completes the proof of (d). □

Corollary 9.14. *Let (R, m, k) be a CM local ring which possesses a canonical module ω_R. Then*

$$\mu_i(p, \omega_R) = \delta_{i \operatorname{ht}(p)}.$$

Proof. Recall from the definition of a canonical module (Definition 9.4)) that $\mu_i(m, \omega_R) = \delta_{id}$ where $d = \dim R = \operatorname{ht} m$. Let $p \in \operatorname{Spec} R$. Since $(\omega_R)_p$ is a canonical module for R_p by the previous result, we have $\mu_i(p, \omega_R) = \mu_i(pR_p, (\omega_R)_p) = \delta_{i \operatorname{ht} p}$. □

Proposition 9.15. *Let (R, m, k) be a CM local ring which has a canonical module ω_R. The following are equivalent:*

(a) ω_R is isomorphic to an ideal of R;

(b) R_p is Gorenstein for all $p \in \operatorname{Min}_R R$. (In this case, R is said to be generically Gorenstein.*)*

Proof. *(a)* $\implies$ *(b)* : Suppose $\omega_R \cong I$ where I is an ideal of R. Let $p \in \operatorname{Min}_R R$. By Proposition 9.13, $(\omega_R)_p \cong \omega_{R_p} \cong E_{R_p}(k(p))$. As $(\omega_R)_p \cong I_p \subseteq R_p$, we have

$$\lambda_{R_p}(E_{R_p}(k(p)) = \lambda_{R_p}(I_p) \leqslant \lambda_{R_p}(R_p) = \lambda_{R_p}(E_{R_p}(k(p)),$$

where the last equality follows from Lemma 6.3. Thus, $\lambda_{R_p}(I_p) = \lambda_{R_p}(R_p)$, which implies $I_p = R_p$. Thus, $\omega_{R_p} \cong R_p$ and hence R_p is Gorenstein.

Conversely, assume R is generically Gorenstein. Let W be the set of non-zero-divisors on R. Then R_W is an Artinian local ring and $\operatorname{Spec} R_W = \{p_W \mid p \in \operatorname{Min}_R R\}$. Let $p \in \operatorname{Min}_R R$. As R_p is Gorenstein, $((\omega_R)_W)_{p_W} \cong (\omega_R)_p \cong R_p \cong (R_W)_{p_W}$. Thus, $(\omega_R)_W$ is a projective R_W-module of constant rank one. By Lemma 8.30, $(\omega_R)_W \cong R_W$. As any non-zero-divisor on R is a non-zero-divisor on ω_R (as ω_R is MCM), the localization map $\omega_R \to (\omega_R)_W$ is injective. Composing with the isomorphism $(\omega_R)_W \to R_W$, we see that ω_R is isomorphic to a finitely generated R-submodule of R_W. But any finitely generated R-submodule of R_W is isomorphic to an ideal of R (by multiplying by a suitable element of W to clear denominators).

□

Theorem 9.16. *Let (S, n, k) be a Gorenstein local ring and I an ideal of S such that $R := S/I$ is CM. Let $t = \dim S - \dim R$. Then $\omega_R \cong \operatorname{Ext}^t_S(R, S)$. In particular, any CM local ring which is the quotient of a Gorenstein ring possesses a canonical module.*

Proof. Let $\mathbf{x} = x_1, \dots, x_g \in I$ be an S-sequence, where $g = \operatorname{grade} I$. Let $\overline{S} = S/(\mathbf{x})$ and $\overline{I} = I/(\mathbf{x})$. Then $\overline{S}$ is a Gorenstein local ring and $R \cong \overline{S}/\overline{I}$. Note, as S is Gorenstein, $\dim R = \dim S/I = \dim S - \operatorname{ht} I = \dim S - g = \dim \overline{S}$. Also, by Theorem 6.10,

$$\operatorname{Ext}^t_S(R, S) \cong \operatorname{Ext}^0_{\overline{S}}(R, \overline{S}) = \operatorname{Hom}_{\overline{S}}(R, \overline{S}).$$

Thus, it suffices to show that $\operatorname{Hom}_{\overline{S}}(R, \overline{S})$ is a canonical module for R. Resetting notation, we may assume that $\dim R = \dim S$. Note that R is an MCM S-module. Let $\mathbf{y} = y_1, \dots, y_d$ be a maximal S-sequence, where $d = \dim S = \dim R$. Then $\mathbf{y}$ is also an R-sequence. Since S has finite injective dimension, we have $\operatorname{Ext}^i_S(R, S) = 0$ for $i > 0$ by Proposition 9.8. Consequently, by Proposition 9.7, $C := \operatorname{Hom}_S(R, S)$ is an MCM S-module, and thus an MCM R-module. By Lemma 9.10, it suffices to prove $C/(\mathbf{y})C$ is a canonical module for $R/(\mathbf{y})R$. Again by Proposition 9.7, we have $C/(\mathbf{y})C \cong \operatorname{Hom}_{S/(\mathbf{y})}(R/(\mathbf{y})R, S/(\mathbf{y}))$. Note that $S/(\mathbf{y})$ is a zero-dimensional Gorenstein local ring. So resetting notation again, it suffices to prove that if S is a zero-dimensional Gorenstein local ring and $R = S/I$, then $\operatorname{Hom}_S(R, S)$ is a canonical module for R. As S is Gorenstein, $S \cong E_S(k)$. But

$$\operatorname{Hom}_S(R, S) \cong \operatorname{Hom}_S(S/I, E_S(k)) \cong E_{S/I}(k) = E_R(k),$$

and $E_R(k)$ is a canonical module for R. This completes the proof. □

Corollary 9.17. *Let (S, n, k) be a regular local ring and I and ideal of S. Suppose $R := S/I$ is Cohen-Macaulay and let $F_\bullet$ be a minimal free S-resolution of R. Then $\Sigma^t \operatorname{Hom}_S(F_\bullet, S)$ is a minimal free S-resolution of ω_R, where $t = \dim S - \dim R = \operatorname{pd}_S R$.*

Proof. First note that by the Auslander-Buchsbaum formula, $\operatorname{pd}_S R = \operatorname{depth} S - \operatorname{depth} R = \dim S - \dim S/I$. Let $F_i = S^{\beta_i}$ for $i = 0, \dots, t$. Then $F_\bullet$ has the form

$$0 \to S^{\beta_t} \xrightarrow{\phi_t} S^{\beta_{t-1}} \to \cdots \to S^{\beta_1} \xrightarrow{\phi_1} S \to 0,$$

where $\phi_i \otimes S/n = 0$ for all i. Applying $\operatorname{Hom}_S(-, S)$, we have the complex

$$0 \to S \xrightarrow{\phi_0^*} S^{\beta_1} \to \cdots \to S^{\beta_{t-1}} \xrightarrow{\phi_t^*} S^{\beta_t} \to 0,$$

where the complex sits in cohomological degrees 0 to t. We have $\operatorname{H}^i(\operatorname{Hom}_S(F_\bullet, S)) = \operatorname{Ext}^i_S(R, S)$ for all i. By Proposition 9.8, $\operatorname{Ext}^i_S(R, S) = 0$ for all $i \neq t$. (Note $\dim R = \dim S - t$.) By Theorem

9.16, $\operatorname{Ext}^t_S(R,S) \cong \omega_R$. Thus, the complex $\operatorname{Hom}_S(F_\bullet, S)$ is exact except in cohomological degree t. Note also that $\phi_i^* \otimes S/n = 0$ for all i. Hence $\operatorname{Hom}_S(F_\bullet, S)$ is a minimal free S-resolution of ω_R, once properly shifted. Switching to homological degrees, $\operatorname{Hom}_S(F_\bullet, S)$ sits in degrees 0 to $-t$, so we need to shift the complex t units to the left. This is accomplished by applying the functor Σ^t (see Definition 4.5). Thus, $\Sigma^t \operatorname{Hom}_S(F_\bullet, S)$ is a minimal free resolution for ω_S.

□

Example 9.18. Let k be a field and $R = k[[t^3, t^4, t^5]]$. As R is the quotient of a Gorenstein local ring (namely, $S = k[[x,y,z]]$), we know that R has a canonical module ω_R. Let's find a resolution of ω_R over S. Let $\phi : S \to R$ be given by $\phi(x) = t^3$, $\phi(y) = t^4$, and $\phi(z) = t^5$. Clearly, ϕ is a surjective ring homomorphism. Using Macaulay2 (or some results by Herzog and others), one can find that $P := \ker\phi = (x^3 - yz, y^2 - xz, z^2 - x^2y)S$. By the Hilbert-Burch theorem (or again, Macaulay2), one obtains the following minimal resolution for R over S:

$$0 \to S^2 \xrightarrow{\begin{pmatrix} y & z \\ z & x^2 \\ x & y \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^3 - yz & y^2 - xz & z^2 - x^2y \end{pmatrix}} S \to R \to 0.$$

Hence, by Corollary 9.17, by applying $\operatorname{Hom}_S(-, S)$ and shifting two degrees to the left we have a resolution of ω_R:

$$0 \to S \xrightarrow{\begin{pmatrix} x^2 - yz \\ y^2 - xz \\ z^2 - x^2y \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} y & z & x \\ z & x^2 & y \end{pmatrix}} S^2 \to \omega_R \to 0.$$

Corollary 9.19. *Let S and R be as above. For a finitely generated S-module M, let $\beta_i(M) := \dim_k \operatorname{Tor}_i^S(k, M)$; i.e., $\beta_i(M)$ is the rank of the free module in degree i of a minimal free resolution of M. (These are called the Betti numbers of M.)*

(a) For all i, $\beta_i(\omega_R) = \beta_{t-i}(R)$, where $t = \operatorname{pd}_S R$.

(b) R is Gorenstein if and only if the sequence of Betti numbers of R is symmetric.

Proof. Part (a) is an immediate consequence of Corollary 9.17. It remains to show (b). If R is Gorenstein, then $\omega_R \cong R$, and the Betti sequence is symmetric by part (a). Conversely, suppose the Betti sequence of R is symmetric. Then $1 = \beta_0(R) = \beta_t(R) = \beta_0(\omega_R)$. Thus, ω_R is cyclic, so $\omega_R \cong R/J$ for some ideal J of R. But by Proposition 9.13(a), $\operatorname{Ann}_R \omega_R = (0)$. Hence, $J = 0$ and $\omega_R \cong R$. Thus, R is Gorenstein. □

The following theorem generalizes Matlis duality in the zero-dimensional case to maximal CM modules over a CM ring:

Theorem 9.20. *Let (R, m, k) be a d-dimensional CM local ring which possesses a canonical module ω_R. Let $(-)^\dagger$ denote the functor $\operatorname{Hom}_R(-, \omega_R)$. Then $(-)^\dagger$ is a dualizing functor on the category of finitely generated maximal CM modules. That is,*

1. *For any finitely generated MCM module M, $M^\dagger$ is also MCM.*

2. *Given any s.e.s. $0 \to A \to B \to C \to 0$ of finitely generated maximal CM modules, the sequence $0 \to C^\dagger \to B^\dagger \to A^\dagger \to 0$ is exact.*

3. *For any f.g. maximal CM module M the evaluation homomorphism $M \to M^{\dagger\dagger}$ is an isomorphism.*

Proof. Part (a) follows from Proposition 9.8. By the same proposition, we know $\mathrm{Ext}^1_R(C, \omega_R) = 0$, where C is as in part (b). Hence the s.e.s. is exact.

For part (c), let $\mathbf{x}$ be a maximal R-sequence. Then $\mathbf{x}$ is a regular sequence on any MCM. Let $\phi : M \to M^{\dagger\dagger}$ be the evaluation homomorphism. Since $M^{\dagger\dagger}$ is an MCM, it suffices to show $\phi \otimes R/(\mathbf{x})$ is an isomorphism by Lemma 9.9. Now, by Propositions 9.7 and 9.8,

$$M^{\dagger\dagger} \otimes_R R/(\mathbf{x}) \cong \mathrm{Hom}_{R/(\mathbf{x})}(\mathrm{Hom}_{R/(\mathbf{x})}(M/\mathbf{x}M, \omega_R/\mathbf{x}\omega_R), \omega_R/\mathbf{x}\omega_R).$$

Note that as $\mathbf{x}$ is a maximal R-sequence, $\omega_R/\mathbf{x}\omega_R \cong \omega_{R/(\mathbf{x})} \cong E_{R/(x)}(k)$. Let $(-)^v := \mathrm{Hom}_{R/(\mathbf{x})}(-, E_{R/(\mathbf{x})}(k))$ Then we have a commutative diagram

$$\begin{array}{ccc}
M \otimes_R R/(\mathbf{x}) & \xrightarrow{\phi \otimes R/(\mathbf{x})} & M^{\dagger\dagger} \otimes_R R/(\mathbf{x}) \\
\downarrow \cong & & \downarrow \cong \\
M/\mathbf{x}M & \longrightarrow & \mathrm{Hom}_{R/(\mathbf{x})}(\mathrm{Hom}_{R/(\mathbf{x})}(M/\mathbf{x}M, \omega_R/\mathbf{x}\omega_R), \omega_R/\mathbf{x}\omega_R) \\
\downarrow \mathrm{id} & & \downarrow \cong \\
M/\mathbf{x}M & \xrightarrow{\cong} & (M/\mathbf{x}M)^{vv}
\end{array}$$

Here, the bottom arrow is the evaluation homomorphism with $E_{R/(\mathbf{x})}(k)$, which is an isomorphism by Matlis duality. Thus, $\phi \otimes R/(\mathbf{x})$ is an isomorphism, and hence so is ϕ.

□

Corollary 9.21. *Let (R, m, k) be a CM local ring which possesses a canonical modules ω_R. For any f.g. MCM R-module M let $r(M)$ denote the CM type of M and $\mu(M)$ the minimal number of generators of M. Let $(-)^\dagger$ denote $\mathrm{Hom}_R(-, \omega_R)$. Then*

1. $r(M) = \mu(M^\dagger)$.
2. $\mu(M) = r(M^\dagger)$.

In particular, the type of R is equal to the minimal number of generators of ω_R.

Proof. We'll prove (b) first. Let $\mathbf{x}$ be a maximal regular sequence on R. Then

$$\begin{aligned}
r(M^\dagger) &= \dim_k \mathrm{Ext}^d_R(k, \mathrm{Hom}_R(M, \omega_R)) \\
&= \dim_k \mathrm{Hom}_{R/(\mathbf{x})}(k, \mathrm{Hom}_{R/(\mathbf{x})}(M/\mathbf{x}M, \omega_R/\mathbf{x}\omega_R)) \\
&= \dim_k \mathrm{Hom}_{R/(\mathbf{x})}(k \otimes_{R/(\mathbf{x})} M, E_{R/(\mathbf{x})}(k)) \\
&= \dim_k \mathrm{Hom}_{R/(\mathbf{x})}(k^{\mu(M)}, E_{R/(\mathbf{x})}(k)) \\
&= \dim_k k^{\mu(M)} \\
&= \mu(M).
\end{aligned}$$

For (a), note that $r(M) = r(M^{\dagger\dagger}) = \mu(M^\dagger)$ by part (b) applied to $M^\dagger$ in place of M.

□

Discussion 9.22. The duality induced by the canonical module is best appreciated on the level of complexes. To state the duality in that context, we first need to define the Hom complex of two complexes M and N. The (homological) degree n component of $\mathrm{Hom}_R(M,N)$ is given by

$$\mathrm{Hom}_R(M,N)_n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(M_p, N_{p+n}).$$

An element $\alpha \in \mathrm{Hom}_R(M,N)_n$ is called a homomorphism of degree n, and we write $|\alpha| = n$. The differential ∂^H on $\mathrm{Hom}_R(M,N)_n$ is defined as follows: For $\alpha \in \mathrm{Hom}_R(M,N)_n$,

$$\partial^H(\alpha) = \partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M.$$

One can easily check that $\partial^H \circ \partial^H = 0$. Note that α is a *chain map* from M to N if $|\alpha| = 0$ and $\partial^H(\alpha) = 0$. Thus, the cycles of degree 0 in $\mathrm{Hom}_R(M,N)$ are the chain maps of M to N. Also, if $|\alpha| = 0$, then α is null-homotopic if and only if there exists $s \in \mathrm{Hom}_R(M,N)_1$ such that $\partial^H(s) = \alpha$, i.e., α is a degree 0 boundary. Hence, $\mathrm{H}_0(\mathrm{Hom}_R(M,N))$ is the homotopy equivalence classes of chain maps from M to N.

An R-complex M is called *finite* and (homologically) *bounded* if $\mathrm{H}_i(M)$ is finitely generated for all i and $\mathrm{H}_i(M) = 0$ for all but finitely many i. (That is, the nonzero homology of M is finitely generated and concentrated in a finite interval.)

Theorem 9.23. *Let (R, m, k) be a CM local ring which possesses a canonical module ω_R. Let D be a minimal injective resolution for ω_R. Then the evaluation chain map*

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,D), D)$$

induces an isomorphism on homology for all finite bounded complexes M.

Proof. See Hartshorne, *Residues and Duality.* □

Thus we see that the duality holds much more generally than just for MCMs. In fact, one doesn't even need R to be CM. Suppose $R = S/I$ (not necessarily CM) where S is a Gorenstein local ring with $\dim S = \dim R$. Let D_S be a minimal injective resolution for S and let $D_R := \mathrm{Hom}_S(R, D_S)$. Then the above theorem holds with D_R in place of D (without the CM hypothesis).

10 The Frobenius functor

Throughout this section, when we say a ring R *has characteristic* p, we always mean p is a (positive) prime integer. Recall that in such rings, $(a+b)^p = a^p + b^p$ for all $a, b \in R$.

Definition 10.1. Let R be a ring of characteristic p. The ring homomorphism $f : R \to R$ given by $f(r) = r^p$ for all $r \in R$ is called the *Frobenius endomorphism* (or simply, the *Frobenius map*) on R.

Notation: For $e \geqslant 1$, $f^e : R \to R$ be the map f composed with itself e times. Hence, $f^e(r) = r^{p^e}$ for all $r \in R$. For an ideal I and $e \geqslant 1$, we let $I^{[p^e]}$ denote the ideal generated by $\{i^{p^e} \mid i \in I\}$, i.e., the ideal generated by $f^e(I)$ in R. For ease of notation and to avoid overuse of double superscripts, we will often denote a power of p by q. Thus, in the context of Frobenius map, q

will denote p^e for some e. For example, instead of writing $r^{p^e} \in I^{[p^e]}$ for all e, we will often write $r^q \in I^{[q]}$ for all q.

Tensoring along a ring homomorphism: Let $\phi : R \to S$ be a ring homomorphism (of any characteristic). For an R-module M and an S-module N, we let $M \otimes_\phi N$ denote the S-module $M \otimes_R N$, where N is viewed as an R-module via ϕ. So $(rm) \otimes n = m \otimes \phi(r)n$ for all $r \in R, m \in M, n \in N$. And just as we consider $M \otimes_R N$ as an S-module through N, we view $M \otimes_\phi N$ as an S-module through N as well. So for $s \in S$ we have $s(m \otimes n) = m \otimes (sn)$. Of course, this is how we normally interpret $M \otimes_R N$ in the situation where S is an R-algebra. However, this new notation becomes advantageous when we have *multiple actions* of R on S. For example, when $S = R$, ϕ could be the identity map, the Frobenius map, or some other endomorphism of R.

For an R-module homomorphism $g : L \to M$, the map $g \otimes 1_N : L \otimes_\phi N \to M \otimes_\phi N$ given by $(g \otimes 1_N)(\ell \otimes n) = g(\ell) \otimes n)$ is a homomorphism of S-modules. In this way for a fixed S-module N, $(-) \otimes_\phi N$ is a covariant functor from the category of R-modules to the category of S-modules.

The Frobenius functor: Now let R be a ring of characteristic p and $f : R \to R$ be the Frobenius map. The functor $\mathrm{F}_R(-) := (-) \otimes_f R$ is a covariant functor from the category of R-modules to itself and is called the *Frobenius functor* on R. For $e \geqslant 1$, $\mathrm{F}_R^e(-) = (-) \otimes_{f^e} R$ is the eth iteration of the Frobenius functor. Note that as F_R^e is defined as a tensor product, it is additive and right exact.

The Frobenius functor can be thought of this way: Let $f : R \to S$ be the Frobenius map, where $S = R$. The Frobenius functor is simply base change with S (where S is viewed as an R-algebra via f), followed by the forgetful functor where we ignore the R-action on S and recall that $S = R$ as rings (i.e., identifying the category of S-modules with the category of R-modules).

Lemma 10.2. *Let R be a ring of characteristic p and F_R^e the eth iteration of the Frobenius functor. Then*

(a) For any free R-module G, $\mathrm{F}_R^e(G) \cong G$ for all $e \geqslant 1$.

(b) Let $\phi : R^m \to R^n$ be given by multiplication by the matrix $A = (a_{ij})$. Then $\mathrm{F}_R^e(\phi) : R^m \to R^n$ is given by multiplication by $A^{[q]} := (a_{ij}^q)$, where $q = p^e$.

(c) For an ideal I of R, $\mathrm{F}_R^e(R/I) \cong R/I^{[q]}$.

Proof. It suffices to prove each part in the case $e = 1$. For (a), let $f : R \to S$, where $S = R$. Let $G = \oplus_\alpha R$. Then $\mathrm{F}_R(G) = G \otimes_R S \cong (\oplus_\alpha R) \otimes_R S \cong \oplus_\alpha S$. Since $S = R$, we see that $\mathrm{F}_R(G) \cong G$. For part (b), $\mathrm{F}_R(\phi) : S^m \to S^n$ is given by multiplication by the matrix $(a_{ij})S = (a_{ij}^p)$. Again recalling $S = R$, we obtain that $\mathrm{F}_R(\phi)$ is multiplication by $A^{[p]}$. For part (c), this is just a property of change of rings: $\mathrm{F}_R(R/I) = R/I \otimes_R S \cong S/IS$. Note that $IS = I^{[p]}$ as an ideal in S. Identifying S with R, we obtain the desired result. □

We next want to show that applying F_R^e to a finite free resolution of a finitely generated R-module M yields a finite free resolution of $\mathrm{F}_R^e(M)$. This is a powerful result with many important consequences. To prove this, we first need a celebrated lemma from the thesis of Peskine and Szpiro:

Proposition 10.3. [Peskine-Szpiro, *Lemma d'acyclicitè*, 1972] *Let (R, m) be a local ring and $T_\bullet$ a finite complex of finitely generated R-modules:*

$$0 \to T_s \xrightarrow{f_s} T_{s-1} \to \cdots \xrightarrow{f_1} T_0 \to 0.$$

Suppose

(i) $\operatorname{depth} T_i \geqslant i$ *for all* i*;*

(ii) For $i \geqslant 1$*, we have* $\mathrm{H}_i(T) = 0$ *or* $\operatorname{depth} \mathrm{H}_i(T) = 0$.

Then $\mathrm{H}_i(T) = 0$ *for all* $i \geqslant 1$.

Proof. For each $1 \leqslant i \leqslant s$, let $C_i = \operatorname{coker} f_{i+1} \cong T_i / \operatorname{im} f_{i+1}$. Let $1 \leqslant r \leqslant s$. We'll prove by descending induction on i that $\operatorname{depth} C_i \geqslant i$ and $\mathrm{H}_i(T) = 0$.

Base case: Suppose $i = s$. As $f_{s+1} = 0$, $C_s = T_s$. Hence, $\operatorname{depth} C_s = \operatorname{depth} T_s \geqslant s$ by assumption (i). Also, $\mathrm{H}_s(T) = \ker f_s \subseteq T_s$. Since $\operatorname{depth} T_s \geqslant s \geqslant 1$, any regular element on T_s is regular on $\mathrm{H}_s(T)$. Thus, $\operatorname{depth} \mathrm{H}_s(T) \geqslant 1$. But by (ii), $\operatorname{depth} \mathrm{H}_s(T) = 0$ or $\mathrm{H}_s(T) = 0$. Therefore, $\mathrm{H}_s(T) = 0$.

Inductive step: Suppose $1 \leqslant i < s$ and $\operatorname{depth} C_{i+1} \geqslant i + 1$ and $\mathrm{H}_{i+1}(T) = 0$. Note that as $\mathrm{H}_{i+1}(T) = 0$, $\operatorname{im} f_{i+2} = \ker f_{i+1}$. Thus,

$$C_{i+1} = T_{i+1} / \operatorname{im} f_{i+2} = T_{i+1} / \ker f_{i+1} \cong \operatorname{im} f_{i+1} \subseteq T_i.$$

Hence, we have an exact sequence

$$0 \to C_{i+1} \to T_i \to C_i \to 0,$$

where the first (nonzero) map is the composition $C_{i+1} \cong \operatorname{im} f_{i+1} \hookrightarrow T_i$. Applying $\operatorname{Hom}_R(R/m, -)$, we obtain the long exact sequence

$$\cdots \to \operatorname{Ext}^j_R(R/m, T_i) \to \operatorname{Ext}^j_R(R/m, C_i) \to \operatorname{Ext}^{j+1}_R(R/m, C_{i+1}) \to \cdots .$$

Since $\operatorname{depth} T_i \geqslant i$ and $\operatorname{depth} C_{i+1} \geqslant i+1$, we have $\operatorname{Ext}^j_R(R/m, T_i) = \operatorname{Ext}^{j+1}_R(R/m, C_{i+1}) = 0$ for all $j \leqslant i - 1$. Hence, $\operatorname{Ext}^j_R(R/m, C_i) = 0$ for all $j \leqslant i - 1$, which implies $\operatorname{depth} C_i \geqslant i$. Now let $K_i = \ker f_i$. Recalling that $C_{i+1} \cong \operatorname{im} f_{i+1}$, we have a short exact sequence

$$0 \to C_{i+1} \to K_i \to \mathrm{H}_i(T) \to 0.$$

Since $K_i \subseteq T_i$ and $\operatorname{depth} T_i \geqslant i \geqslant 1$, we see that $\operatorname{depth} K_i \geqslant 1$. (Any element regular on T_i is regular on K_i.) Thus, $\operatorname{Hom}_R(R/m, K_i) = 0$. Hence, we have an exact sequence

$$0 \to \operatorname{Hom}_R(R/m, \mathrm{H}_i(T)) \to \operatorname{Ext}^1_R(R/m, C_{i+1}) \to \operatorname{Ext}^1_R(R/m, K_i) \to \cdots .$$

Suppose $\mathrm{H}_i(T) \neq 0$. Then, by assumption (ii), $\operatorname{depth} \mathrm{H}_i(T) = 0$. Thus, $\operatorname{Hom}_R(R/m, \mathrm{H}_i(T)) \neq 0$. From the last exact sequence, we obtain that $\operatorname{Ext}^1_R(R/m, C_{i+1}) \neq 0$. This implies $\operatorname{depth} C_{i+1} = 1$, contradicting that $\operatorname{depth} C_{i+1} \geqslant i + 1 \geqslant 2$. Thus, we must have $\mathrm{H}_i(T) = 0$.

□

Corollary 10.4. *Let* (R, m) *be a local ring of depth* r *and suppose* $F_\bullet$ *is a complex of free* R*-modules of finite rank*

$$0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to F_0 \to 0$$

such that for all $i \geqslant 1$*,* $\mathrm{H}_i(F)$ *has finite length. If* $s \leqslant r$ *then* $\mathrm{H}_i(F) = 0$ *for all* $i \geqslant 1$.

Proof. Note that for any (nonzero) free R-module G, $\operatorname{depth} G = \operatorname{depth} R = r$. Hence, as $r \geqslant s$, $\operatorname{depth} F_i \geqslant i$ for all $1 \leqslant i \leqslant s$. Also, as $\mathrm{H}_i(F)$ has finite length for all $i \geqslant 1$, we have $\operatorname{depth} \mathrm{H}_i(F) = 0$ or $\mathrm{H}_i(F) = 0$ for all $i \geqslant 1$. The conclusion now follows from *Lemma d'acyclicitè*. □

Theorem 10.5. *Let (R, m) be a local ring of characteristic p and $f : R \to S$ the Frobenius map. Suppose M is a finitely generated R-module of finite projective dimension and $G_\bullet$ a minimal free resolution of M. Then*

(a) $\operatorname{Tor}_i^R(M, S) = 0$ *for all* $i > 0$.

(b) $\mathrm{F}_R(G_\bullet)$ *is a minimal free resolution of* $\mathrm{F}_R(M)$.

(c) $\operatorname{pd}_R \mathrm{F}_R(M) = \operatorname{pd}_R M$.

(d) $\operatorname{Ass}_R \mathrm{F}_R(M) = \operatorname{Ass}_R M$.

Proof. Suppose by way of contradiction that $\operatorname{Tor}_i^R(M, S) \neq 0$ for some $i \geqslant 1$. Note that $\operatorname{Tor}_i^R(M, S) = 0$ for $i > \operatorname{pd}_R M$ and let

$$N = \bigoplus_{i=1}^{\operatorname{pd}_R M} \operatorname{Tor}_i^R(M, S).$$

Observe that N is a nonzero finitely generated S-module. Let $p \in \operatorname{Min}_S N$. Then $N_p \neq 0$ and $\dim N_p = 0$. Hence, N_p has finite length (as an S_p-module). Note that $f \otimes_R R_p : R_p \to S_p$ is the Frobenius map for R_p and that $\operatorname{Tor}_i^{R_p}(M_p, S_p) \cong \operatorname{Tor}_i^R(M, S)_p$ has finite length for all $i \geqslant 1$. As $\operatorname{pd}_{R_p} M_p < \infty$ we can reset notation and assume $\operatorname{Tor}_i^R(M, S)$ has finite length for all $i \geqslant 1$ and is nonzero for at least one such i. Let T be a minimal free R-resolution of M. Then $F = T \otimes_R S$ is a complex of finitely generated free S-modules, $\mathrm{H}_i(F) = \operatorname{Tor}_i^R(M, S)$ has finite length for all $i \geqslant 1$, and $\mathrm{H}_i(F) \neq 0$ for some $i \geqslant 1$. Furthermore, the length of the F is $\operatorname{pd}_R M \leqslant \operatorname{depth} R = \operatorname{depth} S$. By Corollary 10.4, $\mathrm{H}_i(F) = 0$ for all $i \geqslant 1$, a contradiction. This proves (a).

By part (a), we have $\mathrm{H}_i(\mathrm{F}_R(G)) = \mathrm{H}_i(G \otimes_R S) = \operatorname{Tor}_i^R(M, S) = 0$ for all $i \geqslant 1$. Also, $\mathrm{F}_R(G)$ consists of finitely generated free R-modules in each degree. Thus, $\mathrm{F}_R(G)$ is a free R-resolution of $\mathrm{H}_0(\mathrm{F}_R(G)) = \mathrm{F}_R(M)$. To see that it is minimal, note that since G is minimal,

$$(\partial_i \otimes S)(G_i \otimes_R S) \subseteq mG_{i-1} \otimes_R S = m^{[p]}(G_{i-1} \otimes_R S)$$

for all i. Hence, $\mathrm{F}_R(\partial_i)(\mathrm{F}_R(G)_i) \subseteq m^{[p]} \mathrm{F}_R(G)_{i-1}$ for all i. This proves (b).

Part (c) follows immediately from (b). For (d), note that for any finitely generated R-module N of finite projective dimension, $m \in \operatorname{Ass}_R N$ if and only if $\operatorname{pd}_R N = \operatorname{depth} R$. Thus, for $p \in \operatorname{Spec} R$

$$\begin{aligned} p \in \operatorname{Ass}_R M &\iff pR_p \in \operatorname{Ass}_{R_p} M_p \\ &\iff \operatorname{pd}_{R_p} M_p = \operatorname{depth} R_p \\ &\iff \operatorname{pd}_{R_p} \mathrm{F}_{R_p}(M_p) = \operatorname{depth} R_p \\ &\iff \operatorname{pd}_{R_p} \mathrm{F}_R(M)_p = \operatorname{depth} R_p \\ &\iff pR_p \in \operatorname{Ass}_{R_p} \mathrm{F}_R(M)_p \\ &\iff p \in \operatorname{Ass}_R \mathrm{F}_R(M) \end{aligned}$$

□

Corollary 10.6. *Let R be a regular ring of characteristic p. Then F_R^e is an exact functor for all e. Equivalently, $f^e : R \to R$ is flat ring homomorphism for all e.*

Proof. It suffices to prove the case $e = 1$. Let $f : R \to S$ be the Frobenius map. We wish to show S is a flat R-module. To show S is flat, $\mathrm{Tor}_1^R(M, S) = 0$ for all R-modules M. In fact, one just needs to show this for finitely generated R-modules M. (This is left as an exercise.) It suffices to show $\mathrm{Tor}_1^R(M, S)_p = 0$ for all prime ideals p. Thus, we may assume R is a regular local ring, in which case M has finite projective dimension. Thus, $\mathrm{Tor}_1^R(M, S) = 0$ by Theorem 10.5. □

Remark 10.7. The converse of the above Corollary is also true in a strong form: If f^e is a flat ring homomorphism for some $e \geqslant 1$ then R is regular. This was proved by E. Kunz in 1969. It was greatly generalized in a theorem by Avramov, Hochster, Iyengar and Yao published in 2012.

Lemma 10.8. *Let $f : R \to S$ be a flat ring homomorphism.*

(a) For any ideal I of R, $I \otimes_R S \cong IS$.

(b) For any finitely generated R-module M, $\mathrm{Ann}_S(M \otimes_R S) = (\mathrm{Ann}_R M)S$.

(c) Let I and J be ideals of R, with J finitely generated. Then $(I :_R J)S = (IS :_S JS)$.

Proof. Applying $- \otimes_R S$ to the exact sequence $0 \to I \to R \to R/I \to 0$, we have

$$0 \to I \otimes_R S \xrightarrow{f} S \xrightarrow{g} S/IS \to 0$$

is exact. Thus, $I \otimes_R S \cong \mathrm{im}\, f = \ker g = IS$.

For (b), let $M = Rx_1 + \cdots + Rx_n$. Then there is an exact sequence $0 \to R/\mathrm{Ann}_R M \xrightarrow{f} M^n$ given by $f(\overline{1}) = (x_1, \ldots, x_n)$. Tensoring with S and noting that $M \otimes_R S$ is generated by $x_1 \otimes 1, \cdots, x_n \otimes 1$, we obtain the desired result.

For part (c), let $M = (J + I)/I$. Then M is finitely generated and $\mathrm{Ann}_R M = (I :_R J)$. By (b), $\mathrm{Ann}_S(M \otimes_R S) = (I :_R J)S$. On the other hand, applying $- \otimes_R S$ to the exact sequence $0 \to I \to I + J \to M \to 0$ and using part (a), we have $0 \to IS \to (I + J)S \to M \otimes_R S \to 0$ is exact. Thus, $M \otimes_R S \cong (IS + JS)/IS$. Hence, $\mathrm{Ann}_S M \otimes_R S = (IS :_S JS)$. This completes the proof.

□

Proposition 10.9. *Let (R, m) be a regular local ring of characteristic p and I an ideal of R. Then*

(a) For any $x \in R$, $(I^{[q]} :_R x^q) = (I :_R x)^{[q]}$ for all $q = p^e$.

(b) $\mathrm{Ass}_R R/I^{[q]} = \mathrm{Ass}_R R/I$ for all q. In particular, for any prime ideal P of R, $P^{[q]}$ is P-primary for all q.

Proof. Let $q = p^e$. Since R is regular, the Frobenius map $f^e : R \to S$ is flat. Since $IS = I^{[q]}$ for all ideals I of R, we have by Lemma 10.8 that $(I :_R x)^{[q]} = (I :_R x)S = (IS :_S xS) = (I^{[q]} :_R x^q)$. This proves (a). For (b), note that $F_R^e(R/I) \cong R/I^{[q]}$ by Lemma 10.2. One now invokes part (d) of Theorem 10.5. □

Definition 10.10. Let R be a ring and I an ideal of R. An element $u \in R$ is *integral* over I if there exists an equation of the form

$$u^n + r_1 u^{n-1} + \cdots + r_{n-1} u + r_n = 0$$

for some n and with $r_i \in I^i$ for $i = 1, \ldots, n$. The set of all elements integral over I is called the *integral closure* of I and is denoted $\overline{I}$.

Notation: For a ring R, let $R^o := \{c \in R \mid c \notin p \text{ for all } p \in \operatorname{Min}_R R\}$.

Remark 10.11. *Let R be a ring and I an ideal of R. Then*

(a) $\overline{I}$ is an ideal of R containing I.

(b) $\overline{\overline{I}} = \overline{I}$.

(c) If R is Noetherian, then $u \in \overline{I}$ if and only if there exists $c \in R^o$ such that $cu^n \in I^n$ for all n sufficiently large.

Proof. See, for example, *Integral Closure of Ideals, Rings, and Modules*, by I. Swanson and C. Huneke. □

Definition 10.12. Let R be a Noetherian ring of characteristic p and I an ideal. An element $u \in R$ is in the *tight closure* of I, written $u \in I^*$, if there exists $c \in R^o$ such that $cu^q \in I^{[q]}$ for q sufficiently large.

It is easily seen from the definition that I^* is an ideal of R containing I. And by part (d) of Remark 10.11, we have $I^* \subseteq \overline{I}$. It is left as an exercise to prove $(I^*)^* = I^*$ for any ideal I of R.

One of the most important properties comes from the following remarkable observation:

Theorem 10.13. *Let (R, m) be a regular local ring and I an ideal of R. Then $I = I^*$ for every ideal I of R.*

Proof. Let $u \in I^*$. Suppose $u \notin I$. Then $(I :_R u) \subseteq m$. Now by definition, there exists $c \in R^o$ such that $cu^q \in I^{[q]}$ for all q sufficiently large. By Proposition 10.9, for q sufficiently large, $c \in (I^{[q]} :_R u^q) = (I :_R u)^{[q]} \subseteq m^{[q]} \subseteq m^q$. By Krull's intersection theorem, this implies $c = 0$, a contradiction. □

We give an application to the containment problem for symbolic powers. Recall that for a prime p in a Noetherian ring R, the nth symbolic power of p, denoted $p^{(n)}$, is defined to be the p-primary component of p^n; equivalently, $p^{(n)} = \phi^{-1}(p^n R_p)$ where $\phi : R \to R_p$ is the localization map. It is elementary to see that $p^n \subseteq p^{(n)}$ for all n and that $p^{(m)} p^{(n)} \subseteq p^{(m+n)}$ for all integers n and m. In many cases, it is known that there exists an integer k (depending on p) such that $p^{(kn)} \subseteq p^n$ for all n. However, it's not always easy to find such a k (if it exists), and also answer the question of whether there exists a single k which works for all primes p. However, there is a satisfying answer for regular local rings containing a field which says that k may be taken to be the height of the prime p. This was initially proved in the characteristic zero case by Ein, Lazersfeld, and Smith in 2000. Hochster and Huneke proved the characteristic p case shortly afterward:

Theorem 10.14. [Hochster-Huneke, 2002] *Let (R, m) be a regular local ring of prime characteristic and let p a prime of height h. Then for all $n \geqslant 1$, $p^{(hn)} \subseteq p^n$. In particular, if $d = \dim R$ the $p^{(dn)} \subseteq p^n$ for all n.*

We first need a couple elementary facts:

Lemma 10.15. *Let R be a Noetherian ring of characteristic p and I an ideal. Then*

(a) For all n and $q = p^e$, $(I^{[q]})^n = (I^n)^{[q]}$.

(b) If I is generated by ℓ elements then $I^{\ell q} \subseteq I^{[q]}$ for all q.

Proof. For (a), suppose $I = (a_1, \ldots, a_\ell)$. Then $I^{[q]}$ is generated by elements of the form a_i^q. Hence, $(I^{[q]})^n$ is generated by monomials of the form $(a_1)^{qm_1} \cdots (a_\ell)^{qm_\ell}$ where $m_1 + \cdots + m_\ell = n$. On the other hand, I^n is generated by monomials of the form $a_1^{m_1} \cdots a_\ell^{m_\ell}$ such that $\sum_i m_i = n$. Then $(I^n)^{[q]}$ is generated by the qth powers of these monomials, which gives us the same generators as for $(I^{[q]})^n$.

For (b), let $I = (a_1, \ldots, a_\ell)$. Note that $I^{\ell q}$ is generated by monomials of the form $u = a^{r_1} \cdots a^{r_\ell}$ with $\sum_i r_i = \ell q$. But this implies that $r_j \geqslant q$ for some j. Hence, $u \in (a_j^q) \subseteq I^{[q]}$. □

Proof of Theorem 10.14: We may assume $p \neq (0)$. Let $u \in p^{(hn)}$ for some n. Let q be given, and write $q = an + r$ where $a, r \in \mathbb{Z}$ and $0 \leqslant r \leqslant n - 1$. Then $u^a \in (p^{(hn)})^a \subseteq p^{(ahn)}$. So $p^{hn} u^a \subseteq p^{(ahn+hn)} \subseteq p^{(ahn+hr)} = p^{(hq)}$.

Claim: $p^{(hq)} \subseteq p^{[q]}$.

Proof of Claim: Note that $\operatorname{Ass}_R(p^{(hq)} + p^{[q]})/p^{[q]} \subseteq \operatorname{Ass}_R R/p^{[q]} = \{p\}$ by Proposition 10.9(b). Thus, it suffices to show the Claim holds locally at p. But pR_p is generated by h elements (as R_p is a RLR), so $p^{(hq)} R_p = (pR_p)^{hq} \subseteq (pR_p)^{[q]} = (p^{[q]}) R_p$ by Lemma 10.15. This proves the Claim.

Thus, we have $p^{hn} u^a \in p^{[q]}$. Taking nth powers, we have $p^{hn^2} u^{na} \in (p^{[q]})^n = (p^n)^{[q]}$ by Lemma 10.15. As $q \geqslant na$, we have $p^{hn^2} u^q \in (p^n)^{[q]}$. Now, as R is a domain and $p \neq 0$, this implies $u \in (p^n)^* = p^n$ by Theorem 10.13.

11 Completions

Let R be a ring and M an R-module. A *filtration* $\mathcal{F} = \{M_n\}_{n \geqslant 0}$ of M is a descending chain of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots.$$

Such a filtration induces an inverse system of R-modules

$$M/M_0 \overset{\pi_1}{\longleftarrow} M/M_1 \overset{\pi_2}{\longleftarrow} M/M_2 \leftarrow \cdots$$

where $\pi_n(m + M_n) = m + M_{n-1}$ for all n. The completion $\widehat{M}^{\mathcal{F}}$ of M with respect to $\mathcal{F}$ is defined by

$$\widehat{M}^{\mathcal{F}} := \varprojlim M/M_n.$$

Recall that from Grifo's 915 notes (Theorem 1.67) that

$$\begin{aligned}
\varprojlim M/M_n &= \{(m_n + M_n) \in \prod_{n \geqslant 0} M/M_n \mid \pi_n(m_n + M_n) = m_{n-1} + M_{n-1},\ \forall\, n\} \\
&= \{(m_n + M_n) \in \prod_{n \geqslant 0} M/M_n \mid m_n - m_{n-1} \in M_{n-1},\ \forall\, n\}.
\end{aligned}$$

For each filtration $\mathcal{F}$ there is a canonical map $\phi_M^{\mathcal{F}} : M \to \widehat{M}^{\mathcal{F}}$ given by $\phi_M^{\mathcal{F}}(m) = (m+M_n)_{n\geqslant 0}$. If $\phi_M^{\mathcal{F}}$ is injective, i.e. $\cap_n M_n = 0$, then M is said to be *separated* with respect to $\mathcal{F}$. If ϕ_M^F is an isomorphism, we say that M is *complete* with respect to $\mathcal{F}$.

The following is an elementary exercise on inverse limits:

Lemma 11.1. *Let M be an R-module and $\mathcal{F} = \{M_n\}$ a filtration of M.*

(a) If $M_n = 0$ for some n then $\phi_M^{\mathcal{F}} : M \to \widehat{M}^{\mathcal{F}}$ is an isomorphism.

(b) If $\mathcal{F}' = \{M'_n\}$ is a filtration of M which is cofinal with $\mathcal{F}$, i.e., for all n there exists k such that $M'_{n+k} \subseteq M_n$ and $M_{n+k} \subseteq M'_n$, then $\widehat{M}^{\mathcal{F}} \cong \widehat{M}^{\mathcal{F}'}$.

Proposition 11.2. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules and $\mathcal{F} = \{B_n\}$ a filtration of B. Let $\mathcal{F}' = \{f^{-1}(B_n)\}$ and $\mathcal{F}'' = \{g(B_n)\}$ be the induced filtations of A and C, respectively. Then*

$$0 \to \widehat{A}^{\mathcal{F}'} \to \widehat{B}^{\mathcal{F}} \to \widehat{C}^{\mathcal{F}''} \to 0$$

is exact.

Proof. Let M an R-module and $\mathcal{G} = \{M_n\}$ a filtration of M. Let $\widetilde{M}^{\mathcal{G}} := \prod_n M/M_n$ and define $d^{\widetilde{M}} : \widetilde{M}^{\mathcal{G}} \to \widetilde{M}^{\mathcal{G}}$ by $d^{\widetilde{M}}((m_n+M_n)) = ((m_n - m_{n+1}) + M_n)$. One can check that $d^{\widetilde{M}}$ is a surjective R-module homomorphism and $\ker d^{\widetilde{M}} = \widehat{M}^{\mathcal{G}}$. We then have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A}^{\mathcal{F}'} & \xrightarrow{\tilde{f}} & \widetilde{B}^{\mathcal{F}} & \xrightarrow{\tilde{g}} & \widetilde{C}^{\mathcal{F}''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{\widetilde{A}}} & & \downarrow{\scriptstyle d^{\widetilde{B}}} & & \downarrow{\scriptstyle d^{\widetilde{C}}} & & \\
0 & \longrightarrow & \widetilde{A}^{\mathcal{F}'} & \xrightarrow{\tilde{f}} & \widetilde{B}^{\mathcal{F}} & \xrightarrow{\tilde{g}} & \widetilde{C}^{\mathcal{F}''} & \longrightarrow & 0
\end{array}$$

where $\tilde{f}((a_n + f^{-1}(B_n))) = ((f(a_n) + B_n))$ and $\tilde{g}((b_n + B_n)) = (g(b_n) + g(B_n)))$. It is easily seen that the rows are exact (one can check this component-wise). The result now follows by the snake lemma.

□

Definition 11.3. Let M be an R-module and I an ideal of R. Then $\mathcal{F} = \{I^n M\}_{n\geqslant 0}$ is called the *I-adic filtration* of M, and $\widehat{M}^I := \varprojlim M/I^n M$ is called the *I-adic completion* of M. When there is no possibility of confusion, we'll denote $\widehat{M}^I$ by $\widehat{M}$. When the canonical map $\phi_M^I : M \to \widehat{M}$ is an isomorphism, we say that M is *I-adically complete*.

Example 11.4. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in d variables over a field k and let $m = (x_1, \ldots, x_d)$. Then the m-adic completion of R is the ring of formal power series $k[[x_1, \ldots, x_d]]$. The proof is left as an exercise.

Example 11.5. Let p be a prime integer. Then the (p)-adic completion of $\mathbb{Z}$ is the ring of *p-adic integers*.

Remark 11.6. Let I be an ideal of R and $f : M \to N$ a surjective homomorphism of R-modules. Then $\hat{f} : \widehat{M}^I \to \widehat{N}^I$ is surjective. This follows from Proposition 11.2, since the image of the I-adic filtration of M under f is the I-adic filtration on N. However, I-adic completion need not be right exact, even over Noetherian rings. (Examples are difficult, though.) Also, I-adic completion need not preserve injections. Consider the inclusion $\mathbb{Z} \to \mathbb{Q}$ and $I = 2\mathbb{Z}$. Then $\widehat{\mathbb{Q}}^I = \varprojlim \mathbb{Q}/2\mathbb{Q} = 0$, but $\widehat{\mathbb{Z}}^I \neq 0$. (One can see this by noting $\widehat{\mathbb{Z}}^I/\widehat{2\mathbb{Z}}^I \cong \mathbb{Z}/2\mathbb{Z} \neq 0$ using the Proposition below.)

Proposition 11.7. *Let M be an R-module and I and ideal. Let $\widehat{(-)}$ denote the I-adic completion functor.*

(a) If $N \supseteq I^k M$ for some R-submodule N of M and some $k \geqslant 0$, then $\widehat{M/N} \cong M/N$.

(b) $\widehat{M}/\widehat{I^k M} \cong M/I^k M$ for all $k \geqslant 0$.

(c) $\widehat{M}$ is complete with respect to the filtration $\{\widehat{I^n M}\}$.

Proof. Part (a) follows from part (a) of Lemma 11.1, since $I^k(M/N) = 0$.

For part (b), for each $k \geqslant 0$ we have an exact sequence $0 \to I^k M \xrightarrow{f} M \xrightarrow{g} M/I^k M \to 0$. Let $\mathcal{F} = \{I^n M\}$ be the I-adic filtration of M. Then $\mathcal{F}' = \{f^{-1}(I^n M)\} = \{I^k M \cap I^n M\}$, is cofinal with the I-adic filtration of $I^k M$. Therefore, $\widehat{I^k M}^{\mathcal{F}'} \cong \widehat{I^k M}$ by part (b) of Lemma 11.1. By part (a), we have $\widehat{M/I^k M} \cong M/I^k M$. Thus, by Proposition 11.2, we have

$$0 \to \widehat{I^k M} \to \widehat{M} \to M/I^k M \to 0,$$

is exact. This proves (b). Part (c) follows from part (b) and the definition of completion. □

Remark 11.8. Let R be a ring and $\mathcal{F} = \{I_n\}$ a filtration on R. Since the projection maps $\pi_n : R/I_n \to R/I_{n-1}$ are ring homomorphisms, it is easy to see that $\widehat{R}^{\mathcal{F}} = \varprojlim R/I_n$ is a (commutative) ring and the canonical map $\phi_R^{\mathcal{F}} : R \to \widehat{R}^{\mathcal{F}}$ is a ring homomorphism. For any R-module M and ideal I, it is straightforward to show that $\widehat{M}^I$ is an $\widehat{R}^I$-module. Thus, I-adic completion is a (covariant) functor from the category of R-modules to the category of $\widehat{R}^I$-modules.

Proposition 11.9. *Let R be a ring and I an ideal. Let $\widehat{(-)}$ denote I-adic completion. Then*

(a) For any R-module, $\widehat{I^m} \cdot \widehat{M} \subseteq \widehat{I^m M}$ for all m.

(b) For all nonnegative integers m and n, $\widehat{I^m} \cdot \widehat{I^n} \subseteq \widehat{I^{m+n}}$.

(c) $\widehat{I}$ is contained in the Jacobson radical of $\widehat{R}$.

Proof. Note that $M/I^m M$ is an R/I^m-module. Thus, $\widehat{M/I^m M} \cong \widehat{M}/\widehat{I^m M}$ is an $\widehat{R/I^m} \cong \widehat{R}/\widehat{I^m}$-module (where we have used (a) and (b) of Proposition 11.7). Thus, $\widehat{I^m} \cdot \widehat{M}/\widehat{I^m M} = 0$, and so $\widehat{I^m} \cdot \widehat{M} \subseteq \widehat{I^m M}$. This proves (a). Part (b) follows from (a) by letting $M = I^n$.

To prove (c), it suffices to show that $1 - x$ is a unit for every $x \in \widehat{I}$. So let $x \in \widehat{I}$. By part (c) of Proposition 11.7, the canonical map $\phi : \widehat{R} \to \varprojlim \widehat{R}/\widehat{I^n}$ is an isomorphism. It suffices to prove $\phi(1 - x) = (1 - x + \widehat{I^n})_n$ is a unit in $\varprojlim \widehat{R}/\widehat{I^n}$. Since $x \in \widehat{I}$, $x^j \in (\widehat{I})^j \subseteq \widehat{I^j}$ for all j by part (b). Thus, the element $u = (1 + x + \cdots x^{n-1} + \widehat{I^n})_n \in \varprojlim \widehat{R}/\widehat{I^n}$. Then $\phi(x)u = (1 - x^n + \widehat{I^n})_n = (1 + \widehat{I^n})_n$, showing that $\phi(x)$ is a unit.

□

We next show that, under the hypothesis that R is Noetherian, I-adic completion is exact on the category of finitely generated R-modules. To prove this, we need the following important result:

Theorem 11.10. [The Artin-Rees Lemma] *Let R be a Noetherian ring, I an ideal, M a finitely generated R-module and N a submodule of M. Then there exists an integer k such that*

$$I^nM \cap N = I^{n-k}(I^kM \cap N)$$

for all $n \geqslant k$.

Proof. Let $I = (a_1, \dots, a_r)$ and t an indeterminate over R. Consider the subring of $S = R[a_1t, \dots, a_rt]$ of $R[t]$. Setting $\deg t = 1$ and the degree of R equal to 0, we see that S is a graded subring of $R[t]$ generated over R by homogenous of elements of degree 1. As R is Noetherian and S is a finitely generated R-algebra, we have that S is Noetherian. Observe that the degree n component of S is I^nt^n. Thus,

$$S = R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$$

S is called the *Rees ring* of I. Now consider the module $M[t] = M \otimes_R R[t]$. Then $M[t]$ is a the graded $R[t]$-module $M \oplus Mt \oplus Mt^2 \oplus \cdots$. Of course, $M[t]$ is also a graded S-module by restriction of scalars. Note that $N[t]$ is a graded S-submodule of $M[t]$. Let A be the S-submodule of $M[t]$ generated by the elements of M (i.e., the degree zero component of $M[t]$:

$$A := R[It]M = M \oplus IMt \oplus I^2Mt^2 \oplus \cdots,$$

which is called the *Rees module* of I and M. Since M is a finitely generated R-module, A is a finitely generated S-module and thus is Noetherian (as an S-module). Now consider the S-module $B = A \cap N[t]$. Then

$$B = M \oplus (IM \cap N)t \oplus (I^2M \cap N)t^2 \cdots.$$

Since A is a Noetherian S-module and B is an S-submodule of A, we obtain that B is finitely generated as an S-module. As B is graded, B can be generated over S by finitely many homogeneous elements, say $u_1t^{m_1}, \dots, u_\ell t^{m_\ell}$. Without loss of generality, we may assume $m_i \leqslant m_{i+1}$ for all i. Let $k = m_\ell$. We claim that $I^nM \cap N = I^{n-k}(I^kM \cap N)$ for all $n \geqslant k$. It is easy to see that $I^{n-k}(I^kM \cap N) \subseteq I^nM \cap N$. Let $u \in I^nM \cap N$ where $n \geqslant k$. Then $ut^n \in B$. Thus, we can write ut^n in terms of the homogeneous generators of B as an S-module. Hence, there exists $s_it^{n-m_i} \in S$ for $i = 1, \dots, \ell$ such that

$$\begin{aligned} ut^n &= (s_1t^{n-m_1})(u_1t^{m_1}) + \cdots + (s_\ell t^{n-m_\ell})(u_\ell t^{m_\ell}) \\ &= (s_1u_1 + \cdots + s_\ell u_\ell)t^n \end{aligned}$$

Hence, $u = s_1u_1 + \cdots + s_\ell u_\ell$. Now, as $s_it^{n-m_i} \in S$, we have $s_i \in I^{n-m_i}$ for each i. Similarly, as $u_it^{m_i} \in B$, we have $u_i \in I^{m_i}M \cap N$. Consequently,

$$u \in \sum_{i=1}^{\ell} I^{n-m_i}(I^{m_i}M \cap N).$$

But since $I^r(I^pM \cap N) \subseteq I^{r-1}(I^{p+1}M \cap N)$ for all integers r and p, we have

$$I^{n-m_i}(I^{m_i}M \cap N) \subseteq I^{n-m_\ell}(I^{m_\ell}M \cap N) = I^{n-k}(I^kM \cap N).$$

Therefore, $u \in I^{n-k}(I^kM \cap N)$.

□

Theorem 11.11. *Let R be Noetherian and I an ideal of R. Let $\widehat{(-)}$ denote I-adic completion. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence of finitely generated R-modules. Then*

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

is exact.

Proof. By identifying A with its image in B, we may assume A is a submodule of B. The I-adic filtration of B induces the filtration $\{I^n B \cap A\}$ on A, which is cofinal with the I-adic filtration of A by the Artin-Rees lemma. Applying Proposition 11.2, we obtain the desired exact sequence. □

Proposition 11.12. *Let R be a Noetherian ring, I an ideal of R, and $\widehat{(-)}$ the I-adic completion functor. For any finitely generated R-module, $\widehat{R} \otimes_R M \cong \widehat{M}$.*

Proof. Let $F \cong R^n$ be a finitely generated free R-module. Then

$$\widehat{F} = \varprojlim F/I^n F \cong \bigoplus_{i=1}^{n} \varprojlim R/I^n \cong (\widehat{R})^n \cong \widehat{R} \otimes_R R^n \cong \widehat{R} \otimes_R F.$$

Now let M be a finitely generated R-module and $F \to G \to M \to 0$ a presentation of M by finitely generated free R-modules. Then we have a commutative diagram

$$\begin{array}{ccccccc} \widehat{F} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \\ \big\downarrow{\cong} & & \big\downarrow{\cong} & & & & \\ \widehat{R} \otimes_R F & \longrightarrow & \widehat{R} \otimes_R G & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \end{array}$$

where the top row is exact by Theorem 11.11. By a diagram chase, there exists a unique map $\psi : \widehat{M} \to \widehat{R} \otimes_R M$ which makes the diagram commute. By the Five Lemma, ψ is an isomorphism. □

Lemma 11.13. *Let R be a ring and M an R-module. Then M is flat if and only if for every ideal I of R, the map $I \otimes_R M \to R \otimes_R M$ is injective.*

Proof. The "only if" direction is clear. Suppose $I \otimes_R M \to R \otimes_R M$ is injective for every ideal I of R. Let $0 \to A \xrightarrow{f} B$ be an injective homomorphism of arbitrary R-modules and let $K = \ker f \otimes 1_M$. We wish to show $K = 0$. As it suffices to show $K_m = 0$ for every maximal ideal of R, we may assume R is a quasi-local ring with maximal ideal m. Let $E = E_R(R/m)$ and $(-)^{\vee} = \mathrm{Hom}_R(-, E)$. Applying $(-)^{\vee}$ to the exact sequence $0 \to K \to A \otimes_R M \to B \otimes_R M$, we have $(B \otimes_R M)^{\vee} \to (A \otimes_R M)^{\vee} \to K^{\vee} \to 0$ is exact. By adjunction, this sequence is naturally isomorphic to

$$\mathrm{Hom}_R(B, M^{\vee}) \to \mathrm{Hom}_R(A, M^{\vee}) \to K^{\vee} \to 0.$$

By assumption, $K = 0$ when $B = R$ and $A = I$ is an ideal of B. Hence, $\mathrm{Hom}_R(R, M^{\vee}) \to \mathrm{Hom}_R(I, M^{\vee}) \to 0$ is exact for all ideals I of R. Equivalently, $\mathrm{Ext}^1_R(R/I, M^{\vee}) = 0$ for all ideals I. By Baer's Criterion, this means that $M^{\vee}$ is injective. Hence, $\mathrm{Hom}_R(B, M^{\vee}) \to \mathrm{Hom}_R(A, M^{\vee})$ is surjective. Therefore, $K^{\vee} = 0$, which implies $K = 0$. Hence, M is flat.

□

Corollary 11.14. *Let R be a Noetherian ring, I an ideal, and $\widehat{R}$ the I-adic completion of R.*

(a) $\widehat{R}$ *is a flat* R*-module.*

(b) $\widehat{R}$ *is faithfully flat if and only if* I *is contained in the Jacobson radical of* R*.*

Proof. Let J be an ideal of R. Then $0 \to \widehat{J} \to \widehat{R}$ is exact by Theorem 11.11. Thus, $0 \to \widehat{R} \otimes_R J \to \widehat{R} \otimes_R R$ by Proposition 11.12. Thus, $\widehat{R}$ is flat by Lemma 11.13.

For part (b), suppose I is contained in the Jacobson radical of R. Let m be a maximal ideal of R. Then

$$\widehat{R}/m\widehat{R} \cong \widehat{R} \otimes_R R/m \cong \widehat{R/m} \cong R/m,$$

where the last isomorphism is by part (a) of Proposition 11.7. Thus, $m\widehat{R} \neq \widehat{R}$ for all maximal ideals m of R. Hence $\widehat{R}$ is faithfully flat over R. (See Exercise 61 of Grifo's 915 Notes.) Conversely, suppose I is not contained in some maximal ideal m of R. Then $I^n + m = R$ for all n. Hence, $\widehat{R}/m\widehat{R} \cong \widehat{R/m} \cong \varprojlim R/(I^n + m) = 0$, so $m\widehat{R} = \widehat{R}$. □

Convention: For the rest of this section, we will narrow our focus to the situation where (R, m) is a local ring and $\widehat{(-)}$ denotes the m-adic completion functor. In this context, when we say a ring or a module is *complete*, we mean complete with respect to the m-adic filtation.

Lemma 11.15. *Let* (R, m) *be a local ring,* J *an ideal, and* M *a finitely generated* R*-module. As above, let* $\widehat{(-)}$ *denote* m*-adic completion. Then* $J\widehat{M} = \widehat{J}\widehat{M} = \widehat{JM}$*.*

Proof. Tensoring the exact sequence $0 \to J \to R \to R/J \to 0$ with $\widehat{R}$ we obtain the exact sequence $0 \to \widehat{J} \to \widehat{R} \to \widehat{R}/J\widehat{R} \to 0$. From this, we deduce that $J\widehat{R} = \widehat{J}$. Hence, $\widehat{J}\widehat{M} = J\widehat{R}\widehat{M} = J\widehat{M}$. Now applying $(-) \otimes_{\widehat{R}} \widehat{M}$ to the second exact sequence above, we obtain

$$\widehat{J} \otimes_{\widehat{R}} \widehat{M} \to \widehat{M} \to \widehat{M}/\widehat{JM} \to 0,$$

where here we have used that

$$\widehat{R}/\widehat{J} \otimes_{\widehat{R}} \widehat{M} \cong (R/J \otimes_R \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R M) \cong \widehat{R} \otimes_R R/J \otimes_R M \cong \widehat{M}/\widehat{JM}.$$

From the exact sequence, we conclude that $\widehat{J}\widehat{M} = \widehat{JM}$. □

Proposition 11.16. *Let* (R, m) *be a Noetherian local ring. Then* $\widehat{R}$ *has a unique maximal ideal, namely* $\widehat{m} = m\widehat{R}$*. Furthermore:*

(a) $m^n\widehat{R} = (\widehat{m})^n = \widehat{m^n}$ *for all* n*;*

(b) $\widehat{R}/\widehat{m}^n \cong R/m^n$ *for all* n*.*

(c) $\widehat{m}^n/\widehat{m}^{n+1} \cong m^n/m^{n+1}$ *for all* n*.*

(d) $\widehat{R}$ *is complete with respect to the* $\widehat{m}$*-adic topology.*

Proof. As $\widehat{R}/\widehat{m} \cong \widehat{R/m} \cong R/m$ is a field, we see that $\widehat{m}$ is a maximal ideal of $\widehat{R}$. By part (c) of Proposition 11.9, $\widehat{m}$ is contained in the Jacobson radical of $\widehat{R}$. Hence, $\widehat{m}$ is the unique maximal ideal of $\widehat{R}$.

Part (a) follows from Lemma 11.15 and induction (with $J = m$ and $M = m^{n-1}$).

For part (b), observe that

$$\widehat{R}/\widehat{m}^n \cong \widehat{R}/\widehat{m^n} \cong \widehat{R/m^n} \cong R/m^n$$

with the last isomorphism following from part (a) of Proposition 11.7.

For part (c), note that

$$\widehat{m}^n/\widehat{m}^{n+1} \cong \widehat{m^n}/\widehat{m^{n+1}} \cong \widehat{m^n/m^{n+1}} \cong m^n/m^{n+1}.$$

Finally, part (d) follows immediately from Proposition 11.7(c). □

Theorem 11.17. *Let (R, m) be a local ring. Then $\widehat{R}$ is local.*

Proof. As we already have proved $\widehat{R}$ has a unique maximal ideal, it suffices to prove that $\widehat{R}$ is Noetherian. See Theorem 10.26 of Atiyah-Macdonald. □

Proposition 11.18. *Let (R, m) be a local ring and M a finitely generated R-module.*

(a) If $\dim M = 0$ then $M \cong \widehat{M}$.

(b) If R is complete then $\widehat{M} \cong M$. In particular, R/I is a complete local ring for any ideal I of R.

Proof. For part (a), we have $m^n M = 0$ for some n. Hence, $\widehat{M} \cong M$ by Lemma 11.1(a). For part (b), as R is complete, $R \cong \widehat{R}$. Thus, $\widehat{M} \cong \widehat{R} \otimes_R M \cong R \otimes_R M \cong M$. □

Theorem 11.19. *Let (R, m) be a local ring and M a finitely generated R-module. Then*

(a) $\dim M = \dim \widehat{M}$.

(b) $\beta_i^R(M) = \beta_i^{\widehat{R}}(\widehat{M})$ *for all i.*

(c) $\mu_i(m, M) = \mu_i(\widehat{m}, \widehat{M})$ *for all i.*

(d) $\operatorname{depth}_R M = \operatorname{depth}_{\widehat{R}} \widehat{M}$.

(e) $\operatorname{id}_R M = \operatorname{id}_{\widehat{R}} \widehat{M}$.

(f) $\operatorname{pd}_R M = \operatorname{pd}_{\widehat{R}} \widehat{M}$.

(g) $\mu_R(M) = \mu_{\widehat{R}}(\widehat{M})$.

Proof. Note that $k := R/m \cong R/m \otimes_R \widehat{R} \cong \widehat{R}/\widehat{m}$. Hence, as $\widehat{R}$ is faithfully flat over R, $\lambda_R(N) = \lambda_{\widehat{R}}(N \otimes_R \widehat{R})$ for any R-module N of finite length. Likewise, if $mN = 0$, then $\dim_k N = \dim_k N \otimes_R \widehat{R}$. Hence for all i we have

$$\dim_k \operatorname{Tor}_i^R(k, M) = \dim_k \operatorname{Tor}_i^R(k, M) \otimes_R \widehat{R} = \dim_k \operatorname{Tor}_i^{\widehat{R}}(k, \widehat{M}).$$

Similarly, for all i

$$\dim_k \operatorname{Ext}^i_R(k, M) = \dim_k \operatorname{Ext}^i_R(k, M) \otimes_R \widehat{R} = \dim_k \operatorname{Ext}^i_{\widehat{R}}(k, \widehat{M}).$$

Parts (b)-(f) follow.

Part (g) follows from

$$\mu_R(M) = \dim_k M/mM = \dim_k \widehat{M/mM} = \dim_k \widehat{M}/\widehat{mM} = \dim_k \widehat{M}/\widehat{m}\widehat{M} = \mu_{\widehat{R}}(\widehat{M}).$$

For part (a), one approach is to use that $\dim M = \sup\{i \mid \mathrm{H}^i_m(M) \neq 0\}$, where $\mathrm{H}^i_m(M)$ is the ith local cohomology module of M with support in m. As $\widehat{R}$ is flat over R and using the change of rings principle, $\mathrm{H}^i_m(M) \otimes_R \widehat{R} \cong \mathrm{H}^i_m(\widehat{M}) \cong \mathrm{H}^i_{\widehat{m}}(\widehat{M})$ for all i. As tensoring with $\widehat{R}$ is faithful, we have

$$\dim M = \sup\{i \mid \mathrm{H}^i_m(M) \neq 0\} = \sup\{i \mid \mathrm{H}^i_{\widehat{m}}(\widehat{M}) \neq 0\} = \dim \widehat{M}.$$

Alternatively, one can use the fact that $\dim M$ is the degree of the polynomial (called the Hilbert polynomial of M) which coincides with $\lambda_R(M/m^nM)$ for n sufficiently large. Since $\lambda_R(M/m^nM) = \lambda_{\widehat{R}}(\widehat{M}/\widehat{m}^n\widehat{M})$ for all n (see the remarks at the beginning of this proof), M and $\widehat{M}$ have the same Hilbert polynomial. Thus, $\dim M = \dim \widehat{M}$.

Corollary 11.20. *Let (R, m) be a Noetherian local ring and M a finitely generated R-module. Then*

(a) M is CM (resp., MCM) if and only if $\widehat{M}$ is CM (resp., MCM).

(b) R is Gorenstein if and only if $\widehat{R}$ is Gorenstein.

(c) R is regular local ring if and only if $\widehat{R}$ is a regular local ring.

(d) $\operatorname{edim} R = \operatorname{edim} \widehat{R}$.

(e) M is a canonical module for R if and only if $\widehat{M}$ is a canonical module for $\widehat{R}$.

Proof. All of these follow immediately from Theorem 11.19 and the definitions. □

□

We end this section with a statement of the Cohen Structure Theorem for complete local rings (without proof). A proof can be found in Matsumura's *Commutative ring theory* or in Cohen's original paper. Recall that every local domain (R, m) falls into one of three categories:

(i) $\operatorname{char} R = 0$ and $\operatorname{char} R/m = 0$.

(ii) $\operatorname{char} R = p > 0$ and $\operatorname{char} R/m = p$.

(iii) $\operatorname{char} R = 0$ and $\operatorname{char} R/m = p$.

An example of a local domain of type (i) or (ii) is any field k of characteristic 0 or p. An example of type (iii) is $\mathbb{Z}_{(p)}$. Local domains of type (*i*) and (*ii*) are called *equicharacteristic* local domains, while those of type (iii) are called *mixed characteristic*. It is easily proved that a local domain is equicharacteristic if and only if it contains a field.

Theorem 11.21. (Cohen Structure Theorem) *Let R be a complete local ring. Then*

(a) $R \cong S/J$ for some complete regular local ring S and ideal J.

(b) Let (S, n) be a complete regular local ring of dimension d. Then:

- *(i) If S contains a field then S is isomorphic to a formal power series ring in d variables over a field.*
- *(ii) If $\operatorname{char} S = 0$ and $\operatorname{char} S/n = p > 0$ and $p \notin n^2$, then S is isomorphic to a formal power series ring in $d-1$ variables over a complete DVR V and where the maximal ideal of V is generated by p.*
- *(iii) If $\operatorname{char} S = 0$ and $\operatorname{char} S/n = p > 0$ and $p \in n^2$, then S is isomorphic to a quotient of a formal power series ring in d variables over a complete DVR by a nonzero polynomial.*

Remark 11.22. Complete regular local rings of types (i) and (ii) above are called *unramified*. Those of type (iii) are called *ramified*.

Corollary 11.23. *Let (R, m) be a complete local ring. Then R is catenary.*

Proof. Recall that CM local rings are catenary (Corollary 5.20 and that regular local rings are CM. As quotients of catenary rings are catenary, we have that any complete local ring is catenary from part (a) of the Cohen Structure Theorem. □

Corollary 11.24. *A complete CM local ring possesses a canonical module.*

Proof. We've proved that CM local rings which are the quotient of a Gorenstein ring possess canonical modules (Theorem 9.16). As regular local rings are Gorenstein, the result follows from part (a) of the Cohen Structure Theorem. □

Example 11.25. Let $R = \mathbb{C}[x, y]/(y^2 - x^2(x+1))$ and $m = (x, y)R$. It is easily seen that $y^2 - x^2(x+1)$ is irreducible in $\mathbb{C}[x, y]$, and so R is a domain. Let $\widehat{R}$ be the m-adic completion of R. Then $\widehat{R} \cong \mathbb{C}[[x, y]]/(y^2 - x^2(x+1))$. Using the binomial series, one obtains that $\sqrt{x+1} \in \mathbb{C}[[x]] \subset \mathbb{C}[[x, y]]$. Thus $y^2 - x^2(x+1) = (y - x\sqrt{x+1})(y + x\sqrt{x+1})$ in $\mathbb{C}[[x, y]]$. Hence, $\widehat{R}$ is not a domain.

Remark 11.26. In fact, there are examples of local domains R such that $\widehat{R}$ is not even reduced. Examples of this behavior can be found in Nagata's *Local Rings* and in a paper by Ferrand and Raynaud.

12 Exercises

Math 906

Homework # 1

1. Let (R, m) be a local PID which is not a field. (Such rings are called *discrete valuation rings* or *DVR*s for short.) Prove that $E_R(R/m) \cong Q/R$, where Q is the field of fractions of R. (Hint: Let $m = (x)$. First show that $Q = R_x$. Then show that R/m can be naturally embedded in Q/R.)

2. Let R be a ring, I an ideal of R, and M an R/I-module. Prove that $\operatorname{Hom}_R(R/I, E_R(M)) \cong E_{R/I}(M)$.

3. Let R be a local ring which has a non-zero finitely generated injective module E. Prove that R is Artinian (equivalently, $\dim R = 0$). (Hint: First, one can assume that E is indecomposable. Using the previous exercise, one can reduce (with some work) to the case that R is a domain and $E = E_R(R/m)$.)

4. Let (R, m) be a local ring and suppose R is injective. Prove that $R \cong E_R(R/m)$.

5. Let $R = k[x, y]_{(x,y)}$ where $k[x, y]$ is a polynomial ring over a field k. Let $Q = k(x, y)$ be the field of fractions of R. Prove that Q/R is divisible but not injective. (Hint: Show that there exists an R-module M such that $\operatorname{Ext}^2_R(M, R) \neq 0$. For instance, you can let $M = R/(x, y)$ and use the Koszul complex on x and y; alternatively, one can let $M = R/I$ as in Example 5.26 of Grifo's 915 notes and use the resolution there.)

6. Let R be a Noetherian ring and M an R-module. Prove that the following are equivalent:

   (a) M is injective;

   (b) M_p is injective for all prime ideals p;

   (c) M_m is injective for all maximal ideals m.

7. Let R be a Noetherian domain (but not a field) which is locally a DVR at every nonzero prime ideal. (Such rings are called *Dedekind domains*.) Prove that every divisible R-module is injective.

8. Give an example of a Noetherian ring R and a nonzero injective module I such that $\operatorname{Ann}_R I \neq 0$.

9. Show that $\operatorname{Ann}_R E_R(R/m) = 0$. (Hint: You may use that for any nonzero R-module M, $\operatorname{Hom}_R(M, E_R(R/m)) \neq 0$ – something we'll eventually show in class.)

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Homework # 2

Note: All rings are assumed to be commutative with identity. Local rings are assumed to be Noetherian.

1. Let k be a field and $R = k[x]_{(x)}$. Prove $\mu_1(m, R) = 1$ where $m = (x)R$.

2. Let M be a finitely generated R-module and I an ideal such that $IM = M$. Prove that there exists $s \in I$ such that $(1 - s)M = 0$. (Hint: There are at least two approaches: one uses the determinant trick (see Grifo's 905 notes, Lemma 1.34) and the other uses localization together with Nakayama's lemma to show $I + \operatorname{Ann}_R M = R$.)

3. Let (R, m) be a local ring such that $\operatorname{pd}_R R/m < \infty$. Prove that $\operatorname{id}_R M < \infty$ for all finitely generated R-modules M.

4. Let k be a field and $R = k[x, y, z]$. Prove that $x, y - xy, z - zx$ in an R-sequence but $y - xy, z - zx, x$ is not an R-sequence.

5. Let (R, m) be a local PID which is not field and let $m = (x)$. Consider the ring $S = R[y]$, where y is a variable. Prove that $\{x, y\}$ and $\{1 - xy\}$ are both maximal S-sequences.

6. Let (R, m) be a local ring and M a finitely generated R-module. Suppose $x \in m$ is a regular element on M. Prove that $\operatorname{id}_R M = \operatorname{id}_R M/xM$.

7. Let $\phi : R \to S$ be a faithfully flat ring homomorphism, M an R-module, and $\mathbf{x} = x_1, \dots, x_n$ an M-sequence. Prove that $\phi(\mathbf{x})$ is a $M \otimes_R S$-sequence.

8. Let R be a Noetherian ring and $p \in \operatorname{Spec} R$. Prove that $\operatorname{ht} p \geqslant \operatorname{grade} p$. (Hint: You may use the following consequence of Krull's Principal Ideal Theorem: for any $x \in p$, $\operatorname{ht}(p/(x)) \geqslant \operatorname{ht}(p) - 1$ with equality if x is not contained in any minimal prime of R; cf. Theorem 8.17 of Grifo's 905 notes.)

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Homework # 3

Note: All rings are assumed to be commutative with identity. Local rings are assumed to be Noetherian.

1. Let R be a Noetherian ring, $\mathbf{x} = x_1, \ldots, x_n \in J(R)$, and M a finitely generated R-module. Suppose $\mathrm{H}_i(\mathbf{x}; M) = 0$ for some $i \geqslant 1$. Prove that $\mathrm{H}_j(\mathbf{x}; M) = 0$ for all $j \geqslant i$.

2. Let R be a Noetherian ring, $\mathbf{x} = x_1, \ldots, x_n \in J(R)$, and M a (nonzero) finitely generated R-module. Let $I = (\mathbf{x})$. Prove that

$$\operatorname{grade}(I, M) = n - \sup\{i \mid \mathrm{H}_i(\mathbf{x}; M) \neq 0\}.$$

   (Hint: Use induction on $\operatorname{grade}(I, M)$. Corollary 4.15 and Proposition 4.16 are useful here.)

3. Let (R, m) be a Cohen-Macaulay local ring and $I = (x_1, \ldots, x_n) \subseteq m$. Suppose $\operatorname{ht}(I) = n$. Prove that $x_1, \ldots, x_n$ is a regular sequence.

4. Let (R, m) be a local ring of dimension d and I an ideal of R. Prove that there exists $x_1, \ldots, x_d \in I$ such that $\sqrt{I} = \sqrt{(x_1, \ldots, x_d)}$. (Hint: By induction, show that for $1 \leqslant i \leqslant d$ there exists $x_1, \ldots, x_i \in I$ such that for all primes p of height at most $i-1$, if $p \supseteq (x_1, \ldots, x_i)$ then $p \supseteq I$.)

5. Let (R, m) be a local UFD. Suppose $f, g \in m$. Prove that $\{f, g\}$ is an R-sequence if and only if $\gcd(f, g) = 1$.

6. Decide whether the following rings are CM (assume that k is a field and that all the rings below are localized at the "obvious" maximal ideal):
   (a) $k[x, y, z]/(x^3 - yz, z^3)$
   (b) $k[x, y, z]/(xz, yz)$
   (c) $k[x, y, z, w]/(x, y) \cap (z, w)$

7. Give an example of an ideal I in a Noetherian ring R and prime ideal p such that $\operatorname{grade}(I_p, R_p) > \operatorname{grade}(I, R)$.

8. Let R be a Noetherian ring, $\mathbf{x} \in R$, and M a finitely generated R-module. Let $I = \operatorname{Ann}_R M/\mathbf{x}M$. Prove that the set $\{p \in \operatorname{Spec} R \mid \mathbf{x} \text{ is an } M_p\text{-sequence}\}$ is an open subset of $\mathrm{V}(I)$. (Hint: the Koszul complex may be helpful here.)

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Homework # 4

Note: All rings are assumed to be commutative with identity. Local rings are assumed to be Noetherian.

1. Prove that if R is a Gorenstein ring so is $R[x_1, \dots, x_n]$.

2. Prove that if R is a regular ring so is $R[x_1, \dots, x_n]$.

3. Let R be a Noetherian regular ring of infinite dimension. (Nagata has shown such rings exist.) Let M be a finitely generated R-module. Prove that $\operatorname{pd}_R M < \infty$. (Hint: Let M be a finitely generated R-module. For each $n \geqslant 0$, show that the set of all primes such that $\operatorname{pd}_{R_p} M_p \leqslant n$ is an open subset of $\operatorname{Spec} R$. This gives us an increasing chain of open sets whose union is $\operatorname{Spec} R$ as R is regular. Now use that $\operatorname{Spec} R$ is Noetherian.)

4. Let R be a Noetherian regular ring of infinite dimension. Show that there exists an R-module M such that $\operatorname{pd}_R M = \infty$.

5. Let R be a Noetherian ring and M an R-module. Prove that $\operatorname{fd}_R M \leqslant \operatorname{pd}_R M$ with equality if M is finitely generated. (Here, $\operatorname{fd}_R M$ denotes the flat dimension of M, i.e., the length of the shortest resolution of M by flat modules.)

6. Let R be a Noetherian ring. Prove that $\operatorname{gl-dim} R \leqslant n$ if and only if $\operatorname{fd}_R M \leqslant n$ for all R-modules M.

7. Let (R, m) be a zero-dimensional local ring. Prove that R is Gorenstein if and only if $(0 :_R (0 :_R I)) = I$ for all ideals I of R.

8. Let $(R.m)$ be a local ring. An ideal I is called *perfect* if $\operatorname{pd}_R R/I = \operatorname{grade} I$.

   (a) Let I be an ideal. Prove that $\operatorname{grade} I \leqslant \operatorname{pd}_R R/I$. (Theorem 3.16 is helpful here.)

   (b) Assume R is CM and $\operatorname{pd}_R R/I < \infty$. Prove that I is perfect if and only if R/I is CM.

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Homework # 5

Note: All rings are assumed to be commutative with identity. Local rings are assumed to be Noetherian.

1. Decide whether the following ring is normal: $\mathbb{C}[x,y,z]/(x^5+y^5+z^5)$. Is $x^5+y^5+z^5$ irreducible in $\mathbb{C}[x,y,z]$? Why or why not?

2. Let R be an Artinian ring and let $p_1,\ldots,p_s$ be its prime ideals. Prove that $R \cong R_{p_1} \times \cdots \times R_{p_s}$. (Hint: Use CRT.)

3. Let D be a Dedekind domain. Prove that every ideal of D can be generated by two elements. (Hint: For any nonzero $a \in I$, $R/(a)$ is an Artinian ring.)

4. Prove that a Noetherian ring R satisfies S_n if and only if R_p is CM for all primes p such that $\operatorname{depth} R_p < n$.

5. Prove that a regular ring is isomorphic to a finite product of regular domains. (Hint: See Theorem 8.7.)

6. Let $S = \mathbb{R}[x,y]/(x^2+y^2-1)$ (the coordinate ring for the real circle). Define $f : S \to S^2$ by $f(s) = (s\overline{x}, s\overline{y})$. Let $P = \operatorname{coker} f$. Prove that P is free.

7. Let $(R,m) \subseteq (S,n)$ be local rings. Suppose R is an RLR and S is finitely generated as an R-module. Prove that S is CM if and only if S is a free R-module. (Hint: Use the hypothesis to show that $\operatorname{depth}_R S = \operatorname{depth}_S S$. Then apply the Auslander-Buchsbaum formula.)

8. Let R be a Noetherian ring such that every finitely generated R-module has an FFR. Prove that R is a UFD. (Hint: Use Problem #5, Propositions 8.17 and 8.24, and Theorem 8.28.)

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Homework # 6

Note: All rings are assumed to be commutative with identity. Local rings are assumed to be Noetherian.

1. Let R be a d-dimensional CM local ring with canonical module ω_R. Suppose M is a an R-module of finite length. Prove that $\lambda_R(M) = \lambda_R(\operatorname{Ext}^d_R(M, \omega_R))$.

2. Let R be a CM local ring with canonical module ω_R. Prove that the following are equivalent:

   (a) R is Gorenstein.

   (b) For every finitely generated R-module M there exists a surjection $\omega_R^n \to M$ for some n.

3. Let (R, m, k) be a CM local ring with canonical module ω_R. Suppose M is a finitely generated MCM which has finite injective dimension. Prove that $M \cong \omega_R^n$ for some n. (Hint: Let $n = r(M) = \mu_R(M^\dagger)$ by Corollary 9.21. Then there exists an exact sequence $0 \to K \to R^n \to M^\dagger \to 0$. This gives a s.e.s. $0 \to M \xrightarrow{\phi} \omega_R^n \to K^\dagger \to 0$. Let $\mathbf{x}$ be a maximal R-sequence. Show that $\phi \otimes_R R/(\mathbf{x})$ is an isomorphism.)

4. Let (R, m, k) be a Gorenstein local ring and M a finitely generated R-module. Prove that $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R M < \infty$. (Hint: For the backward direction, induct on $\dim R - \operatorname{depth} M$. Use the previous problem for the base case. If M is not MCM, let $0 \to K \to R^n \to M \to 0$ be exact. Show that $\operatorname{depth} K = \operatorname{depth} M + 1$.)

5. Let $\phi : (R, m, k) \to (S, n, \ell)$ be a faithfully flat homomorphism of CM local rings. (In particular, this implies $mS \subseteq n$.) Let C be a finitely generated R-module such that $C \otimes_R S$ is a canonical module for S. Prove that C is a canonical module for R. (Hint: Start by localizing at a prime minimal over mS to reduce to the situation that mS is n-primary. Then S/mS is a finite length S-module. Lemma 2.22 is useful.)

6. Let (S, n, k) be a regular local ring and I an ideal such that $R := S/I$ is CM. Prove that $\operatorname{Hom}_R(\omega_R, R) \cong \operatorname{Tor}^S_t(R, R)$ where $t = \operatorname{pd}_S R$. (Hint: Use Corollary 9.17.)