Thick subcategories and Gorenstein projective modules

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$\mathrm{D}_+(R)$ and $\mathrm{D}_b(R)$ will denote the subcategories of $\mathrm{D}(R)$ consisting of the (homologically) bounded below and bounded complexes, respectively.

If A is a subcategory of $\mathrm{D}(R)$, A^f will denote the subcategory consisting of all complexes C in A such that $\mathrm{H}_n(C)$ is finitely generated for all n.

Thick subcategories

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- The subcategories consisting of all complexes of finite projective/injective/flat dimension.

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- The subcategory consisting of all complexes C such that H(C) has finite length.
- The subcategories consisting of all complexes of finite projective/injective/flat dimension.
- For any subset U of Spec R, the subcategory consisting of all complexes C such that $C \otimes^{\mathbf{L}}_R k(\mathfrak{p}) \simeq 0$ for all $\mathfrak{p} \in U$.

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- $\mathrm{thick}_R(R)$ is the subcategory of perfect complexes; i.e., the subcategory consisting of complexes in $\mathrm{D}(R)$ which are isomorphic to bounded complexes of f.g. projective modules.
- $\mathrm{thick}_R(k)$ is the subcategory consisting of complexes isomorphic in $\mathrm{D}(R)$ to a bounded complex with finite length homology.

Thickenings

We can filter $\text{thick}_R(S)$ using subcategories $\text{thick}_R^n(S)$ defined as follows:

- $\text{thick}_R^0(S) := \{0\}$;
- For $n \geq 1$, $M \in \text{thick}_R^n(S)$ if and only if M can be built from complexes in S using finite direct sums, shifts, retracts, and at most $n-1$ mapping cones.

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If S is a collection of modules closed under direct summands and finite sums then $M \in \text{thick}_R^1(S)$ if and only if M is isomorphic in $\mathrm{D}(R)$ to a bounded complex of modules from S with zero differentials.

Levels

Given a complex M and a collection of complexes S, we define the level of M with respect to S by

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Example

Let M be the complex $0 \to R \xrightarrow{0} R \to 0$ situated in any homological degree. Then $\operatorname{pd}_R M = \sup M$ while $\operatorname{level}_R^R M = 1$.

Applications

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Theorem (Avramov-Buchweitz-Iyengar-Miller, 2010)

Let F be a finite complex of free R-modules such that $\mathrm{H}(F)$ has nonzero finite length. Then

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \, \mathrm{H}_n(F) \geq \mathrm{level}_R^k \, F \geq \text{cf-rank}\, R + 1,$$

where $\ell\ell_R(-)$ denotes Loewy length and cf-rank R is the conormal free rank of R.

Level and projective dimension

For a nonzero complex M in $D_b^f(R)$ one can show:

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Theorem (Avramov-Buchweitz-Iyengar-Miller, 2010)

The following are equivalent:

1. R is regular;
2. $\operatorname{level}_R^R k < \infty$;
3. $\operatorname{level}_R^R k = \dim R + 1$;
4. $\operatorname{level}_R^R M \leq \dim R + 1$ for any M in $\mathrm{D}_b^f(R)$.

Gorenstein projective modules

Definition

A finitely generated module is called Gorenstein projective if $M \cong M^{**}$ and $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for all $i > 0$, where $(-)^*$ denotes the functor $\operatorname{Hom}_R(-, R)$.

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- All finitely generated projective modules are Gorenstein projective.
- If R is Gorenstein then M is Gorenstein projective if and only M is a maximal CM module.

The Gorenstein projective dimension $\operatorname{Gpd}_R M$ of a f.g. module M is the shortest length of a resolution by Gorenstein projectives.

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It is straightforward to see that for $M \in \mathsf{D}^f_+(R)$

$$\operatorname{level}^{\mathsf{G}}_R M \leq \operatorname{Gpd}_R M - \inf M + 1.$$

Main Results

Theorem (Awadalla - M)

For M in $\mathsf{D}_b^f(R)$ we have $\operatorname{level}_R^{\mathsf{G}} M \geq \operatorname{Gpd}_R M - \sup M + 1$.

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Outline of proof:

- Let $n = \sup M$ and $X \in \mathsf{D}_b^f(R)$ be isomorphic to M such that X_i is projective for all $i \neq n$ and X_n is Gorenstein projective. (L.W. Christensen- Iyengar, 2007)

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- $\phi_{g-1}\phi_{g-2}\cdots\phi_n$ is nonzero in $\mathsf{D}_b^f(R)$, where $g = \operatorname{Gpd}_R M$.
- By the Ghost lemma, $\operatorname{level}_R^{\mathsf{G}} M \geq g - n + 1$.

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The following are equivalent:

1. R is Gorenstein.
2. $\operatorname{level}_R^{\mathsf{G}} k < \infty$
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Note: The bound in condition (4) is obtained in some examples. However, when R is regular, $\operatorname{level}_R^{\mathsf{G}} M \leq \dim R + 1$.

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- As Z and B have zero differentials, and every f.g. module has Gpd at most $\dim R$, we have $\operatorname{level}_R^{\mathsf{G}} Z$ and $\operatorname{level}_R^{\mathsf{G}} B$ are at most $\dim R + 1$.
- Hence, $\operatorname{level}_R^{\mathsf{G}} M \leq 2(\dim R + 1)$.

The End

Thank you!