THE FROBENIUS FUNCTOR AND INJECTIVE MODULES

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Abstract. We investigate commutative Noetherian rings of prime characteristic such that the Frobenius functor applied to any injective module is again injective. We characterize the class of one-dimensional local rings with this property and show that it includes all one-dimensional F-pure rings. We also give a characterization of Gorenstein local rings in terms of $\mathrm{Tor}_i^R(R^f, E)$, where E is the injective hull of the residue field and R^f is the ring R whose right R-module action is given by the Frobenius map.

1. Introduction

Let R be a commutative Noetherian ring of prime characteristic p and $f : R \longrightarrow R$ the Frobenius ring homomorphism (i.e., $f(r) = r^p$ for $r \in R$). We let R^f denote the ring R with the R-R bimodule structure given by $r \cdot s := rs$ and $s \cdot r := sf(r)$ for $r \in R$ and $s \in R^f$. Then $F_R(-) := R^f \otimes_R -$ is a right exact functor on the category of (left) R-modules and is called the *Frobenius functor* on R. This functor has played an essential role in the solution of many important problems in commutative algebra for local rings of prime characteristic (e.g., Hochster and J. Roberts [11], Peskine and Szpiro [16], P. Roberts [17]). Of particular interest is how properties of the Frobenius map (or functor) characterize classical properties of the ring. The most important result of this type, proved by Kunz [13], says that F_R is exact if and only if R is a regular ring. As another example, Iyengar and Sather-Wagstaff prove that a local ring R is Gorenstein if and only if R^f (viewed as a right R-module) has finite G-dimension [14, Theorems 6.2 and 6.6].

As F_R is additive and $F_R(R) \cong R$, it is easily seen that F_R preserves projective (in fact, flat) modules. In this paper, we consider rings for which F_R preserves injective modules, i.e., rings R having the property that $F_R(I)$ is injective for every injective R-module I. A result of Huneke and Sharp [12, Lemma 1.4] shows that Gorenstein rings have this property, and in fact this is true for quasi-Gorenstein rings as well (Proposition 3.6). In Section 3, we show that if F_R preserves injectives then R satisfies Serre's condition S_1 and that $F_R(I) \cong I$ for every injective R-module I. Moreover, if R is a homomorphic image of a Gorenstein local ring, then F_R preserves all injectives if and only if $F_R(E)$ is injective, where E is the injective hull of the residue field. We also give a criterion (Theorem 3.14) for a local ring R to be Gorenstein in terms of $\mathrm{Tor}_i^R(R^f, E)$:

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Theorem 1.1. *Let (R, m) be a local ring and $E = E_R(R/m)$. Then the following are equivalent:*
(a) R is Gorenstein;
(b) $\operatorname{Tor}_0^R(R^f, E) \cong E$ and $\operatorname{Tor}_i^R(R^f, E) = 0$ for $i = 1, \ldots, \operatorname{depth} R$.

In Section 4, we study one-dimensional rings R such that F_R preserves injectives. In particular, we give the following characterization (Theorem 4.1) in the case where R is local:

Theorem 1.2. *Let (R, m) be a one-dimensional local ring and $E = E_R(R/m)$. The following conditions are equivalent:*
(a) $F_R(E)$ is injective;
(b) $F_R(I) \cong I$ for all injective R-modules I;
(c) R is Cohen-Macaulay and has a canonical ideal ω_R such that $\omega_R \cong \omega_R^{[p]}$.

Using this characterization, we show that every one-dimensional F-pure ring preserves injectives. We also prove, using a result of Goto [7], that if R is a complete one-dimensional local ring with algebraically closed residue field and has at most two associated primes then R is Gorenstein if and only if F_R preserves injectives. We remark that Theorems 1.1 and 1.2 are dual to results appearing in [7] in the case where the Frobenius map is finite. This duality is made explicit in Proposition 3.10.

In Section 2, we summarize several results concerning the Frobenius functor and canonical modules which will be needed in the later sections. Most of these are well-known, but for some we could not find a reference in the literature.

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2. Some properties of the Frobenius functor and canonical modules

Throughout this paper R denotes a commutative Noetherian ring of prime characteristic p. For an R-module M, $E_R(M)$ will denote the injective hull of M. If I is an ideal of R then $H_I^i(M)$ will denote the ith local cohomology module of M with support in I. If R is local with maximal ideal m, we denote the m-adic completion of R by $\widehat{R}$. We refer the reader to [2] or [15] for any unexplained terminology or notation.

Let M be a finitely generated R-module with presentation $R^r \xrightarrow{\varphi} R^s \longrightarrow M \longrightarrow 0$, where φ is represented (after fixing bases) by an $s \times r$ matrix A. Then $F_R(M)$ has the presentation $R^r \xrightarrow{F_R(\varphi)} R^s \longrightarrow F_R(M) \longrightarrow 0$ and the map $F_R(\varphi)$ is represented by the matrix $A^{[p]}$ obtained by raising the corresponding entries of A to the pth power. For an ideal I of R and $q = p^e$, we let $I^{[q]}$ denote the ideal generated by the set $\{i^q \mid i \in I\}$. Note that, by above, $F_R^e(R/I) \cong R/I^{[q]}$, where F_R^e is the functor F_R iterated e times.

The following proposition lists a few properties of the Frobenius functor which we will use in the sequel. Most of these are well known:

Proposition 2.1. *Let M be an R-module and S a multiplicatively closed set of R.*

(a) If T is an R-algebra then there is a T-isomorphism $F_T(T \otimes_R M) \cong T \otimes_R F_R(M)$. In particular, $F_R(M)_S \cong F_{R_S}(M_S)$ as R_S-modules.
(b) For any R_S-module N, the map $h : F_R(N) \longrightarrow F_{R_S}(N)$ given by $h(r \otimes n) = \frac{r}{1} \otimes n$ for $r \in R^f$ and $n \in N$ is an R_S-isomorphism.
(c) $\operatorname{Supp}_R F_R(M) = \operatorname{Supp}_R M$.
(d) M is Artinian if and only if $F_R(M)$ is Artinian.
(e) If (R, m) is local, and M is finitely generated of dimension s, then $F_R(H^s_m(M)) \cong H^s_m(F_R(M))$.

Proof. Part (a) follows easily from properties of tensor products. For part (b), one first observes that the map $g : R^f \otimes_R R_S \longrightarrow (R_S)^f$ given by $g(r \otimes \frac{a}{s}) = \frac{ra^p}{s^p}$ is an isomorphism of R_S-R_S bimodules. Tensoring with N (over R_S) on the right gives the desired isomorphism. For (c), it suffices to show $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R F_R(M)$. By part (a), it is enough to prove that if R is a complete local ring and $M \neq 0$ then $F_R(M) \neq 0$. Then $R = S/I$ where S is a regular local ring of characteristic p. Let $Q \in \operatorname{Ass}_S M$. Then there is an exact sequence $0 \longrightarrow S/Q \longrightarrow M$. As S is regular, F_S is exact and we have an injection $S/Q^{[p]} \longrightarrow F_S(M)$. Hence, $F_S(M) \neq 0$. By part (a), $F_R(M) \cong S/I \otimes_S F_S(M)$. As $IM = 0$, $I^{[p]} F_S(M) = 0$. Let t be an integer such that $I^t \subseteq I^{[p]}$. Now, if $F_R(M) = 0$ then $F_S(M) = IF_S(M)$. Iterating, we have $F_S(M) = I^t F_S(M) = 0$, a contradiction. Hence, $F_R(M) \neq 0$.

For (d), since $\operatorname{Supp}_R M = \operatorname{Supp}_R F_R(M)$ and the support of an Artinian module is finite, it suffices to consider the case when (R, m) is a local ring. Thus, if either M or $F_R(M)$ is Artinian, $\operatorname{Supp}_R M = \operatorname{Supp}_R F_R(M) \subseteq \{m\}$. We note that for any R-module N with $\operatorname{Supp}_R N \subseteq \{m\}$, we have that $\widehat{R} \otimes_R N \cong N$. Consequently, for any such N, $F_R(N) \cong \widehat{R} \otimes_R F_R(N) \cong F_{\widehat{R}}(N)$, where the latter isomorphism follows from part (a). Hence, we may assume that R is complete. Then $R = S/I$ where S is a regular local ring of characteristic p. As $F_R(M) \cong S/I \otimes_S F_S(M)$, it is clear that if $F_S(M)$ is Artinian then so is $F_R(M)$. Conversely, if $S/I \otimes_S F_S(M)$ is Artinian, then $F_S(M)$ is Artinian since we also have $I^{[p]} F_S(M) = 0$. Thus, it is enough to prove the result in the case where R is a regular local ring. Recall that an R-module is Artinian if and only if $\operatorname{Supp}_R M \subseteq \{m\}$ and $(0 :_M m)$ is finitely generated. Since $\operatorname{Supp}_R M = \operatorname{Supp}_R F_R(M)$, it suffices to prove that $(0 :_M m)$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated. As F_R is exact, $F_R((0 :_M m)) \cong (0 :_{F_R(M)} m^{[p]})$. But $(0 :_M m)$ is finitely generated if and only if $F_R((0 :_M m))$ is finitely generated, and $(0 :_{F_R(M)} m^{[p]})$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated.

For (e), let $I = \operatorname{Ann}_R M$ and choose $x_1 \dots, x_s \in m$ such that their images in R/I form a system of parameters. Set $J = (x_1, \dots, x_s)$. Then $H^s_m(M) \cong H^s_J(M)$. Since $H^i_J(R) = 0$ for all $i > s$, $T \otimes_R H^s_J(M) \cong H^s_{TJ}(T \otimes_R M)$ for any R-algebra T. Then

$$F_R(H^s_m(M)) \cong R^f \otimes_R H^s_J(M) \cong H^s_{J^{[p]}}(F_R(M)).$$

Finally, as $I^{[p]} \subseteq \operatorname{Ann}_R F_R(M)$ and $J^{[p]} + I^{[p]}$ is m-primary, we have $H^s_{J^{[p]}}(F_R(M)) \cong H^s_m(F_R(M))$. □

We need one more (likely well-known) result concerning the Frobenius:

Lemma 2.2. *Let (R,m) be a local ring of dimension d. If R is Cohen-Macaulay then for all $i \geq 1$ we have* $\operatorname{Tor}_i^R(R^f, H_m^d(R)) = 0$.

Proof. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters for R and $C(\mathbf{x})$ the Čech cochain complex with respect to $\mathbf{x}$. Note that $F_R(C(\mathbf{x})) \cong C(\mathbf{x}^p)$ where $\mathbf{x}^p = x_1^p, \ldots, x_d^p$. Since R is Cohen-Macaulay, $\mathbf{x}$ is a regular sequence and thus $C(\mathbf{x})$ is a flat resolution of $H_m^d(R)$. Hence for $i \geq 1$,

$$\operatorname{Tor}_i^R(R^f, H_m^d(R)) \cong \mathrm{H}^{d-i}(R^f \otimes_R C(\mathbf{x})) \cong \mathrm{H}^{d-i}(C(\mathbf{x}^p)) = 0.$$

□

For a nonzero finitely generated R-module M we let $U_R(M)$ be the intersection of all the primary components Q of 0 in M such that $\dim M/Q = \dim M$. It is easily seen that $U_R(M) = \{x \in M \mid \dim Rx < \dim M\}$. A local ring R is said to be *quasi-unmixed* if $U_{\widehat{R}}(\widehat{R}) = 0$.

Let (R,m) be a local ring of dimension d, $E = E_R(R/m)$, and $(-)^{\mathrm{v}} := \operatorname{Hom}_R(-,E)$ the Matlis duality functor. A finitely generated R-module K is called a *canonical module* of R if there is an $\widehat{R}$-isomorphism $K \otimes_R \widehat{R} \cong H_m^d(R)^{\mathrm{v}}$. If a canonical module exists, it is unique up to isomorphism and denoted by ω_R. Any complete local ring possesses a canonical module. More generally, R possesses a canonical module if R is a homomorphic image of a Gorenstein ring. Proofs of these facts can be found in Aoyama [1] (or the references cited there). We summarize some additional properties of canonical modules in the following proposition:

Proposition 2.3. *Let R be a local ring which possesses a canonical module ω_R and let $h : R \longrightarrow \operatorname{Hom}_R(\omega_R, \omega_R)$ be the natural map. The following hold:*

(a) $\operatorname{Ann}_R \omega_R = U_R(R)$.
(b) $(\omega_R)_P \cong \omega_{R_P}$ *for every prime* $P \in \operatorname{Supp}_R \omega_R$.
(c) $\omega_R \otimes_R \widehat{R} \cong \omega_{\widehat{R}}$.
(d) $\ker h = U_R(R)$.
(e) h is an isomorphism if and only if R satisfies Serre's condition S_2.
(f) If R is complete, $\operatorname{Hom}_R(M, \omega_R) \cong H_m^d(M)^{\mathrm{v}}$ *for any R-module M.*

Proof. The proofs of parts (a)-(e) can be found in [1]. Part (f) is just local duality, but it can also be seen directly from the definition of ω_R and adjointness:

$$\operatorname{Hom}_R(H_m^d(M), E) \cong \operatorname{Hom}_R(M \otimes_R H_m^d(R), E) \cong \operatorname{Hom}_R(M, H_m^d(R)^{\mathrm{v}}).$$

□

If R is a local ring possessing a canonical module ω_R such that $\omega_R \cong R$, then R is said to be *quasi-Gorenstein*. Equivalently, R is quasi-Gorenstein if and only if $H_m^d(R) \cong E$. By the proposition above, if R is quasi-Gorenstein then R is S_2 and R_P is quasi-Gorenstein for every $P \in \operatorname{Spec} R$. It is easily seen that R is Gorenstein if and only if R is Cohen-Macaulay and quasi-Gorenstein. Finally, there exist quasi-Gorenstein rings which are not Cohen-Macaulay. Examples of such rings can be constructed using Theorem 2.11 and Corollary 2.12 of [1].

3. Rings for which Frobenius preserves injectives

To facilitate our discussion we make the following definition:

Definition 3.1. The ring R is said to be *FPI* (i.e., 'Frobenius Preserves Injectives') if $F_R(I)$ is injective for every injective R-module I. We say that R is *weakly FPI* if $F_R(I)$ is injective for every Artinian injective R-module I.

By Matlis's decomposition theory for injective modules ([15, Theorems 18.4 and 18.5]), every injective R-module is a direct sum of modules of the form $E_R(R/P)$ for various prime ideals P of R. Further, $E_R(R/P)$ is Artinian if and only if P is a maximal ideal of R. Accordingly, we see that R is FPI (respectively, weakly FPI) if and only if $F_R(E_R(R/P))$ is injective for every prime (respectively, maximal) ideal P of R.

Lemma 3.2. *Let P be a prime ideal of R and S a multiplicatively closed set of R such that $S \cap P = \emptyset$. Then the following hold:*

(a) $F_R(E_R(R/P)) \cong F_{R_S}(E_{R_S}(R_S/PR_S))$ as R_S-modules.

(b) $F_R(E_R(R/P))$ is injective as an R-module if and only if $F_{R_S}(E_{R_S}(R_S/PR_S))$ is injective as an R_S-module.

Proof. To prove (a), note that by Lemma 10.1.12 and Proposition 10.1.13 of [3], there are R_S-isomorphisms $E_R(R/P) \cong E_R(R/P)_S \cong E_{R_S}(R_S/PR_S)$. Hence, by Proposition 2.1(b) we have that $F_R(E_R(R/P)) \cong F_{R_S}(E_{R_S}(R_S/PR_S))$ as R_S-modules. Part (b) follows from (a) and the fact that an R_S-module is injective as an R-module if and only if it is injective as an R_S-module ([3, Lemma 10.1.11]). □

Combining this Lemma with the remarks in the previous paragraph, we obtain the following:

Proposition 3.3. *The following hold for the ring R:*

(a) R is weakly FPI if and only if R_m is weakly FPI for every maximal ideal m of R.

(b) R is FPI if and only if R_m is FPI for every maximal ideal m of R.

(c) R is FPI if and only if R_P is weakly FPI for every prime ideal P of R.

Proof. We have that R is weakly FPI if and only if $F_R(E_R(R/m))$ is injective for every maximal ideal m of R. For a given maximal ideal m, we have by part (b) of Lemma 3.2 that $F_R(E_R(R/m))$ is injective if and only if $F_{R_m}(E_{R_m}(R/m))$ is injective as an R_m-module. As the latter condition holds if and only if R_m is weakly FPI, we see that (a) holds. Parts (b) and (c) are proved similarly. □

We summarize some properties of FPI rings in the following proposition. Recall that a ring R is said to be *generically Gorenstein* if R_P is Gorenstein for every $P \in \operatorname{Min}_R R$.

Proposition 3.4. *Let R be a Noetherian ring.*

(a) If R is FPI and S is a multiplicatively closed set of R then R_S is FPI.

(b) If R is FPI then R is generically Gorenstein.

(c) If R is local then R is weakly FPI if and only if $\widehat{R}$ is weakly FPI.

(d) Let S be a faithfully flat R-algebra which is FPI and suppose that the fibers $k(P) \otimes_R S$ are generically Gorenstein for all $P \in \operatorname{Spec} R$. Then R is FPI.

(e) Suppose R is a quotient of a Gorenstein local ring. If $\widehat{R}$ is FPI then so is R.

Proof. Part (a) follows from Lemma 3.2(b). To prove (b), let $P \in \operatorname{Min}_R R$. By part (a), R_P is FPI. Resetting notation, it suffices to show that if (R, m) is a zero-dimensional local FPI ring then R is Gorenstein. In this situation, note that if M is a finitely generated R-module then $F^e_R(M)$ is free for sufficiently large e. To see this, consider a minimal presentation for M: $R^s \xrightarrow{A} R^t \longrightarrow M \longrightarrow 0$, where A is a matrix all of whose entries are in m. Applying $F^e_R(-)$ to this presentation and noting that $A^{[q]} = 0$ for sufficiently large $q = p^e$ (since m is nilpotent), we obtain that $F^e_R(M) \cong R^t$ for sufficiently large e. Now let $E = E_R(R/m)$, which is a finitely generated R-module. Since R is FPI, $F^e_R(E)$ is injective for all e. Also, $F^e_R(E) \neq 0$ for all e by Proposition 2.1(c). Hence, there exists a nonzero free R-module which is injective. This implies R is injective and therefore Gorenstein.

For part (c), let $E = E_R(R/m)$. Then $E \cong \widehat{R} \otimes_R E \cong E_{\widehat{R}}(\widehat{R}/\widehat{m})$ as $\widehat{R}$-modules. By Proposition 2.1(d), $F_R(E)$ is Artinian and consequently $\widehat{R} \otimes_R F_R(E) \cong F_R(E)$. By part (a) of Proposition 2.1, we have $\widehat{R}$-isomorphisms $F_{\widehat{R}}(E) \cong \widehat{R} \otimes_R F_R(E) \cong F_R(E)$. Now, $F_R(E)$ is R-injective if and only if there is an R-isomorphism $F_R(E) \cong E^n$ for some n. However, every R-homomorphism between Artinian modules is also an $\widehat{R}$-homomorphism. Hence, $F_R(E) \cong E^n$ if and only if $F_{\widehat{R}}(E) \cong E^n$ as $\widehat{R}$-modules. Thus, $F_R(E)$ is R-injective if and only if $F_{\widehat{R}}(E)$ is $\widehat{R}$-injective.

To prove (d), let P be an arbitrary prime ideal of R. By Proposition 3.3(c), it suffices to show that R_P is weakly FPI. Let $Q \in \operatorname{Spec} S$ which is minimal over PS. Then S_Q is a faithfully flat R_P-algebra and is FPI by part (a). Hence, we may assume (R, m) and (S, n) are local, $P = m$, and the fiber S/mS is zero-dimensional Gorenstein. Let $E = E_R(R/m)$. By [6, Theorem 1], $S \otimes_R E$ is an injective S-module. As S is FPI, $S \otimes_R F_R(E) \cong F_S(S \otimes_R E)$ is injective. Since S is faithfully flat over R, this implies $F_R(E)$ is an injective R-module.

Part (e) follows from (d) since the hypothesis implies that the formal fibers of R are Gorenstein (see [15, Exercise 23.1]). □

We now show that F_R preserves injectives if and only if F_R fixes injectives:

Proposition 3.5. *Let I be an injective R-module and suppose $F_R(I)$ is injective. Then $F_R(I) \cong I$.*

Proof. By Matlis's decomposition theorem, it suffices to prove the result in the case when $I = E_R(R/P)$ where P is a prime ideal. By Lemma 3.2(a), we may assume (R, m) is local and $P = m$. Furthermore, by the arguments used in the proof of Proposition 3.4(c), we may also assume R is complete. Let $E = E_R(R/m)$ and $d = \dim R$. Since E is Artinian, $F_R(E)$ is Artinian by Proposition 2.1(d). Hence, $F_R(E) \cong E^n$ for some integer $n \geq 1$. It suffices to show that $n = 1$. Let $U = U_R(R)$. By parts (d) and (f) of Proposition 2.3, we have an exact sequence $0 \longrightarrow R/U \longrightarrow H^d_m(\omega_R)^{\mathrm{v}}$. Dualizing, we obtain a surjection $H^d_m(\omega_R) \longrightarrow E_{R/U}$, where $E_{R/U} := E_{R/U}(R/m) \cong \operatorname{Hom}_R(R/U, E)$. Since ω_R is a finitely generated R-module, we also have a surjection $R^s \longrightarrow \omega_R$ for some s. This yields an exact sequence $H^d_m(R)^s \longrightarrow H^d_m(\omega_R) \longrightarrow 0$. Composing, we obtain an exact sequence

$$(*) \qquad H^d_m(R)^s \longrightarrow E_{R/U} \longrightarrow 0.$$

Now consider the short exact sequence $0 \longrightarrow U \longrightarrow R \longrightarrow R/U \longrightarrow 0$. Applying Matlis duality, we have that

$$0 \longrightarrow E_{R/U} \longrightarrow E \longrightarrow U^{\mathrm{v}} \longrightarrow 0$$

is exact. Combining with (*), we have an exact sequence

$$H^d_m(R)^s \longrightarrow E \longrightarrow U^{\mathrm{v}} \longrightarrow 0.$$

Applying $F^e_R(-)$ and using that $F_R(E) \cong E^n$ and $F_R(H^d_m(R)) \cong H^d_m(R)$, we obtain an exact sequence

$$H^d_m(R)^s \longrightarrow E^{n^e} \longrightarrow F^e_R(U^{\mathrm{v}}) \longrightarrow 0.$$

Dualizing once again yields an exact sequence

$$(**) \qquad 0 \longrightarrow F^e_R(U^{\mathrm{v}})^{\mathrm{v}} \longrightarrow R^{n^e} \longrightarrow (\omega_R)^s.$$

Let $I = \mathrm{Ann}_R U = \mathrm{Ann}_R U^{\mathrm{v}}$. Since $\dim R/I < d$ and $I^{[q]} \subseteq \mathrm{Ann}_R F^e_R(U^{\mathrm{v}}) = \mathrm{Ann}_R F^e_R(U^{\mathrm{v}})^{\mathrm{v}}$ (where $q = p^e$), we have $\dim F^e_R(U^{\mathrm{v}})^{\mathrm{v}} < d$. Let P be a prime ideal of R such that $\dim R/P = d$. Localizing (**) at P, we have the exactness of $0 \longrightarrow R_P^{n^e} \longrightarrow (\omega_{R_P})^s$. If $n > 1$, we easily obtain a contradiction by comparing lengths, as e can be arbitrarily large. □

The following result is essentially [12, Proposition 1.5]:

Proposition 3.6. *Let (R, m) be a quasi-Gorenstein local ring. Then R is FPI.*

Proof. It suffices to prove that R_P is weakly FPI for every prime ideal P. As R_P is quasi-Gorenstein, we may assume $P = m$. Let $E = E_R(R/m)$. Then $E \cong H^d_m(R)$ where $d = \dim R$. Hence, by Proposition 2.1(e), $F_R(E) \cong F_R(H^d_m(R)) \cong H^d_m(R) \cong E$. □

Next, we show that for a large class of rings, weakly FPI implies FPI. We first establish a few preliminary results.

Lemma 3.7. *Let $\varphi : R \longrightarrow S$ be a homomorphism of local rings such that S is finitely generated as an R-module. Let k and ℓ denote the residue fields of R and S, respectively, and set $E_R = E_R(k)$ and $E_S = E_S(\ell)$. Then $\mathrm{Hom}_R(S, E_R) \cong E_S$ as S-modules.*

Proof. Let m and n be the maximal ideals of R and S, respectively. As S is finitely generated over R, $\varphi^{-1}(n) = m$ and $\ell \cong k^t$ as R-modules for some positive integer t. Clearly, $\mathrm{Hom}_R(S, E_R)$ is injective as an S-module. Also $\mathrm{Supp}_S \mathrm{Hom}_R(S, E_R) = \{n\}$, since $\mathrm{Supp}_R E_R = \{m\}$. Hence, $\mathrm{Hom}_R(S, E_R) \cong (E_S)^r$ where $r = \dim_\ell \mathrm{Hom}_S(\ell, \mathrm{Hom}_R(S, E_R))$. It suffices to show that $r = 1$. We have the following R-module isomorphisms:

$$\begin{aligned} \mathrm{Hom}_S(\ell, \mathrm{Hom}_R(S, E_R)) &\cong \mathrm{Hom}_R(\ell \otimes_S S, E_R) \\ &\cong \mathrm{Hom}_R(k^t, E_R) \\ &\cong k^t. \end{aligned}$$

Thus, $r = \dim_\ell \mathrm{Hom}_S(\ell, \mathrm{Hom}_R(S, E_R)) = \frac{1}{t} \dim_k \mathrm{Hom}_S(\ell, \mathrm{Hom}_R(S, E_R)) = 1$. □

Proposition 3.8. *Let $\varphi : R \longrightarrow S$ be a homomorphism of local rings such that S is finitely generated as an R-module. Let E_R and E_S be as in Lemma 3.7. Then for any R-module M we have an S-module isomorphism*

$$\mathrm{Hom}_S(S \otimes_R M, E_S) \cong \mathrm{Hom}_R(S, \mathrm{Hom}_R(M, E_R)).$$

Proof. Using adjointness and Lemma 3.7, we have S-module isomorphisms

$$\begin{aligned}\mathrm{Hom}_S(S\otimes_R M, E_S) &\cong \mathrm{Hom}_S(S\otimes_R M, \mathrm{Hom}_R(S,E_R))\\ &\cong \mathrm{Hom}_R(S\otimes_R M, E_R)\\ &\cong \mathrm{Hom}_R(S, \mathrm{Hom}_R(M,E_R)).\end{aligned}$$

□

Given an arbitrary R-module M, we let $\mathrm{Hom}_R(R^f, M)$ denote the abelian group of *right* R-module homomorphisms from R^f to M. (As R is commutative, we can conveniently view M as either a left or right R-module as the situation warrants.) Note that this group has a natural structure as a left R-module via the left R-action on R^f; i.e., $rh(s) = h(rs)$ for all $r \in R, s \in R^f$, and $h \in \mathrm{Hom}_R(R^f, M)$.

Recall that a ring R of characteristic p is called *F-finite* if the Frobenius map $f : R \longrightarrow R$ is a finite morphism; i.e., R^f is finitely generated as a right R-module. If, in Proposition 3.8, we take S to be R and φ to be the Frobenius homomorphism f, we obtain the following:

Corollary 3.9. *Let R be an F-finite local ring. For any R-module M we have an isomorphism of left R-modules*

$$F_R(M)^{\vee} \cong \mathrm{Hom}_R(R^f, M^{\vee}).$$

Proposition 3.10. *Suppose R is an F-finite local ring and E the injective hull of the residue field of R. The following are equivalent:*
(a) $\mathrm{Hom}_R(R^f, R) \cong R$;
(b) $F_R(E) \cong E$.

Proof. We first claim that it suffices to prove the result in the case when R is complete. By Propositions 3.4(c) and 3.5, we have that $F_R(E) \cong E$ as R-modules if and only if $F_{\widehat{R}}(E) \cong E$ as $\widehat{R}$-modules. Let S be the ring R considered as an R-module via f, and let T be the ring $\widehat{R}$ viewed as an $\widehat{R}$-module via f. Since S is finitely generated as an R-module, $\widehat{R}\otimes_R S \cong \widehat{S} = T$ as $\widehat{R}$-modules. Thus, $\widehat{S}\otimes_S \mathrm{Hom}_R(S,R) \cong \widehat{R}\otimes_R \mathrm{Hom}_R(S,R) \cong \mathrm{Hom}_{\widehat{R}}(T,\widehat{R})$. Since $\mathrm{Hom}_R(S,R)$ is finitely generated as an S-module, $\mathrm{Hom}_R(S,R)\cong S$ as S-modules if and only if $\mathrm{Hom}_{\widehat{R}}(T,\widehat{R}) \cong T$ as T-modules ([5, Exercise 7.5]). In other words, $\mathrm{Hom}_R(R^f,R)\cong R$ if and only if $\mathrm{Hom}_{\widehat{R}}(\widehat{R}^f,\widehat{R}) \cong \widehat{R}$. This proves the claim.

Now suppose R is complete. By Corollary 3.9, we have $F_R(E)^{\vee} \cong \mathrm{Hom}_R(R^f, R)$. The result now follows by Matlis Duality. □

Theorem 3.11. *Let R be a homomorphic image of a Gorenstein ring. Then R is FPI if and only if R is weakly FPI.*

Proof. By parts (a) and (b) of Proposition 3.3, it suffices to prove this in the case when R is a local ring with maximal ideal m. We first prove the theorem in the case when R is F-finite. Suppose $F_R(E) \cong E$ and let $P \in \operatorname{Spec} R$. By Proposition 3.10, we have $\mathrm{Hom}_R(R^f, R) \cong R$. As R^f is finitely generated as a right R-module and $(R_P)^f \cong (R^f)_P$, we obtain that $\mathrm{Hom}_{R_P}((R_P)^f, R_P) \cong R_P$ as R_P-modules. Since R_P is also F-finite, we have again by Proposition 3.10 that $F_{R_P}(E_{R_P}(k(P))) \cong E_{R_P}(k(P))$, where $k(P) \cong R_P/PR_P$; i.e., R_P is weakly FPI. As P was arbitrary, we obtain that R is FPI by Proposition 3.3(c).

To prove the general case, first note that by parts (c) and (e) of Proposition 3.4 we may assume that R is complete. Let k be the residue field of R. By the Cohen Structure Theorem, $R \cong A/I$ where $A = k[[T_1, \dots, T_n]]$, $T_1, \dots, T_n$ are indeterminates, and I is an ideal of A. Let ℓ be the algebraic closure of k, $B = \ell[[T_1, \dots, T_n]]$, and $S = B/IB$. Note that as B is faithfully flat over A ([15, Theorem 22.4(i)]), S is faithfully flat over R. Now suppose that R is weakly FPI. Then, by [6, Theorem 1], we have $E_S(\ell) \cong S \otimes_R E_R(k)$. Thus, $F_S(E_S(\ell)) \cong F_S(S \otimes_R E_R(k)) \cong S \otimes_R F_R(E_R(k)) \cong S \otimes_R E_R(k) \cong E_S(\ell)$, where the second isomorphism is by Proposition 2.1(a). Hence, S is weakly FPI. As S is F-finite ([4, Lemma 1.5]), we have that S is FPI. Finally, since the fibers of S over R are Gorenstein ([15, Theorem 23.4]), we have that R is FPI by Proposition 3.4(d). □

We next show that a weakly FPI ring has no embedded associated primes:

Proposition 3.12. *Let R be a weakly FPI ring. Then R satisfies Serre's condition S_1.*

Proof. By Propositions 3.3(a) and 3.4(c) and [15, Theorem 23.9(iii)], we may assume that R is local and complete. Let $P \in \operatorname{Spec} R$ and $s = \dim R/P$. Since $\omega_{R/P}$ is a rank one torsion-free R/P-module, there exists an exact sequence $0 \longrightarrow \omega_{R/P} \longrightarrow R/P$. By Matlis duality, we obtain an exact sequence

$$E_{R/P} \longrightarrow H^s_m(R/P) \longrightarrow 0,$$

where $E_{R/P} := E_{R/P}(R/m)$. Applying F^e_R to this sequence, we obtain the exactness of

$$F^e_R(E_{R/P}) \longrightarrow H^s_m(R/P^{[q]}) \longrightarrow 0,$$

where $q = p^e$. By Proposition 2.3(a), $\operatorname{Ann}_R H^s_m(R/P^{[q]}) = U_R(R/P^{[q]})$. Note that as $\operatorname{Min}_R R/P^{[q]} = \{P\}$, $U_R(R/P^{[q]}) = \psi^{-1}(P^{[q]}R_P)$, where $\psi : R \longrightarrow R_P$ is the natural map. Hence, for all $q = p^e$ we have

$$(\#) \qquad \operatorname{Ann}_R F^e_R(E_{R/P}) \subseteq \psi^{-1}(P^{[q]}R_P).$$

Now suppose that $P \in \operatorname{Ass}_R R$. Then there exists an exact sequence $0 \longrightarrow R/P \longrightarrow R$. Dualizing, we have $E \longrightarrow E_{R/P} \longrightarrow 0$ is exact, where $E = E_R(R/m)$. Applying F^e_R and using that $F_R(E) \cong E$, we have an exact sequence

$$E \longrightarrow F^e_R(E_{R/P}) \longrightarrow 0.$$

Dualizing again, we obtain an exact sequence

$$0 \longrightarrow F^e_R(E_{R/P})^{\vee} \longrightarrow R.$$

Note that as $PE_{R/P} = 0$, $P^{[q]}F^e_R(E_{R/P}) = P^{[q]}F^e_R(E_{R/P})^{\vee} = 0$ for all $q = p^e$. Hence, for all q we have an exact sequence

$$0 \longrightarrow F^e_R(E_{R/P})^{\vee} \longrightarrow H^0_P(R).$$

Therefore, there exists a positive integer n such that $P^n \subseteq \operatorname{Ann}_R F^e_R(E_{R/P})$ for all e. By (#), this implies $P^n R_P \subseteq P^{[q]}R_P$ for all $q = p^e$. Hence, $P^n R_P = 0$ and $\operatorname{ht} P = 0$. □

In general, if R is weakly FPI and x is a non-zero-divisor on R then $R/(x)$ need not be weakly FPI. Otherwise, using Propositions 3.4(b) and 3.12 and [15, Exercise 18.1], one could prove that every weakly FPI ring is Gorenstein; but there exist quasi-Gorenstein rings (hence weakly FPI rings) which are not Gorenstein. However, it is true that if (R, m) is a local weakly FPI ring and $\operatorname{Tor}_1^R(R^f, E) = 0$ then $R/(x)$ is weakly FPI for every non-zero-divisor $x \in m$. This can be proved by a simple modification of the arguments used in the proof of Theorem 3.14 below, where we provide a criterion for R to be Gorenstein in terms of the modules $\operatorname{Tor}_i^R(R^f, E)$. Before proving this result, we first need the following lemma:

Lemma 3.13. *Let (R, m) be a local ring and I an ideal generated by a regular sequence. Then*

$$R/I \otimes_R E_{R/I^{[p]}}(R/m) \cong E_{R/I}(R/m).$$

Proof. Without loss of generality, we may assume R is complete. Let $E = E_R(R/m)$. Note that $E_{R/I^{[p]}}(R/m) \cong \operatorname{Hom}_R(R/I^{[p]}, E)$ and $E_{R/I}(R/m) \cong \operatorname{Hom}_R(R/I, E)$. Taking Matlis duals it suffices to prove that $\operatorname{Hom}_R(R/I, R/I^{[p]}) \cong R/I$. But this is easily seen to hold as I is generated by a regular sequence. □

The following result is dual to Theorem 1.1 of Goto [7], which holds in the case where the Frobenius map is a finite morphism.

Theorem 3.14. *Let (R, m) be a local ring and $E = E_R(R/m)$. The following conditions are equivalent:*

(1) $\operatorname{Tor}_0^R(R^f, E) \cong E$ and $\operatorname{Tor}_i^R(R^f, E) = 0$ for all $i = 1, \ldots, \operatorname{depth} R$;
(2) R is Gorenstein.

Proof. Condition (2) implies (1) by Lemma 2.2 and Proposition 3.6 (note $E \cong H_m^d(R)$). Conversely, suppose condition (1) holds. Let $\mathbf{x} = x_1, \ldots, x_r \in m$ be a maximal regular sequence on R and $K(\mathbf{x})$ the Koszul complex with respect to $\mathbf{x}$. Then $K(\mathbf{x}) \xrightarrow{\epsilon} R/(\mathbf{x}) \longrightarrow 0$ is exact, where ϵ is the augmentation map. Dualizing, we have $0 \longrightarrow E_{R/(\mathbf{x})}(R/m) \longrightarrow K(\mathbf{x})^{\mathrm{v}}$ is exact. Since $K(\mathbf{x})_j^{\mathrm{v}} \cong E^{\binom{r}{j}}$ for all j and $\operatorname{Tor}_i^R(R^f, E) = 0$ for $1 \le i \le r$, we obtain that $0 \longrightarrow F_R(E_{R/(\mathbf{x})}(R/m)) \longrightarrow F_R(K(\mathbf{x})^{\mathrm{v}})$ is exact. In particular, since $F_R(E) \cong E$, we have an exact sequence

$$0 \longrightarrow F_R(E_{R/(\mathbf{x})}(R/m)) \longrightarrow E \xrightarrow{[x_1^p \ldots x_r^p]} E^r.$$

Hence,

$$F_R(E_{R/(\mathbf{x})}(R/m)) \cong \operatorname{Hom}_R(R/(\mathbf{x})^{[p]}, E) \cong E_{R/(\mathbf{x})^{[p]}}(R/m).$$

Using Lemma 3.13, we have

$$\begin{aligned} F_{R/(\mathbf{x})}(E_{R/(\mathbf{x})}(R/m)) &\cong R/(\mathbf{x}) \otimes_R F_R(E_{R/(\mathbf{x})}(R/m)) \\ &\cong R/(\mathbf{x}) \otimes_R E_{R/(\mathbf{x})^{[p]}}(R/m) \\ &\cong E_{R/(\mathbf{x})}(R/m). \end{aligned}$$

This says that $R/(\mathbf{x})$ is weakly FPI. Since $\operatorname{depth} R/(\mathbf{x}) = 0$, we must have $\dim R/(\mathbf{x}) = 0$ by Proposition 3.12. But then $R/(\mathbf{x})$ is Gorenstein by Proposition 3.4(b). Hence, R is Gorenstein. □

4. One-dimensional FPI rings

We now turn our attention to the one-dimensional case. If R is a local ring possessing an ideal which is also a canonical module of R, this ideal is referred to as a *canonical ideal* of R. If (R, m) is a one-dimensional Cohen-Macaulay local ring, then R has a canonical ideal (necessarily m-primary) if and only if $\widehat{R}$ is generically Gorenstein ([10, Satz 6.21]). The following result can viewed as a generalization of Lemma 2.6 of [7], which holds in the case where the Frobenius map is finite:

Theorem 4.1. *Let (R, m) be a one-dimensional local ring. The following conditions are equivalent:*

(a) R is weakly FPI;

(b) R is FPI;

(c) R is Cohen-Macaulay and has a canonical ideal ω_R such that $\omega_R \cong \omega_R^{[p]}$.

Proof. Since (b) trivially implies (a), it suffices to prove (a) implies (c) and (c) implies (b).

We first prove (a) implies (c): As R is weakly FPI, R is Cohen-Macaulay by Proposition 3.12. Furthemore, $\widehat{R}$ is weakly FPI and thus FPI by Theorem 3.11. Thus, $\widehat{R}$ is generically Gorenstein, which implies R possesses a canonical ideal ω_R. To show $\omega_R \cong \omega_R^{[p]}$, it suffices to show that $\omega_R^{[p]}$ is a canonical ideal of R. Since $\omega_R^{[p]}$ is a canonical ideal for R if and only if $\omega_R^{[p]} \otimes_R \widehat{R} \cong (\omega_R \widehat{R})^{[p]}$ is a canonical ideal for $\widehat{R}$, we may assume without loss of generality that R is complete. Since R is Cohen-Macaulay, $H^1_m(\omega_R) \cong E$, where $E = E_R(R/m)$. Applying local cohomology to the exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow R/\omega_R \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow R/\omega_R \longrightarrow E \longrightarrow H^1_m(R) \longrightarrow 0.$$

Applying F_R, we have an exact sequence

$$0 \longrightarrow R/\omega_R^{[p]} \longrightarrow E \longrightarrow H^1_m(R) \longrightarrow 0,$$

where we have used Proposition 2.1(e) and Lemma 2.2. Dualizing, we have an exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow \operatorname{Hom}_R(R/\omega_R^{[p]}, E) \longrightarrow 0.$$

From the exactness of $0 \longrightarrow \operatorname{Hom}_R(R/m, R/\omega_R^{[p]}) \longrightarrow \operatorname{Hom}_R(R/m, E)$, we see that the socle of $R/\omega_R^{[p]}$ is one-dimensional and hence $R/\omega_R^{[p]}$ is Gorenstein. Thus, $\operatorname{Hom}_R(R/\omega_R^{[p]}, E) \cong R/\omega_R^{[p]}$ and we obtain an exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow R/\omega_R^{[p]} \longrightarrow 0.$$

This implies that $\omega_R \cong \omega_R^{[p]}$.

Next, we prove (c) implies (b): Since R is Cohen-Macaulay and possesses a canonical ideal, R is a homomorphic image of a Gorenstein ring (see [2, Theorem 3.3.6]). Hence, by Theorem 3.11, it suffices to prove that R is weakly FPI. Let $\pi : F_R(\omega_R) \longrightarrow \omega_R^{[p]}$ be the natural surjection given by $\pi(r \otimes u) = ru^p$ and let $C = \ker \pi$. Since R_P is Gorenstein for all primes $P \neq m$, $\dim C = 0$. Consequently, $H^1_m(C) = 0$ and $H^1_m(F_R(\omega_R)) \cong H^1_m(\omega_R^{[p]})$. Since $E \cong H^1_m(\omega_R)$ and $\omega_R \cong \omega_R^{[p]}$, we have

$$F_R(E) \cong F_R(H^1_m(\omega_R)) \cong H^1_m(F_R(\omega_R)) \cong H^1_m(\omega_R^{[p]}) \cong H^1_m(\omega_R) \cong E.$$

Hence, R is weakly FPI. □

We remark that there exist one-dimensional local FPI rings which are not Gorenstein. In fact, the next result shows that every one-dimensional F-pure ring is FPI. Recall that a homomorphism $A \longrightarrow B$ of commutative rings is called *pure* if the map $M \longrightarrow B \otimes_A M$ is injective for every A-module M. A ring R of prime characteristic is called *F-pure* if the Frobenius map $f : R \longrightarrow R$ is pure.

Proposition 4.2. *Let R be a one-dimensional F-pure ring. Then R is FPI.*

Proof. By Proposition 3.3(b) and since F-purity localizes, we may assume that R is local. By Theorem 4.1, it suffices to show that R is weakly FPI. By Proposition 3.4(c) and [11, Corollary 6.13], we may assume R is complete. Let k be the residue field of R. By the Cohen Structure Theorem, $R \cong A/I$ where $A = k[[T_1, \ldots, T_n]]$, $T_1, \ldots, T_n$ are indeterminates, and I is an ideal of A. Let ℓ be the algebraic closure k, $B = \ell[[T_1, \ldots, T_n]]$, and $S = B/IB$. Note that as B is faithfully flat over A, S is faithfully flat over R. Since R is F-pure we have that S is F-pure by Fedder's criterion [4, Theorem 1.12]. Finally, by [6, Theorem 1], $E_S(\ell) \cong E_R(k) \otimes_R S$. Hence, S is weakly FPI if and only if R is weakly FPI. Thus, resetting notation, we may assume R is complete and its residue field k is algebraically closed. By [8, Theorem 1.1], $R \cong k[[T_1, \ldots, T_n]]/I$ where $I = (\{T_iT_j \mid 1 \leq i < j \leq n\})$. By Goto [7, Example 2.8], $\omega_R = (T_2 - T_1, \ldots, T_n - T_1)R$ is a canonical ideal of R, $T_1 + \cdots + T_n$ is a non-zerodivisor on R, and $\omega_R^{[p]} = (T_1 + \cdots + T_n)^{p-1}\omega_R$. Hence, $\omega_R^{[p]} \cong \omega_R$ and R is weakly FPI by Theorem 4.1. □

As a specific example of a one-dimensional non-Gorenstein FPI ring, let k be any field of characteristic p and $R = k[[x, y, z]]/(xy, xz, yz)$. Then R is a one-dimensional local ring which is F-pure (and hence FPI) but not Gorenstein. Notice in this example that R has three associated primes. Regarding this we note the following, which is a consequence of Corollary 1.3 of [7]:

Corollary 4.3. *Let R be a one-dimensional complete local ring with algebraically closed residue field and suppose R has at most two associated primes. The following are equivalent:*
(a) R is weakly FPI;
(b) R is Gorenstein.

Proof. Notice that the hypotheses imply that R is F-finite. Hence, (a) is equivalent to the condition that $\mathrm{Hom}_R(R^f, R_R) \cong {}_R R$ by Corollary 3.10. The result now follows from [7, Corollary 1.3]. □

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