THE ASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES OVER RINGS OF SMALL DIMENSION

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Abstract. Let R be a commutative Noetherian local ring of dimension d, I an ideal of R, and M a finitely generated R-module. We prove that the set of associated primes of the local cohomology module $H^i_I(M)$ is finite for all $i \geq 0$ in the following cases: (1) $d \leq 3$; (2) $d = 4$ and R is regular on the punctured spectrum; (3) $d = 5$, R is an unramified regular local ring, and M is torsion-free. In addition, if $d > 0$ then $H^{d-1}_I(M)$ has finite support for arbitrary R, I, and M.

1. Introduction

Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module. An important problem in commutative algebra is determining when the set of associated primes of the ith local cohomology module $H^i_I(M)$ of M with support in I is finite. C. Huneke and R. Sharp [HS] (in the case of positive characteristic) and Lyubeznik [L1] (in characteristic zero) have shown that if R is a regular local ring containing a field then $H^i_I(R)$ has only finitely many associated primes for all $i \geq 0$ and all ideals I of R. Recently, Lyubeznik [L3] has proved this result also holds for unramified regular local rings of mixed characteristic.

On the other hand, A. Singh [Si] has given an example of a six-dimensional (non-local) Noetherian ring R and a 3-generated ideal I such that $H^3_I(R)$ has infinitely many associated primes. However, the question as to whether the set of associated primes of a local cohomology module of a finitely generated module over a Noetherian *local* ring is always finite remains open. This was conjectured to be the case by C. Huneke in [Hu], although not much progress has been made on this conjecture for arbitrary modules. However, see [BF], [BRS], [L2] and [Sa] for some results in this direction.

In this paper, we offer some evidence that Huneke's conjecture may be true in its full generality. Specifically, we establish the finiteness of the set of associated

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primes of local cohomology modules for any module finitely generated over a local ring of dimension three (Corollary 2.6); over a four-dimensional excellent local ring which is regular in codimension two (Theorem 2.8); and over a five-dimensional unramified regular local ring provided the module is torsion-free (Theorem 2.10). An important ingredient in the proofs of all of these results is that the support of the $d-1^{\text{st}}$ local cohomology module (where $d = \dim R$) is always finite. In fact, we show that $\operatorname{Supp}_R H_I^{d-1}(M) \subseteq \overline{A}^*(I) \cup \{m\}$, where $\overline{A}^*(I)$ is the stable value of $\operatorname{Ass}_R R/\overline{I^n}$ for large n (Corollary 2.4). (Here m denotes the maximal ideal of R and $\overline{I^n}$ the integral closure of I^n.)

Throughout this paper all rings are assumed to be commutative, Noetherian, and to have an identity element. For any unexplained notation or terminology, we refer the reader to [Mat] or [BH]. For a ring R, ideal I and R-module M, the ith local cohomology module of M with support in I is defined by

$$H_I^i(M) := \varinjlim \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [G] or [BS] for the basic properties of local cohomology.

Recall that $p \in \operatorname{Spec} R$ is an associated prime of an R-module N if there exists an R-module monomorphism $R/p \hookrightarrow N$. We denote the set of associated primes of N by $\operatorname{Ass}_R N$. Below we summarize some well-known facts concerning the associated primes of local cohomology modules which we will need in the next section:

Proposition 1.1. *Let (R, m) be a Noetherian local ring of dimension d, I an ideal of R, and M a finitely generated R-module. Then*

(a) $\operatorname{Ass}_R H_I^i(M) = \operatorname{Ass}_R \operatorname{Hom}_R(R/I, H_I^i(M))$.
(b) $\operatorname{Ass}_R H_I^g(M) = \operatorname{Ass}_R \operatorname{Ext}_R^g(R/I, M)$ *where* $g = \operatorname{depth}_I M$*; thus,* $\operatorname{Ass}_R H_I^i(M)$ *is finite for all* $i \leq g$.
(c) $\operatorname{Ass}_R H_I^i(M)$ *is finite for* $i = 0, 1$.
(d) $\operatorname{Supp}_R H_I^i(M)$ *is finite for all* i *if* $\dim R/I \leq 1$.
(e) $\operatorname{Supp}_R H_I^d(M) \subseteq \{m\}$.

Proof. Part (a) follows easily from the fact that $\operatorname{Ass}_R H_I^i(M) \subseteq \mathrm{V}(I)$. Part (b) is a consequence of (a) and the isomorphism $\operatorname{Hom}_R(R/I, H_I^g(M)) \cong \operatorname{Ext}_R^g(R/I, M)$. (This isomorphism is readily seen from the spectral sequence $\operatorname{Ext}_R^p(R/I, H_I^q(M)) \Rightarrow \operatorname{Ext}_R^{p+q}(R/I, M)$.) For part (c), let $N = M/H_I^0(M)$; then $\operatorname{depth}_I N \geq 1$ and $H_I^i(M) \cong H_I^i(N)$ for all $i \geq 1$. The result now follows from (b). Part (d) easily follows from the fact that $\operatorname{Supp} H_I^i(M) \subseteq \mathrm{V}(I)$. Part (e) is a trivial consequence of the fact that $H_I^i(M) = 0$ for $i > \dim R$. $\square$

2. Main results

We begin by showing that in order to prove that a module has only finitely many associated primes, we may pass to a faithfully flat ring extension.

Lemma 2.1. *Let R be a Noetherian ring, M an R-module, and S a commutative Noetherian faithfully flat R-algebra. Then* $\operatorname{Ass}_R M \subseteq \{p \cap R \mid p \in \operatorname{Ass}_S(M \otimes_R S)\}$.

Proof. First note that if $p \in \operatorname{Spec} R$ and $Q \in \operatorname{Ass}_S S/pS$ then $Q \cap R = p$. For, if $r \in R$, $r \notin p$ then r is not a zero-divisor on R/p; thus, r is not a zero-divisor on S/pS. Now let $p \in \operatorname{Ass}_R M$. Then there is an injective map $R/p \to M$. Hence, there is an injective map $S/pS \to M \otimes_R S$. Let $Q \in \operatorname{Ass}_S S/pS$. Then $Q \in \operatorname{Ass}_S(M \otimes_R S)$ and $Q \cap R = p$. $\square$

Let (R, m) be a local ring and I and ideal of R. We let $\ell(I)$ denote the analytic spread of I; i.e., $\ell(I)$ is the Krull dimension of the graded ring $R[It]/mR[It]$. If R/m is infinite then $\ell(I)$ is the least number of generators of any minimal reduction of I (see [NR]). We note the following:

Lemma 2.2. *Let (R, m) be a local ring of dimension d and I an ideal of R. Suppose $H_I^d(R) \neq 0$. Then $\ell(I) = d$.*

Proof. Let $S = R[t]_{mR[t]}$ where t is an indeterminate. Then S is a faithfully flat R-algebra and has an infinite residue field. Furthermore, $\ell(I) = \ell(IS)$ and $H_{IS}^d(S) \neq 0$. So we can assume R has an infinite residue field. Thus, if $\ell(I) < d$ then I can be generated up to radical by less than d elements, contradicting that $H_I^d(R) \neq 0$. $\square$

For an ideal I in a ring R, let $\overline{A}^*(I)$ denote $\bigcup_{n \geq 0} \operatorname{Ass}_R R/\overline{I^n}$, where $\overline{I^n}$ denotes the integral closure of I^n. If R is Noetherian then $\overline{A}^*(I)$ is a finite set for all I [Ra]. A theorem of S. McAdam [Mc, Proposition 4.1] states that if $\ell(I_p) = \operatorname{ht} p$ then $p \in \overline{A}^*(I)$.

For an R-module M and $i \geq 0$, let $\operatorname{Supp}_R^i(M) := \{p \in \operatorname{Supp}_R M \mid \operatorname{ht} p = i\}$.

Proposition 2.2. *Let R be a Noetherian ring, I an ideal of R, and M an R-module. Then $\operatorname{Supp}_R^i(H_I^i(M)) \subseteq \overline{A}^*(I)$ for all $i \geq 0$. In particular, $\operatorname{Supp}_R^i(H_I^i(M))$ is a finite set.*

Proof. Let $p \in \operatorname{Supp}_R^i(H_I^i(M))$. Then $H_{IR_p}^i(M_p) \neq 0$ and thus $H_{IR_p}^i(R_p) \neq 0$ since $i = \dim R_p$. By Lemma 2.2 this implies that $\ell(I_p) = \operatorname{ht} p$. By McAdam's result, $p \in \overline{A}^*(I)$. $\square$

Two nice consequences of this are:

Corollary 2.3. *Let R be a Noetherian ring of finite dimension d, I an ideal of R, and M an R-module. Then $\operatorname{Supp}_R H_I^d(M) \subseteq \overline{A}^*(I)$. In particular, $\operatorname{Ass}_R H_I^d(M)$ is finite.*

Corollary 2.4. *Let (R, m) be a local ring of dimension d, I an ideal of R, and M an R-module. Then $\operatorname{Supp}_R H_I^{d-1}(M) \subseteq \overline{A}^*(I) \cup \{m\}$. In particular, $\operatorname{Ass}_R H_I^{d-1}(M)$ is finite.*

Remark 2.5: We note that the fact that $H_I^d(M)$ has only finitely many associated primes (in the non-local case) has also been observed in [BRS, Remark 3.11], but to the best of our knowledge the finiteness of $\operatorname{Ass}_R H_I^{d-1}(M)$ (in the local case) was previously unknown. This latter result can be seen in another and more direct way using M. Brodmann's [Br] result that $A^*(I) := \bigcup_{n\geq 0} \operatorname{Ass}_R R/I^n$ is finite. We sketch this argument here:

First, we may assume that R is complete (using Lemma 2.1) and Gorenstein of dimension d (using the change of ring principle). Assume that $\operatorname{Supp}_R H_I^{d-1}(M)$ is infinite. Then, as R is complete and local, there exists a nonunit $x \in R$ which avoids infinitely many of the primes in $\operatorname{Supp}_R H_I^{d-1}(M)$ (e.g., [Bu]). If we let $S = R_x$, $J = I_x$, and $N = M_x$, we have $\operatorname{Supp}_S H_J^{d-1}(N)$ is infinite. Since $H_J^{d-1}(N) = H_J^{d-1}(S) \otimes_S N$ (as $d-1 = \dim S$), this implies that $\operatorname{Supp}_S H_J^{d-1}(S)$ is infinite.

We now claim that $\operatorname{Supp}_S H_J^{d-1}(S) \subseteq A^*(J)$, which gives the desired contradiction. Suppose $H_J^{d-1}(S)_p \neq 0$. Then $\operatorname{Ext}_{S_p}^{d-1}(S_p/J_p^n, S_p) \neq 0$ for infinitely many n. As S_p is Gorenstein of dimension $d-1$, $H_{p_p}^0(S_p/J_p^n) \neq 0$ for infinitely many n by local duality. Hence, $p \in \operatorname{Ass}_S(S/J^n) \subseteq A^*(J)$.

An immediate consequence of Corollary 2.4 is the following:

Corollary 2.6. *Let (R, m) be a local ring and M a finitely generated R-module of dimension at most three. Then $\operatorname{Ass}_R H_I^i(M)$ is finite for all i and all ideals I.*

Proof. By replacing R with $R/\operatorname{Ann}_R M$ we can assume $\dim R \leq 3$. Now use Proposition 1.1 and Corollary 2.4. $\square$

Over a four-dimensional local ring, the local cohomology modules with support in a height two ideal have finitely many associated primes, as the following proposition shows:

Proposition 2.7. *Let (R, m) be a four-dimensional local ring, M a finitely generated R-module, and I an ideal of R of such that $\operatorname{ht} I \geq 2$. Then $\operatorname{Ass}_R H_I^i(M)$ is finite for all $i \geq 0$.*

Proof. By Corollary 2.6, we may assume $\dim M = 4$. In addition, by Lemma 2.1, we can assume R is a complete Gorenstein local ring of dimension four. By Proposition 1.1 and Corollary 2.4, it is enough to prove that if $\operatorname{ht} I = 2$ then $\operatorname{Ass}_R H_I^2(M)$ is finite. First suppose that $\operatorname{Ass}_R M \subseteq \operatorname{Ass}_R R$. Then by [EG, Theorem 3.5] there exists an exact sequence $0 \to M \to R^n \to C \to 0$ for some n and some R-module C. This yields the exact sequence

$$0 \to H_I^1(C) \to H_I^2(M) \to H_I^2(R^n).$$

Since $\operatorname{Ass}_R H_I^1(C)$ and $\operatorname{Ass}_R H_I^2(R^n)$ are both finite (by Proposition 1.1 and as $\operatorname{depth}_I R = 2$), we see that $\operatorname{Ass}_R H_I^2(M)$ is finite.

Now suppose $\operatorname{Ass}_R M \not\subseteq \operatorname{Ass}_R R$. Let K be a submodule of M maximal with respect to the property that $\dim K \leq 3$. Then $\operatorname{Ass}_R M/K \subseteq \operatorname{Ass}_R R$. Thus, $\operatorname{Supp}_R H_I^2(K)$ is finite by Corollary 2.4 and $\operatorname{Ass}_R H_I^2(M/K)$ is finite by above. The finiteness of $\operatorname{Ass}_R H_I^2(M)$ now follows from the exactness of

$$\cdots \to H_I^2(K) \to H_I^2(M) \to H_I^2(M/K).$$

□

With additional hypotheses we can account for the ideals of height less than two as well:

Theorem 2.8. *Let (R, m) be a four-dimensional local ring satisfying Serre's condition R_2 and assume that the nonsingular locus of R is open. Then $\operatorname{Ass} H_I^i(M)$ is finite for all ideals I of R, finitely generated R-modules M, and $i \geq 0$.*

Proof. In the light of Proposition 2.7 we need only consider the case when $\operatorname{ht} I \leq 1$. We may assume $\sqrt{I} = I$. We first suppose that $\operatorname{ht} p \leq 1$ for all $p \in \operatorname{Min} R/I$. Let $\mathrm{V}(K)$ be the singular locus of R, where K is an ideal of R. Since R satisfies R_2, $\operatorname{ht} K \geq 3$. For $p \not\supseteq K$, I_p is principal as R_p is a UFD. Thus, $H_I^2(M)_p = 0$ for all $p \not\supseteq K$; consequently, $\operatorname{Supp} H_I^2(M) \subseteq V(K)$, which is a finite set.

In the general situation, we have $I = A \cap B$, where A is the intersection of minimal and height one primes and $\operatorname{ht} B \geq 2$; furthermore, $\operatorname{ht}(A + B) \geq 3$. If Q is a prime ideal of R such that $Q \not\supset A + B$ then $H_I^2(M)_Q$ is isomorphic to either $H_A^2(M)_Q$ or $H_B^2(M)_Q$. Therefore,

$$\operatorname{Ass}_R H_I^2(M) \subseteq \operatorname{Ass}_R H_A^2(M) \cup \operatorname{Ass}_R H_B^2(M) \cup \mathrm{V}(A + B),$$

which, by the preceding paragraph and Proposition 2.7, is a finite set. □

We note the following special case of Theorem 2.8:

Corollary 2.9. *Let (R, m) be a four-dimensional local ring such that R_p is regular for all prime ideals $p \neq m$. Then $H_I^i(M)$ has finitely many associated primes for all ideals I of R, finitely generated R-modules M, and all $i \geq 0$.*

We note that V. Sapko [Sa] has observed that the conclusion of Theorem 2.8 also holds for four-dimensional local factorial domains.

Finally, making use of the results of Huneke, Sharp, and Lyubeznik for unramified regular local rings, we obtain the following result in the five-dimensional case:

Theorem 2.10. *Let (R, m) be a five-dimensional unramified regular local ring and M a finitely generated torsion-free R-module. Then $\operatorname{Ass}_R H_I^i(M)$ is finite for all ideals I of R and all $i \geq 0$.*

Proof. By Proposition 1.1 and Corollary 2.4, it suffices to show that $\operatorname{Ass}_R H_I^2(M)$ and $\operatorname{Ass}_R H_I^3(M)$ are finite for all I. We first consider $H_I^2(M)$:

Case 1(a): $\operatorname{ht} I \geq 2$.

Since M is torsion-free there exists an exact sequence of the form

$$0 \to M \to R^n \to C \to 0.$$

As R is Cohen-Macaulay, we obtain the exact sequence

$$0 \to H_I^1(C) \to H_I^2(M) \to H_I^2(R^n).$$

Now $\operatorname{Ass}_R H_I^1(C)$ and $\operatorname{Ass}_R H_I^2(R^n)$ are both finite by Proposition 1.1. Thus, $\operatorname{Ass}_R H_I^2(M)$ is finite.

Case 1(b): $\operatorname{ht} I = 1$.

We may assume I is a radical ideal. Thus, as R is a UFD, $I = (x) \cap J$ for some $x \in R$ and ideal J with $\operatorname{ht} J \geq 2$ (or $J = R$). Since $H_I^i(M) \cong H_J^i(M)_x$ for $i \geq 2$ (e.g., [BS, Exercise 5.1.22]), we see that $\operatorname{Ass}_R H_I^2(M)$ is finite by Case 1(a).

We now consider $H_I^3(M)$:

Case 2(a): $\operatorname{ht} I \geq 3$.

As in Case 1(a) we have an exact sequence

$$0 \to H_I^2(C) \to H_I^3(M) \to H_I^3(R^n).$$

$\operatorname{Ass}_R H_I^3(R^n)$ is finite by Proposition 1.1(b). Thus, it is enough to show $\operatorname{Ass}_R H_I^2(C)$ is finite. We can assume C is torsion (by letting $n = \operatorname{rank} M$), so let $x \in \operatorname{Ann}_R C$, $x \neq 0$. Then C is an $R/(x)$-module and $\operatorname{ht} I(R/(x)) \geq 2$. Hence, $\operatorname{Ass}_R H_I^2(C)$ is finite by Proposition 2.7.

Case2(b): $\operatorname{ht} I \leq 2$.

For this case we use the fact that $\operatorname{Ass}_R H_I^3(R)$ is finite since R is an unramified regular local ring ([HS],[L1], and [L3]). If $\operatorname{ht} I = 1$ then we can employ the same argument used in Case 1(b) to reduce to the case where $\operatorname{ht} I \geq 2$. Suppose that I has height two and is height unmixed (i.e., $\operatorname{ht} P = 2$ for all $P \in \operatorname{Min}_R R/I$). We claim that $\operatorname{Supp}_R H_I^3(M)$ is finite. For, let $Q \in \operatorname{Supp}_R H_I^3(M)$ where $Q \neq m$. By the Hartshorne-Lichtenbaum vanishing theorem [Ha], we must have that $\operatorname{ht} Q = 4$ and $H_I^3(R)_Q \neq 0$. Since Q is minimal in the support of $H_I^3(R)$, $Q \in \operatorname{Ass}_R H_I^3(R)$, which is a finite set.

For an arbitrary height two radical ideal I, we have that $I = J \cap K$ where J is height unmixed, $\operatorname{ht} J = 2$, $\operatorname{ht} K \geq 3$, and $\operatorname{ht}(J + K) \geq 4$. Using the Mayer-Vietoris sequence we have an exact sequence

$$H_{J+K}^3(M) \to H_J^3(M) \oplus H_K^3(M) \to H_I^3(M) \to H_{J+K}^4(M).$$

Now $\operatorname{Supp}_R H^i_{J+K}(M)$ is finite for all i as $\dim R/(J+K) \le 1$; $\operatorname{Supp}_R H^3_J(M)$ is finite by above; and $\operatorname{Ass}_R H^3_K(M)$ is finite by Case 2(a). If $Q \in \operatorname{Ass}_R H^3_I(M)$ but $Q \notin \operatorname{Supp}_R H^3_{J+K}(M) \bigcup \operatorname{Supp}_R H^3_J(M) \bigcup \operatorname{Supp}_R H^4_{J+K}(M)$, then $H^3_I(M)_Q \cong H^3_K(M)_Q$, whence $Q \in \operatorname{Ass}_R H^3_K(M)$. Therefore,

$$\operatorname{Ass}_R H^3_I(M) \subseteq \bigcup_{i\in\{3,4\}} \operatorname{Supp}_R H^i_{J+K}(M) \bigcup \operatorname{Supp}_R H^3_J(M) \bigcup \operatorname{Ass}_R H^3_K(M),$$

which is a finite set. $\square$

References

[Br] M. P. Brodmann, *Asymptotic stability of* $\operatorname{Ass} M/I^nM$, Proc. Amer. Math. Soc. **74** (1979), 16–18.

[BF] M. P. Brodmann and A. Lashgari Faghani, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (2000), 2851–2853.

[BRS] M. P. Brodmann, C. Rotthaus, and R. Y. Sharp, *On annihilators and associated primes of local cohomology modules*, J. Pure Appl. Algebra **153** (2000), 197–227.

[BS] M. P. Brodmann and R. Sharp, *Local Cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics no. 60, Cambridge University Press, Cambridge, 1998.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics no. 39, Cambridge University Press, Cambridge, 1993.

[Bu] L. Burch, *Codimension and analytic spread*, Proc. Camb. Phil. Soc. **72** (1972), 369–373.

[EG] E. G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lect. Notes Ser. no. 106, Cambridge University Press, 1985.

[G] A. Grothendieck, *Local Cohomology*, notes by R. Hartshorne, Lecture Notes in Math. no. 41, Springer-Verlag, 1967.

[Ha] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. Math. **88** (1968), 403–450.

[Hu] C. Huneke, *Problems on local cohomology*, in: Free resolutions in commutative algebra and algebraic geometry (Sundance, Utah, 1990), pp. 93–108, Research Notes in Mathematics 2, Jones and Bartlett Publishers, Boston, MA, 1992.

[HS] C. Huneke and R. Sharp, *Bass numbers of local cohomology modules*, Trans. American Math. Soc. **339** (1993), 765–779.

[L1] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Inv. Math. **113** (1993), 41–55.

[L2] G. Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic* $p > 0$, J. Reine Angew. Math. **491** (1997), 65–130.

[L3] G. Lyubeznik, *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case*, Comm. Alg. (to appear).

[Mat] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge., 1986.

[Mc] S. McAdam, *Asymptotic Prime Divisors*, Lecture Notes in Math. no. 1023, Springer-Verlag, 1983.

[NR] D.G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Phil. Soc. **50** (1954), 145–158.

[Ra] L.J. Ratliff, *On asymptotic prime divisors*, Pacific J. Math. **111** (1984), 395–413.

[Sa] V. Sapko, *Ph.D. Thesis*, University of Nebraska- Lincoln, 2001.

[Si] A. Singh, *p-torsion elements in local cohomology modules*, Math. Res. Lett. **7** (2000), 165–176.

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