Extremal Cases of the Ahlswede-Cai Inequality

A. J. Radcliffe and Zs. Szaniszló
University of Nebraska–Lincoln
Department of Mathematics
810 Oldfather Hall
University of Nebraska-Lincoln
Lincoln, NE 68588

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Proof should be sent:

A. J. Radcliffe
Department of Mathematics
810 Oldfather Hall
University of Nebraska-Lincoln
Lincoln, NE 68588
e-mail: ajr@unlinfo.unl.edu

Abstract

Consider two set systems $\mathcal{A}$ and $\mathcal{B}$ in the powerset $\mathcal{P}(n)$ with the property that for each $A \in \mathcal{A}$ there exists a unique $B \in \mathcal{B}$ such that $A \subset B$. Ahlswede and Cai proved an inequality about such systems which is a generalization of the LYM and Bollobás inequalities. In this paper we characterize the structure of the extremal cases.

§0. Notation

We are concerned with the poset $\mathcal{P}(n) = \mathcal{P}(\{1, 2, \ldots, n\})$. This is the power set of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. A *set system* is simply a subset of $\mathcal{P}(n)$. A set system is an *antichain* if no two of its members are comparable. Conversely a *chain* is a totally ordered set system. We shall often consider *maximal chains*; those chains which cannot be extended. In particular such chains contain exactly one set from each of the *levels* of $\mathcal{P}(n)$. The k^{th} level is the system $[n]^{(k)} = \{A \in \mathcal{P}(n) : |A| = k\}$. There are exactly $n!$ maximal chains in $\mathcal{P}(n)$.

Occasionally we think of $\mathcal{P}(n)$ as a graph with edges AB for all $A, B \in \mathcal{P}(n)$ with $|A \Delta B| = 1$.

An *upset* is a set system $\mathcal{U}$ with the property that $A \supset B \in \mathcal{U}$ implies $A \in \mathcal{U}$. A *downset* is defined similarly. If $\mathcal{A}$ is an arbitrary set system we use $\mathcal{U}(\mathcal{A})$ to denote the upset *generated by* $\mathcal{A}$. I.e., $\mathcal{U}(\mathcal{A}) = \{X \in \mathcal{P}(n) : \exists A \in \mathcal{A}, A \subset X\}$. The downset generated by $\mathcal{A}$, denoted $\mathcal{D}(\mathcal{A})$, is defined similarly.

§1. Introduction

Given an antichain $\mathcal{A}$ in $\mathcal{P}(n)$ the LYM inequality states that

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1.$$

This inequality is named after Lubell, Yamamoto and Meshalkin, who proved it independently (see [5], [6], [8]). A short proof of the inequality follows from noting that the left hand side of the inequality is just the probability that a maximal chain [picked uniformly at random from the collection of all maximal chains] in $\mathcal{P}(n)$ intersects $\mathcal{A}$. (The probability that a randomly chosen maximal chain contains a fixed set $X \in \mathcal{P}(n)$ is $\binom{n}{|X|}^{-1}$ and the events that a chain contains the various elements of an antichain $\mathcal{A}$ are all disjoint.) Since it is a probability it is certainly at most 1.

The Bollobás inequality [4] (stronger than the LYM inequality) has a similar proof. The inequality states that if $\mathcal{A} = (A_i)_1^N$ and $\mathcal{B} = (B_i)_1^N$ have the property that $A_i \subset B_j$ if and only if $i = j$ then

$$\sum_{i=1}^{N} \binom{n - |B_i \setminus A_i|}{|A_i|}^{-1} \leq 1. \tag{1}$$

Here the probability that a randomly chosen maximal chain meets the interval $[A_i, B_i]$ is $\binom{n-|B_i \setminus A_i|}{|A_i|}^{-1}$ and, by the condition on $\mathcal{A}$ and $\mathcal{B}$, these events are disjoint. Thus (1) just states that the probability of a maximal chain meeting $\cup_{i=1}^{N}[A_i, B_i]$ is at most 1.

In [2] Ahlswede and Zhang, again with an essentially probabilistic proof, extended these results by considering not just the event that a maximal chain meets a certain set system $\mathcal{A}$ but also the event that it leaves $\mathcal{U}(\mathcal{A})$. To be precise let us say that a maximal chain $\mathcal{C}$ *leaves an upset* $\mathcal{U}$ *at* $U \in \mathcal{U}$ if $U \in \mathcal{C} \cap \mathcal{U}$ and $C \notin \mathcal{U}$ for all $C \in \mathcal{C}$ strictly below U. Of course if $\mathcal{C}$ leaves $\mathcal{U}$ at U then it leaves along an *exit edge* – an edge of $\mathcal{P}(n)$ joining $\mathcal{U}$ to $\mathcal{P}(n) \setminus \mathcal{U}$. Ahlswede and Zhang define $W_{\mathcal{A}}(X)$ to be the number of exit edges of $\mathcal{U}(\mathcal{A})$ incident with X for $X \in \mathcal{U}(\mathcal{A})$ and 0 otherwise. Equivalently

$$W_{\mathcal{A}}(X) = \begin{cases} 0 & A \not\subset X \text{ for all } A \in \mathcal{A} \\ |\bigcap_{\substack{A \in \mathcal{A} \\ A \subset X}} A| & \text{otherwise} \end{cases}.$$

Note that $W_{\mathcal{U}(\mathcal{A})}(X) = W_{\mathcal{A}}(X)$ for all $X \in \mathcal{P}(n)$.

Ahlswede and Zhang essentially proved the following theorem.

Theorem 1. *Suppose $\mathcal{U}, \mathcal{D} \subset \mathcal{P}(n)$ are an upset and a downset respectively with $\mathcal{U} \neq \emptyset, \mathcal{P}(n)$. If $\mathcal{C}$ is a maximal chain chosen uniformly at random then*

$$\mathbf{P}(\mathcal{C} \text{ meets } \mathcal{U} \cap \mathcal{D}) + \sum_{X \in \mathcal{U} \setminus \mathcal{D}} \frac{W_{\mathcal{U}}(X)}{|X|\binom{n}{|X|}} = 1.$$

Proof. First note that if $\mathcal{C}$ meets $\mathcal{U} \cap \mathcal{D}$ then it leaves $\mathcal{U}$ from $\mathcal{U} \cap \mathcal{D}$. Also, since $\mathcal{U} \neq \emptyset, \mathcal{P}(n)$, the probability that $\mathcal{C}$ leaves $\mathcal{U}$ is 1. Secondly, if X is a set in $\mathcal{U}$ then

$$\begin{aligned} \mathbf{P}(\mathcal{C} \text{ leaves } \mathcal{U} \text{ at } X) &= \mathbf{P}(X \in \mathcal{C})\, \mathbf{P}(\mathcal{C} \text{ leaves } \mathcal{U} \text{ at } X | X \in \mathcal{C}) \\ &= \binom{n}{|X|}^{-1} \frac{W_{\mathcal{U}}(X)}{|X|}. \end{aligned}$$

(The last factor is the proportion of downward edges from X which lead out of $\mathcal{U}$.) Thus

$$\begin{aligned} 1 &= \mathbf{P}(\mathcal{C} \text{ leaves } \mathcal{U}) \\ &= \mathbf{P}(\mathcal{C} \text{ leaves } \mathcal{U} \text{ at } X \in \mathcal{U} \cap \mathcal{D}) + \mathbf{P}(\mathcal{C} \text{ leaves } \mathcal{U} \text{ at } X \in \mathcal{U} \setminus \mathcal{D}) \\ &= \mathbf{P}(\mathcal{C} \text{ meets } \mathcal{U} \cap \mathcal{D}) + \sum_{X \in \mathcal{U} \setminus \mathcal{D}} \frac{W_{\mathcal{U}}(X)}{|X|\binom{n}{|X|}}. \end{aligned}$$

■

This theorem is a little too general to be useful. The following corollaries are much more natural. The first is essentially Theorem 1 in [2].

Corollary 2 (Ahlswede, Zhang [2]). *If $\mathcal{A} \subset \mathcal{P}(n)$ is an antichain then*

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{W_{\mathcal{U}}(X)}{|X|\binom{n}{|X|}} = 1.$$

Proof. In Theorem 1 set $\mathcal{U} = \mathcal{U}(\mathcal{A})$ and $\mathcal{D} = \mathcal{D}(\mathcal{A})$. Note that the first term on the left hand side (as in the proof of the LYM inequality) is the probability that a randomly chosen maximal chain meets $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{A}) = \mathcal{A}$. ■

Corollary 3 (Ahlswede, Zhang [3]). *If $\mathcal{A} = (A_i)_1^N$ and $\mathcal{B} = (B_i)_1^N$ have the property that $A_i \subset B_j$ if and only if $i = j$ then*

$$\sum_{i=1}^{N} \binom{n - |B_i \setminus A_i|}{|A_i|}^{-1} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

Proof. In Theorem 1 set $\mathcal{U} = \mathcal{U}(\mathcal{A})$ and $\mathcal{D} = \mathcal{D}(\mathcal{B})$. ∎

The result we are most interested in this paper is the following result of Ahlswede and Cai [1] which generalizes the Bollobás inequality to pairs of set systems, $\mathcal{A}$ and $\mathcal{B}$, satisfying the following condition:

$$\forall A \in \mathcal{A} \quad \exists!\ B \in \mathcal{B} \text{ such that } A \subset B. \tag{$*$}$$

(If, in addition, each set in $\mathcal{B}$ contains exactly one set in $\mathcal{A}$ then we are in the setup of the Bollobás inequality.)

Corollary 4 (Ahlswede, Cai [1]). *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be two set systems satisfying (*) with $\mathcal{U}(\mathcal{A}) \neq \emptyset, \mathcal{P}(n)$. For $B \in \mathcal{B}$ set $\mathcal{A}_B = \{A \in \mathcal{A} : A \subset B\}$. Also let $\bigcup \mathcal{S} = \bigcup_{S \in \mathcal{S}} S$, and $\bigcap \mathcal{S} = \bigcap_{S \in \mathcal{S}} S$. Then*

$$\sum_{B \in \mathcal{B}} \sum_{\mathcal{S} \subset \mathcal{A}_B} (-1)^{|\mathcal{S}|} \binom{n - |B \setminus \bigcup \mathcal{S}|}{|\bigcup \mathcal{S}|}^{-1} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

In particular, given such systems we have

$$\sum_{B \in \mathcal{B}} \sum_{\mathcal{S} \subset \mathcal{A}_B} (-1)^{|\mathcal{S}|} \binom{n - |B \setminus \bigcup \mathcal{S}|}{|\bigcup \mathcal{S}|}^{-1} \leq 1. \tag{2}$$

Proof. In Theorem 1 let $\mathcal{U} = \mathcal{U}(\mathcal{A})$, $\mathcal{D} = \mathcal{D}(B)$ and note that the first term on the left hand side is simply the probability that a random maximal chain meets $\mathcal{U} \cap \mathcal{D}$, computed using inclusion/exclusion. ∎

In the remainder of the paper we characterize pairs of systems $\mathcal{A}, \mathcal{B}$ satisfying the conditions of Cor.4 for which (2) holds with equality. We call such pairs of systems *extremal pairs*. Section 2 presents most of the analysis of extremal pairs, culminating in Theorem 14, which gives necessary and sufficient conditions for a pair to be extremal. In Section 3 we give a (hopefully) more illuminating characterization in terms of matroids.

§2. Characterization of the Extremal Systems.

We want to understand the structure of the cases of equality in (2). There are two ways of looking at extremality; we can either use the fact that a pair $\mathcal{A}, \mathcal{B}$ is extremal iff every maximal chain meets $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ or that extremality is equivalent to the condition $W_{\mathcal{A}}(X) = 0$ for all $X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})$.

Suppose then that $\mathcal{A}, \mathcal{B}$ are a pair of set systems in $\mathcal{P}(n)$ which satisfy (*), with $\mathcal{U}(\mathcal{A}) \neq \emptyset, \mathcal{P}(n)$, and for which (2) holds with equality. We start with some simple remarks.

Remark. Necessarily $\bigcup \mathcal{B} = [n]$. Otherwise let $b \notin \bigcup \mathcal{B}$. Any maximal chain passing through $\{b\}$ misses $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$, contradicting the extremality of $\mathcal{A}, \mathcal{B}$.

Remark. We may suppose that every set in $\mathcal{B}$ contains at least one set in $\mathcal{A}$, since removing a set in $\mathcal{B}$ which does not contain any set in $\mathcal{A}$ leaves the left hand side of (2) unchanged.

Remark. We may assume that $\mathcal{A}$ is an antichain. The left hand side of (2) depends only on $\mathcal{U}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ so we may safely replace $\mathcal{A}$ by the antichain of its minimal elements. Similarly we may suppose that $\mathcal{B}$ is an antichain.

Remark. There is one rather trivial case of equality; when $\mathcal{B} = \{[n]\}$ and $\mathcal{A}$ is any antichain with $\mathcal{U}(\mathcal{A}) \neq \emptyset, \mathcal{P}(n)$. We shall suppose henceforth that $\mathcal{B} \neq \{[n]\}$.

In summary, we suppose for the remainder of this section that $\mathcal{A}$ and $\mathcal{B}$ are nonempty antichains satisfying (*) and (2) with equality, and in addition $\mathcal{A} \neq \{\emptyset\}$, $\mathcal{B} \neq \{[n]\}$, $\bigcup \mathcal{B} = [n]$.

It turns out that the most important parameter of the pair $\mathcal{A}, \mathcal{B}$ is the size of the smallest sets in $\mathcal{A}$ (which we will show is the common size of all the sets in $\mathcal{A}$). Therefore define

$$k := \min\{|A| \,:\, A \in \mathcal{A}\}.$$

The following simple lemmas will be used repeatedly in this section.

Lemma 5. *If $X \in [n]^{(k+1)} \setminus \mathcal{D}(B)$ then all the k subsets of X belong to $\mathcal{A}$.*

Proof. Suppose $K \subset X$ has size k and $K \notin \mathcal{A}$. Then any maximal chain passing through K and X misses $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(B)$ (In levels 0 through $k-1$ nothing belongs to $\mathcal{U}(\mathcal{A})$ by the definitition of k; K does not belong to $\mathcal{U}(\mathcal{A})$ by assumption; X, and *a fortiori* anything higher on the chain, doesn't belong to $\mathcal{D}(\mathcal{B})$.) This contradicts the extremality of $\mathcal{A}, \mathcal{B}$. ∎

Lemma 6. *If $A \in \mathcal{A}$ has size k, B_A is the unique element of $\mathcal{B}$ containing A, and $b \notin B_A$ then $A \Delta \{a, b\} \in \mathcal{A}$ for all $a \in A$.*

Proof. Let $X = A \cup \{b\}$. Now $X \notin \mathcal{D}(\mathcal{B})$ because $b \notin B_A$ (so $X \not\subset B_A$) and $A \not\subset B$ for all $B \in \mathcal{B}, B \neq B_A$ by (*). By Lemma 5 all k-subsets of X belong to $\mathcal{A}$. ∎

The extremal cases for which $k = 1$ were already characterized in [1] (See Fig. 1). In this case $\mathcal{B}$ is a 'flower'; the intersection of any two sets in $\mathcal{B}$ is the same as the common intersection $\bigcap \mathcal{B}$. Moreover $\mathcal{A} = \{\{a\} : a \in [n] \setminus \bigcap \mathcal{B}\}$.

Theorem 7. *If $k = 1$ then setting $B' = \bigcap \mathcal{B}$ we have $\mathcal{A} = \{\{a\} : a \in [n] \setminus B'\}$, and $\forall B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 = B'$.*

Proof. Let $\{x\}$ be any singleton element of $\mathcal{A}$, with B_x the unique set in $\mathcal{B}$ containing it. If y is any element of $[n] \setminus B_x$ then by Lemma 6 (with $A = \{x\}$ and $b = y$) $\{y\} \in \mathcal{A}$. If $z \neq x$ belongs to $B_x \setminus B'$ then pick a set in $\mathcal{B}$ B with $z \notin B$ (possible since $z \notin \bigcap \mathcal{B}$) and an element $y \in B \setminus B_x$ (possible since $B \not\subset B_x$). By the above argument we have $\{y\} \in \mathcal{A}$ and by applying Lemma 6 again ($A = \{y\}$, $b = z$) we have $\{z\} \in \mathcal{A}$.

Having established that $\{a\} \in \mathcal{A}$ for all $a \in [n] \setminus B'$ it is clear by (*) that $B_1 \cap B_2 = B'$ for all $B_1 \neq B_2$ in $\mathcal{B}$. ■

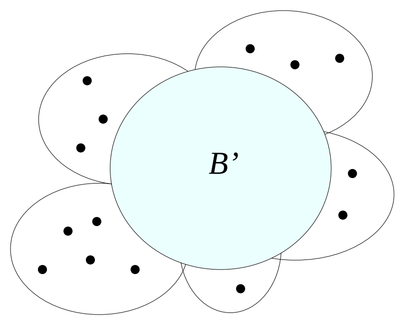

Fig.1.

We turn now to the case $k \geq 2$. The next lemma essentially establishes that if $\mathcal{A}, \mathcal{B}$ is an extremal pair then all of the sets in $\mathcal{A}$ have a common size. Somehow it seems a little surprising that sizes of sets in $\mathcal{A}$ are tightly restricted while sets in $\mathcal{B}$ can have essentially any size.

Lemma 8. *If $A \in \mathcal{A}$ has size k and $B_A \in \mathcal{B}$ with $A \subset B_A$ then every set in $\mathcal{A}$ not contained in B_A is of size k.*

Proof. Given $D \in \mathcal{A}$ with $D \subset B \in \mathcal{B}$, $B \neq B_A$ we want to show that D has size k. Pick $A' \in \mathcal{A}$ subject to the following conditions:

(i) $|A'| = k$

(ii) $A' \subset B' \in \mathcal{B}$, $B' \neq B$

(iii) The distance $d(A', D) = |A' \Delta D|$, is minimal subject to (i) and (ii).

Sets $A' \in \mathcal{A}$ satisfying (i) and (ii) certainly exist since A is one. Now pick $x \in A' \setminus B$ and $y \in D \setminus B'$. By Lemma 5 all k-subsets of $X' = A' \cup \{y\}$ belong to $\mathcal{A}$.

If there exists $c \in A' \setminus (D \cup \{x\})$ then for $A'' = X' \setminus \{c\}$ we have $A'' \in \mathcal{A}$, $|A''| = k$, $A'' \not\subset B$ (since $x \notin B$), and $d(A'', D) < d(A'D)$, contradicting (iii).

Otherwise $A' \subset D \cup \{x\}$, in which case setting $A'' = X' \setminus \{x\}$ we have $A'' \in \mathcal{A}$ and $A'' \subset D$. Since $\mathcal{A}$ is an antichain we must have $A'' = D$ and hence $|D| = k$. ■

Proposition 9. *Every set in $\mathcal{A}$ has size k.*

Proof. Pick a set in $\mathcal{A}$, A, of size k and let B_A be the set in $\mathcal{B}$ containing it. By the previous proposition any set in $\mathcal{A}$ not contained in B_A has size k and, since we have assumed that $\mathcal{B}$ is not trivial there do exist such sets in $\mathcal{A}$. Applying the previous lemma again we see that all sets in $\mathcal{A}$ have size k. ■

We now turn to the system $\mathcal{B}$. The next three lemmas establish that $\mathcal{B}$ covers $[n]^{(k)}$ and that $\mathcal{B}$ determines $\mathcal{A}$.

Lemma 10. *For all $C \in [n]^{(k)}$ there exists $B \in \mathcal{B}$ such that $C \subset B$.*

Proof. Let $C \in [n]^{(k)}$, $A \in \mathcal{A}$ with $d(A, C)$ minimal, say $A \subset B_A \in \mathcal{B}$. We will establish that $C \subset B_A$. Suppose not; then in particular $C \neq A$. Pick $c \in C \setminus B_A$ and $a \in A \setminus C$. Applying Lemma 6 we have that $A' = A\Delta\{a, c\} \in \mathcal{A}$ and moreover $d(A', C) < d(A, C)$. This contradicts the definition of A', hence we must have $C \subset B_A$. ■

The next lemma shows that $\mathcal{B}$ determines $\mathcal{A}$; in fact it turns out that every set that can be a set in $\mathcal{A}$ is in fact a set in $\mathcal{A}$.

Lemma 11. *If $X \in [n]^{(k)}$ and there exists a unique $B \in \mathcal{B}$ such that $X \subset B$ then $X \in \mathcal{A}$.*

Proof. Suppose $X \notin \mathcal{A}$. Pick $c \notin B$, then $X \cup \{c\} \notin \mathcal{D}(\mathcal{B})$ and any maximal chain which passes through X and $X \cup \{c\}$ misses $\mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})$, contradicting the extremality of $\mathcal{A}, \mathcal{B}$. Hence $X \in \mathcal{A}$. ■

The next (technical) lemma says that in some sense $\mathcal{A}$ is "connected".

Lemma 12. *For any X_1 , $X_2 \in [n]^{(k)}$ and $B \in \mathcal{B}$ with $X_1 \in \mathcal{A}$, $X_2 \notin \mathcal{A}$, $X_1 \subset B$, $X_2 \not\subset B$, we have $|X_1 \Delta X_2| > 2$. (See Fig. 2.)*

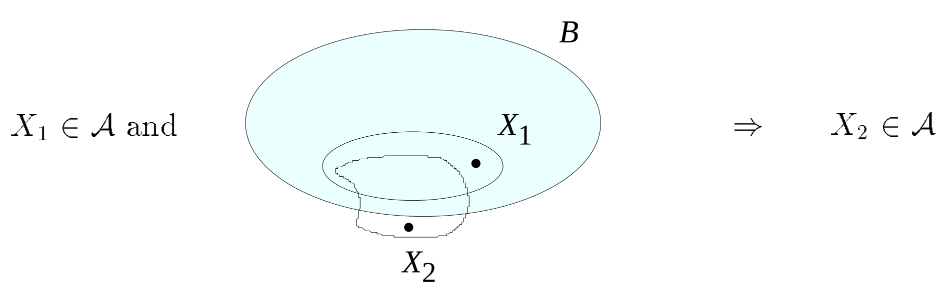

Fig.2.

Proof. Suppose there are some X_1, X_2 , and B satisfying the conditions with $|X_1 \Delta X_2| = 2$. Then $X_1 \cup X_2 \notin \mathcal{D}(\mathcal{B})$. ($X_1 \cup X_2 \not\subset B$ since $X_2 \not\subset B$ and for $B' \in \mathcal{B}$, $B' \neq B$ we have $X_1 \cup X_2 \not\subset B'$ since $X_1 \not\subset B'$.) Applying Lemma 5 with $X = X_1 \cup X_2$ (which has size $k+1$) we must have $X_2 \in \mathcal{A}$, which contradicts our assumptions. ■

It turns out that the condition on extremal pairs proved in Lemma 12 is in fact sufficient for extremality.

Proposition 13. *Suppose $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ are antichains satisfying (*) and the conclusion of Lemma 12 (with $k = \min\{|A| : A \in \mathcal{A}\}$). Then $\mathcal{A}, \mathcal{B}$ is an extremal pair.*

Proof. We must show that for any set X in $\mathcal{U}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{B})$ the deficiency term $W_{\mathcal{A}}(X)$ is zero. We know that there exists $A \in \mathcal{A}$ with $A \subset X$. Let B_A be the unique element of $\mathcal{B}$ containing A. Since $X \notin \mathcal{D}(\mathcal{B})$ there exists an element $x \in X \setminus B_A$. Now for all $a \in A$ we have, by the condition in Lemma 6, that $A\Delta\{a,x\} \in \mathcal{A}$. Hence $W_{\mathcal{A}}(A \cup \{x\}) = 0$. (Since $A \cap \bigcap \{A\Delta\{a,x\} : a \in A\} = \emptyset$.) By the monotonicity of $W_{\mathcal{A}}$, $W_{\mathcal{A}}(X) = 0$. ■

We summarize the results of this section in the following theorem.

Theorem 14. *Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be two antichains satisfying (*), with $\mathcal{B} \neq \{[n]\}$. Let $k = \min\{|A| : A \in \mathcal{A}\}$. The pair $\mathcal{A}, \mathcal{B}$ is extremal iff the following condition holds:*

$$\text{If } X_1, X_2 \in [n]^{(k)}, B \in \mathcal{B} \text{ with } X_1 \in \mathcal{A}, X_2 \notin \mathcal{A}, X_1 \subset B, X_2 \not\subset B \text{ then } |X_1 \Delta X_2| > 2. \tag{**}$$

If the pair $\mathcal{A}, \mathcal{B}$ is extremal then
(i) $\mathcal{A} \subset [n]^{(k)}$.
(ii) For all $C \in [n]^{(k)}$ there exists $B \in \mathcal{B}$ such that $C \subset B$.
(iii) If $X \in [n]^{(k)}$ and there exists a unique $B \in \mathcal{B}$ such that $X \subset B$ then $X \in \mathcal{A}$.

Proof. Lemmas 10–12 and Propositions 9 and 13. ■

§3. Structural Characterization

In this section we show that the rather unpleasant characterization given in Theorem 14 can be replaced by a useful description in terms of matroids. Our notation for matroids is reasonably standard; see e.g. [7], or any standard text, for further reference.

We think of a matroid as a pair $\mathcal{M} = (S, \mathcal{I})$ where S is a finite set and $\mathcal{I} \subset \mathcal{P}(S)$ is the collection of independent sets defining the matroid. We will sometimes write $\mathcal{I}(\mathcal{M})$ for the collection of independent sets associated with the matroid $\mathcal{M}$. We write $\rho_{\mathcal{M}}$, or, if no ambiguity is possible, simply ρ, for the rank function of $\mathcal{M}$, defined on $\mathcal{P}(S)$ by

$$\rho_{\mathcal{M}}(X) = \max\{|I| \,:\, I \in \mathcal{I}, I \subset X\}.$$

We write $\mathcal{E}(\mathcal{M})$ for the collection of all bases of $\mathcal{M}$ where a basis is a maximal independent subset of S.

Given a matroid $\mathcal{M}$ and an integer k, let

$$\begin{aligned} \mathcal{I}_k(\mathcal{M}) &= \{I \in \mathcal{I}(M) \,:\, |I| = k\} \\ \mathcal{F}_k(\mathcal{M}) &= \{\text{maximal sets of rank } k\} \\ &= \{X \subset S \,:\, \rho_{\mathcal{M}}(X) = k \text{ and } \rho(X \cup \{x\}) = k+1 \ \forall x \notin X\}. \end{aligned}$$

We first show that matroids provide us with a plentiful supply of examples of extremal pairs.

Theorem 15. *Let $\mathcal{M} = ([n], \mathcal{I})$ be a matroid with $\rho([n]) \geq k+1$, and set $\mathcal{A} = \mathcal{I}_k(\mathcal{M})$ and $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$. Then the pair $\mathcal{A}, \mathcal{B}$ is extremal.*

Proof. First we must check that $\mathcal{A}, \mathcal{B}$ (both obviously antichains) satisfy (*). Suppose then that $A \subset B_1, B_2$ with $A \in \mathcal{A}$, $B_1, B_2 \in \mathcal{B}, B_1 \neq B_2$. By the submodularity and monotinicity of ρ we have

$$\begin{aligned} k \leq \rho(B_1 \cup B_2) &\leq \rho(B_1) + \rho(B_2) - \rho(B_1 \cap B_2) \\ &\leq \rho(B_1) + \rho(B_2) - \rho(A) \\ &= k \end{aligned}$$

Since $B_1 \neq B_2$ we have that $B_1 \cup B_2$ is a set of rank k strictly containing B_1, contradicting the fact that B_1 is a maximal set of rank k.

To show that $\mathcal{A}, \mathcal{B}$ is extremal, pick a maximal chain $\mathcal{C} = (C_i)_0^n$ in $\mathcal{P}(n)$, where $|C_i| = i$. We must show that $\mathcal{C}$ meets $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$. Since $\rho(C_i)$ increases from 0 to $\rho([n]) > k$ we may define $j = \min\{i \,:\, \rho(C_i) = k\}$. C_j contains an independent set of size k, A say, (by the definition of rank) and is contained in a maximal set of rank k, B say. Thus $A \subset C_j \subset B$ and $\mathcal{C} \cap (\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})) \neq \emptyset$. ■

The next Theorem, the main result of this section, establishes that not only do matroids give many extremal examples, they in fact give all extremal examples.

Theorem 16. *If $\mathcal{A}, \mathcal{B}$ is a pair of non empty antichains satisfying (*) and (2) with equality and $\mathcal{B} \neq \{[n]\}$ then there exists a matroid $\mathcal{M}$ such that*

$$\mathcal{A} = \mathcal{I}_k(\mathcal{M}) \qquad \text{and} \qquad \mathcal{B} = \mathcal{F}_k(\mathcal{M}).$$

Proof. For every extremal system we have to construct a matroid and show, that from this matroid we get back the original extremal system.

Set $\mathcal{E} = [n]^{(k+1)} \setminus \mathcal{D}(\mathcal{B})$. We'll show the following:

(i) $\mathcal{E}$ is the collection of bases for some matroid $\mathcal{M} = ([n], \mathcal{I})$.

(ii) $\mathcal{A} = \mathcal{I}_k(\mathcal{M})$.

(iii) $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$.

To prove that $\mathcal{E}$ is the collection of bases for a matroid we must show (see e.g. [7]) that $\mathcal{E}$ is a non-empty antichain and that for all $E_1, E_2 \in \mathcal{E}$ and $x \in E_1 \setminus E_2$ there exists $y \in E_2 \setminus E_1$ such that $E_1 \Delta \{x, y\} \in \mathcal{E}$. The first condition is certainly satisfied since our $\mathcal{E}$-sets are all the same size and $\mathcal{D}(\mathcal{B})$ cannot cover $[n]^{(k+1)}$ without covering each the set in $[n]^{(k)}$ many times, contradicting (*).

So consider $E_1, E_2 \in \mathcal{E}$ and let $x \in E_1 \setminus E_2$. Then $E_1 \setminus \{x\}$ is a k-element set. Pick a maximal chain, $\mathcal{C}$, passing through E_1 and $E_1 \setminus \{x\}$. By the extremality of $\mathcal{A}, \mathcal{B}$ the chain $\mathcal{C}$ must meet some set-interval $[A, B]$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since $E_1 \notin \mathcal{D}(B)$, $E_1 \setminus \{x\}$ must be a set in $\mathcal{A}$. Thus there exists a unique $B \in \mathcal{B}$ such that $E_1 \setminus \{x\} \subset B$. Now pick $y \in E_2 \setminus (B \cup \{x\}))$. (Such a y exists since $E_2 \notin \mathcal{D}(\mathcal{B})$ and $x \notin E_2$) Then $y \notin B$, so $E_1 \Delta \{x, y\} \not\subset B$. Also $E_1 \Delta \{x, y\} \not\subset B'$ for any $B' \neq B$ with $B' \in \mathcal{B}$ because $E_1 \setminus \{x\}$ is contained in a unique set in $\mathcal{B}$. Hence $E_1 \Delta \{x, y\} \in \mathcal{E}$.

Now that we have defined our matroid $\mathcal{M}$ we want to show that $\mathcal{A} = \mathcal{I}_k(\mathcal{M})$ and that $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$. First note that $\mathcal{I}(\mathcal{M}) = \mathcal{D}(\mathcal{E}) = \mathcal{D}([n]^{(k+1)} \setminus \mathcal{D}(\mathcal{B}))$.

To see that $\mathcal{A} \subset I_k(\mathcal{M})$, consider any $A \in \mathcal{A}$. There exists a unique $B \in \mathcal{B}$ with $A \subset B$. Now pick $x \notin B$. By essentially the argument of Lemma 6 $A \cup \{x\} \in \mathcal{E} = [n]^{(k+1)} \setminus \mathcal{D}(\mathcal{B})$ since it clearly has the right size and $x \notin B$ while $A \not\subset B'$ whenever $B' \neq B, B' \in \mathcal{B}$. Hence $A \in \mathcal{I}_k(\mathcal{M})$.

For the other containment suppose that there exist $E \in \mathcal{E}$ and $I \subset E$ with $|I| = k$ and $I \notin \mathcal{A}$. Now let $\mathcal{C}$ be any maximal chain passing through I and E. $\mathcal{C}$ misses $\mathcal{U}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ because no set of size smaller than k is in $\mathcal{U}(\mathcal{A})$, $I \notin \mathcal{U}(\mathcal{A})$, and no set containing E is in $\mathcal{D}(\mathcal{B})$. This contradicts the extremality of $\mathcal{A}, \mathcal{B}$. Hence $I \in \mathcal{A}$ and thus $\mathcal{A} \subset I_k(\mathcal{M})$.

To see that $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$ we have to show that the elements of $\mathcal{B}$ are exactly the elements, X, of the Boolean algebra that satisfy the following three conditions:

(a) There exists $A \in \mathcal{A}$ such that $A \subset X$.

(b) There is no $E \in \mathcal{E}$ with $E \subset X$.

(c) For all $x \notin X$ there exists $E \in \mathcal{E}$ with $E \subset X \cup \{x\}$.

Claim. For all $B \in \mathcal{B}$, B satisfies conditions (a)–(c).

Condition (a) is satisfied by assumption (see the remarks at the beginning of section 2). Condition (b) is satisfied by the definition of $\mathcal{E}$. For (c) consider $B \in \mathcal{B}$ and $x \notin B$. Pick $A \subset B$ with $A \in \mathcal{A}$. Now $A \cup \{x\}$ is a $(k+1)$-set and it is not in $\mathcal{D}(\mathcal{B})$ (since $x \notin B$ and $A \not\subset B'$ for $B' \in \mathcal{B}$, $B' \neq B$). Thus $E = A \cup \{x\}$ has $E \in \mathcal{E}$ and $E \subset B \cup \{x\}$.

Claim. Every element of the Boolean algebra satisfying conditions (a)–(c) is an element of $\mathcal{B}$.

Pick any set $X \in \mathcal{P}(n)$ satisfying conditions (a), (b) and (c). By (a) we know that there exists $A \in \mathcal{A}$ such that $A \subset X$, and, by (*) , there exists a unique $B \in \mathcal{B}$ with $A \subset B$.

Case 1: $X \subset B$. If $X = B$ then we have shown that $X \in \mathcal{B}$. Otherwise $B \setminus X \neq \emptyset$. Pick any $x \in B \setminus X$. By condition (c) there exists $E \in \mathcal{E}$ such that $E \subset X \cup \{x\} \subset B$, but this contradicts the definition of $\mathcal{E}$.

Case 2: $X \not\subset B$. Pick any $x \in X \setminus B$. By condition (b) $A \cup \{x\}$, being a $(k+1)$-set but not in $\mathcal{E}$ by (b), belongs to $\mathcal{D}(\mathcal{B})$, hence there exists some $B' \in \mathcal{B}$, $B' \neq B$ such that $A \cup \{x\} \subset B'$. But this means that $A \subset B, B'$ which contradicts (*).

Hence $X \in \mathcal{B}$ and therefore $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$. ■

We summarize the results of this section in the following theorem.

Theorem 17. *A pair of antichains $\mathcal{A}, \mathcal{B}$ satisfying (*) with $\mathcal{U}(\mathcal{A}) \neq \emptyset, \mathcal{P}(n)$ and $\mathcal{B} \neq \{[n]\}$ satisfies (2) with equality if and only if there exists a matroid $\mathcal{M}$ such that*

$$\mathcal{A} = \mathcal{I}_k(\mathcal{M}) \qquad \text{and} \qquad \mathcal{B} = \mathcal{F}_k(\mathcal{M}).$$

Proof. Theorems 15 and 16. ■

The case $k = 2$ is particularly nice. In order to describe it we need to recall the notion of a pairwise balanced design. (We supress some parameters for clarity.)

A *pairwise balanced design* of type (v, λ) is a pair $(X, \mathcal{B})$ where X is a v-set of points and $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X (called blocks) with the property that every 2-element subset of X is contained in exactly λ blocks.

Given a (v, λ)-pairwise balanced design $(X, \mathcal{V})$ and a partition of $[n]$ into v parts, $P = (P_i)_1^v$, the *P-expansion* of $(X, \mathcal{V})$ is the set system

$$\mathcal{X}_P(\mathcal{V}) = \{\cup_{i \in B} P_i \,:\, B \in \mathcal{V}\}.$$

Theorem 18. *Given any extremal pair $\mathcal{A}, \mathcal{B}$ with $k = 2$ there exists a partition $P = (P_i)_0^v$ of $[n]$ and a $(v, 1)$-pairwise balanced design $(X, \mathcal{B})$ such that*

$$\mathcal{B} = \{P_0 \cup B \,:\, B \in \mathcal{X}_P(\mathcal{V})\}$$
$$\mathcal{A} = \{\{x, y\} \,:\, x \in P_i, y \in P_j, i \neq j, i, j \in [v]\}$$

Remark. Given an extremal pair $\mathcal{A}, \mathcal{B}$, construct a graph G on $[n]$ by drawing an edge between two vertices exactly if the pair is a set in $\mathcal{A}$, i.e., when the vertices are independent. Then the theorem just says that G is the complete v-paritite graph with parts $P_1, \ldots, P_v$.

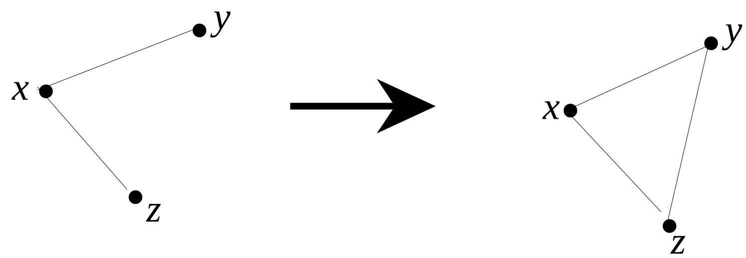

Fig.3.

Proof. Since $\mathcal{A}, \mathcal{B}$ is an extremal pair we know by Theorem 17 that there exists a matroid $\mathcal{M} = ([n], \mathcal{I})$ such that $\mathcal{A} = \mathcal{I}_k(\mathcal{M})$ and $\mathcal{B} = \mathcal{F}_k(\mathcal{M})$. Set $P_0 = \{x \in [n] : \rho(\{x\}) = 0\}$. Clearly we cannot have $P_0 \cap A \neq \emptyset$ for any $A \in \mathcal{A}$ for if $A = \{x, y\}$ and $x \in P_0$ then $\rho(A) \leq \rho(\{x\}) + \rho(\{y\}) \leq 1$, a contradiction. To show that the independent pairs form a complete multipartite graph we'll prove that the dependent pairs from $[n] \setminus P_0$ form cliques. Thus, suppose $\{x, y\}$ and $\{y, z\}$ are *not* independent, with $x, y, z \in [n] \setminus P_0$. By the submodularity of the rank function and the fact that $\rho(\{x\}) = 1$ for all $x \in [n] \setminus P_0$ we have

$$\rho(\{x, y, z\}) \leq \rho(\{x, y\}) + \rho(\{y, z\}) - \rho(\{x\}) = 1.$$

Therefore $\rho(\{x, z\}) = 1$ and we have shown that the dependent pairs form cliques; let the vertex sets of these cliques be $P_1, \ldots, P_v$, so that $P = (P_0)_1^v$ is a partition of $[n]$. It remains to prove that $\mathcal{B}$ is the expansion of some pairwise balanced design by P. To prove that each set in $\mathcal{B}$ contains P_0 note that if $\{x\}$ has rank 0 then $\rho(X \cup \{x\}) = \rho(X)$ for all $X \in \mathcal{P}(n)$ therefore all maximal sets of rank k must contain x. We also have to show that each set in $\mathcal{B}$ is a union of the P_i. Suppose then that $x, y \in P_i$, $i \neq 0$ and $x \in B \in \mathcal{B}$. Set $B' = B \setminus \{x, y\}$. Now

$$\begin{aligned}\rho(B \cup \{y\}) &= \rho(B \cup \{x, y\}) \\ &\leq \rho(B) + \rho(\{x, y\}) - \rho(\{x\}) \\ &= k.\end{aligned}$$

Thus, by the maximality of B, $y \in B$. To finish the proof we need to show that for all $i, j \in [v]$, $i \neq j$, there exists a unique $B \in \mathcal{B}$ such that $P_i \cup P_j \subset B$, but that follows immediately from (*) since if $x \in P_i$ and $y \in P_j$ with $i \neq j$ we know $\{x, y\} \in \mathcal{A}$. ■

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