

# Reconstructing subsets of reals

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## Abstract

We consider the problem of reconstructing a set of real numbers up to translation from the multiset of its subsets of fixed size, given up to translation. This is impossible in general: for instance almost all subsets of  $\mathbb{Z}$  contain infinitely many translates of every finite subset of  $\mathbb{Z}$ . We therefore restrict our attention to subsets of  $\mathbb{R}$  which are *locally finite*; those which contain only finitely many translates of any given finite set of size at least 2.

We prove that every locally finite subset of  $\mathbb{R}$  is reconstructible from the multiset of its 3-subsets, given up to translation.

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# 1 Introduction.

Reconstructing combinatorial objects from information about their subobjects is a long-standing problem. The Reconstruction Conjecture and the Edge Reconstruction Conjecture both deal with the problem of reconstructing a graph from a multiset of subgraphs; in one case the collection of all induced subgraphs with one fewer vertex, in the other the collection of all subgraphs with one fewer edge (see Bondy [2] and Bondy and Hemminger [3]).

The very general problem is that of reconstructing a combinatorial object (up to isomorphism) from the collection of isomorphism classes of its subobjects. Isomorphism plays a crucial rôle. Thus it seems that the natural ingredients for a reconstruction problem are a group action (to provide a notion of isomorphism) and an idea of what constitutes a subobject. Reconstruction problems have been considered from this perspective by, for instance, Alon, Caro, Krasikov and Roditty [1], Radcliffe and Scott [11], [10], Cameron [4], [5], and Mnukhin [7], [8], [9].

In this paper we consider the problem of reconstructing subsets of the groups  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  from the multiset of isomorphism classes of their subsets of fixed size, where two subsets are isomorphic if one subset is a translate of the other. Where the subsets have size  $k$  we call this collection the  $k$ -deck.

Maybe the first thing to notice is that for  $|A| \geq k$  one can reconstruct the  $l$ -deck of  $A$  from the  $k$ -deck for any  $l \leq k$ . This is a straightforward translation of Kelly's lemma (see [2]). On the other hand if  $|A| < k$  then the  $k$ -deck of  $A$  is empty, and therefore  $A$  cannot be distinguished from any other subset of size strictly less than  $k$ . It makes the statement of our theorems slightly easier if we use a definition of deck for which this issue does not arise. The definition we adopt below regards the deck as a function on multisets of size  $k$ . It is straightforward to check that this form of the  $k$ -deck can be determined from the deck as defined above, provided  $|A| \geq k$ .

**Definition 1** Let  $A$  be a subset of  $\mathbb{F}$ , where  $\mathbb{F}$  is one of  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{R}$ . The  $k$ -deck of  $A$  is the function defined on multisets  $Y$  of size  $k$  from  $\mathbb{F}$  by

$$d_{A,k}(Y) = |\{i \in \mathbb{F} : \text{supp}(Y + i) \subset A\}|,$$

where  $\text{supp}(Y)$  is the set of elements of  $Y$ , considered without multiplicity. We say that  $A$  is *reconstructible from its  $k$ -deck* if we can deduce  $A$  up to

translation from its  $k$ -deck; in other words, we have

$$d_{B,k} \equiv d_{A,k} \Rightarrow B = A + i, \text{ for some } i \in \mathbb{F}.$$

More generally we say that a function of  $A$  is *reconstructible from the  $k$ -deck of  $A$*  if its value can be determined from  $d_{A,k}$ .

Certain subtleties arise since the groups involved are infinite. It may be that the  $k$ -deck of  $A \subset \mathbb{F}$  takes the value  $\infty$  on some finite (multi)sets. In fact, for any fixed finite subset  $F \subset \mathbb{Z}$ , almost all subsets of  $\mathbb{Z}$  (with respect to the obvious symmetric probability measure on  $\mathcal{P}(\mathbb{Z})$ ) contain infinitely many translates of  $F$ . Thus it is trivial to find, for all  $k \geq 1$ , two subsets of  $\mathbb{Z}$  with the same  $k$ -deck which are not translates of one another.

For this reason we restrict our attention to subsets  $A \subset \mathbb{F}$  for which the 2-deck (and *a fortiori* the  $k$ -deck for all  $k \geq 2$ ) takes only finite values, or equivalently, every distance occurs at most finitely many times. We shall call such sets *locally finite*.

It is easily seen that every finite subset  $A \subset \mathbb{R}$  can be reconstructed from its 3-deck,  $d_{A,3}$ : indeed, let  $n = \text{diam } A := \max A - \min A$ ; then

$$A \simeq \{0, n\} \cup \{r : d_{A,3}(\{0, r, n\}) > 0\}.$$

The 2-deck is not, however, in general enough. For instance, if  $A$  and  $B$  are finite sets of reals then  $A + B$  and  $A - B$  have the same 2-deck.

Our aim in this note is to prove a reconstruction result for locally finite sets of reals. We begin by proving a result for  $\mathbb{Z}$  and work in stages towards  $\mathbb{R}$ . We shall write  $A \simeq B$  if  $A$  is a translation of  $B$ .

**Theorem 1** *Let  $A \subset \mathbb{Z}$  be locally finite. Then  $A$  is reconstructible from its 3-deck. In other words, if  $A, B \subset \mathbb{Z}$  have the same 3-deck then  $A \simeq B$ .*

We shall first prove a lemma. For subsets  $A, B \subset \mathbb{Z}$ , we define  $A + B$  to be the multiset of all  $a + b$  with  $a \in A$  and  $b \in B$ . (This multiset might of course take infinite values). Thus, for finite  $A$  and  $B$ , if we identify  $A$  with  $a(x) = \sum_{i \in A} x^i$  and  $B$  with  $b(x) = \sum_{i \in B} x^i$ , then  $A + B$  can be identified with  $a(x)b(x)$ , where the coefficient of  $x^i$  in  $a(x)b(x)$  is the multiplicity of  $i$  in the sum  $A + B$ .

If  $L$  is a multiset of  $\mathbb{Z}$  we write  $m_L(i)$  for the multiplicity of  $i$  in  $L$ .

**Lemma 2** *Let  $A, B, C \subset \mathbb{Z}$  be finite and suppose that  $A + C = B + C$ . Then  $A = B$ .*

**Proof.** Straightforward by induction on  $|A|$ , noting that  $\min(A + C) = \min A + \min C$ . ■

**Lemma 3** *If  $A, B \subset \mathbb{Z}$  are locally finite, infinite sets with  $A \Delta B$  finite, and  $C$  is a finite set with  $A + C = B + C$  then  $A = B$ .*

**Proof.** Let  $A_0 = A \setminus B$ , let  $B_0 = B \setminus A$ , and set  $R = A \cap B$ . Now for all  $i$  we have

$$\begin{aligned} m_{A_0+C}(i) &= m_{A+C}(i) - m_{R+C}(i) \\ &= m_{B+C}(i) - m_{R+C}(i) \\ &= m_{B_0+C}(i). \end{aligned}$$

Thus  $A_0 + C = B_0 + C$  and it follows from Lemma 2 that  $A_0 = B_0$  and so  $A = B$ . ■

**Lemma 4** *If  $A, B \subset \mathbb{Z}$  are locally finite, infinite sets, and  $C$  is a finite set with  $A + C = B + C$  then  $A = B$ .*

**Proof.** We may suppose, without loss of generality, that  $0 \in C$ . Now let  $S = \{i : C + i \subset A + C\}$  and  $c = \text{diam}(C)$ . We aim to show that, except for a finite amount of confusion, we have  $S = A$ . To this end, let  $N$  be sufficiently large such that for all distinct  $a, a' \in A$  with  $|a| > N$  we have  $|a' - a| > 4c$  and for all distinct  $b, b' \in B$  with  $|b| > N$  we have  $|b' - b| > 4c$ . (Such an  $N$  exists since  $A$  and  $B$  are locally finite.) Suppose now that  $k$ , with  $|k| > N + 4c$ , belongs to two sets from  $\{C + i : i \in S\}$ , say  $k \in (C + i) \cap (C + j)$ . Define  $D = (C + i) \cup (C + j)$ . Since  $\text{diam}(D) > c$ , while  $D \subset A + C$ , there must be distinct elements  $a_1, a_2 \in A$  such that  $D$  meets both  $C + a_1$  and  $C + a_2$ . But this is impossible, for then  $|a_1 - a_2| \leq 4c$ , while  $|a_1| > N$ . Thus every  $k \in A + C$  with  $|k| > N + 4c$  belongs to exactly one set  $C + i$ . It follows that  $i \in A$ , and by the same reasoning  $i \in B$ .

Now set  $R = \{i \in S : |i| > N + 4c\}$ . We have just established that  $R \subset A$  and  $R \subset B$ , and obviously  $R \supset \{a \in A : |a| > N + 4c\}$  and  $R \supset \{b \in B : |b| > N + 4c\}$ . Thus  $A \Delta B$  is finite, and by Lemma 3 the result is established. ■

**Lemma 5** *Let  $A, B \subset \mathbb{Z}$  be locally finite infinite sets and let  $C, D \subset \mathbb{Z}$  be finite. If  $A + C = B + D$  then  $A \simeq B$  and  $C \simeq D$ .*

**Proof.** We may clearly assume that  $\min C = \min D = 0$ . Under this hypothesis we will prove that  $A = B$  and  $C = D$ .

We will show that  $C$  (and equally  $D$ ) is the largest set such that infinitely many translates of  $C$  are contained in  $A + C = B + D$ . Suppose then that  $A + C$  contains infinitely many translates of some set  $E$  and that no translate of  $E$  is a subset of  $C$ . Let  $E_1, E_2, \dots$  be translates of  $E$ , where  $E_i \subset A + C$  and  $|\min E_i| \rightarrow \infty$  as  $i \rightarrow \infty$ . Since  $E$  is contained in no translate of  $C$ , every  $E_i$  must meet at least two translates of  $C$ , say  $C_{a_i}$  and  $C_{b_i}$ , where  $a_i$  and  $b_i$  are distinct elements of  $A$ . Thus there are distinct  $a_i, b_i \in A$  with

$$|a_i - b_i| \leq 2 \operatorname{diam}(C) + \operatorname{diam}(E)$$

and  $|a_i| \rightarrow \infty$ ; since there are only finitely many possibilities for  $a_i - b_i$  and infinitely many  $a_i$ , some distance must occur infinitely many times, which contradicts the assumption that  $A$  is locally finite.

We conclude that  $C$  is the largest set (uniquely defined up to translation) that has infinitely many translates as subsets of  $A + C$ . Hence we have  $C \simeq D$  and so  $C \equiv D$ , since  $\min C = \min D$ . Thus  $A + C = B + D = B + C$ , and by Lemma 4,  $A = B$ . ■

**Proof. [of Theorem 1]** If  $A$  is finite then it is easily reconstructed from its 3-deck, as noted above. Thus we may assume that  $A$  is infinite.

Let  $k$  be a difference that occurs in  $A$  (i.e. there are  $a_1, a_2 \in A$  with  $a_1 - a_2 = k$ ). We shall show that  $A$  can be reconstructed from its 3-deck; moreover, it can be reconstructed from its 3-deck restricted to multisets of the form  $\{0, k, \alpha\}$ . Indeed, let  $B$  be another set with the same 3-deck. Define

$$X_A = \{a \in A : a + k \in A\}$$

and

$$X_B = \{b \in B : b + k \in B\}.$$

Then, translating if necessary, we may assume that  $\min X_A = \min X_B$ . We claim now that  $A = B$ .

In order to prove our result it is enough to show that  $-A + X_A = -B + X_B$ , for then the result follows immediately from Lemma 5: since  $-A = -B$  we also have  $A = B$ .

Now for  $i \in \mathbb{Z}$ , the multiplicity of  $i$  in  $-A + X_A$  is

$$\begin{aligned} |\{j : j \in X_A, i - j \in -A\}| &= |\{j : j \in X_A, j - i \in A\}| \\ &= |\{j : j, j + k, j - i \in A\}|. \end{aligned}$$

If  $i \neq 0, -k$ , then this is the multiplicity of  $\{0, i, i + k\}$  in the 3-deck of  $A$ ; if  $i = 0$  or  $i = -k$  then this is  $|X_A|$ , the multiplicity of  $\{0, k\}$  in the 2-deck of  $A$ . Clearly, similar calculations hold for  $B$ , so  $-A + X_A = -B + X_B$ . ■

**Theorem 6** *Lemmas 2, 3, 4, and 5 hold in  $\mathbb{Z}^n$  for all positive integers  $n$ . Moreover if  $A, B \subset \mathbb{Z}^n$  have the same 3-deck then  $A \simeq B$ .*

**Proof.** The proofs are almost identical to those for the corresponding results about  $\mathbb{Z}$ . We use the norm  $|a| = \|a\|_2$ , and order  $\mathbb{Z}^n$  lexicographically, so  $a \leq b$  if the first nonzero coordinate of  $b - a$  is positive. The assumptions  $\min C = \min D$  in the proof of Lemma 2 and  $\min X_A = \min X_B$  in the proof of Theorem 1 then make sense. Moreover, the claim in the proof of Lemma 4 that  $\text{diam}(D) > \text{diam}(C)$  is easily seen to hold in  $\mathbb{Z}^n$  also: suppose  $D = (C+i) \cup (C+j)$  and  $x, y \in C$  satisfy  $|x - y| = \text{diam}(C)$ . Let  $v = i - j \neq 0$ . Now  $|(x + i) - (y + j)| = |(x - y) + v|$  and  $|(x + j) - (y + i)| = |(x - y) - v|$  and one of these two norms is strictly greater than  $|x - y| = \text{diam}(C)$  (by the strict convexity of the norm we have chosen). ■

**Theorem 7** *Let  $A, B \subset \mathbb{Q}$  be locally finite and have the same 3-deck, then  $A \simeq B$ .*

**Proof.** Suppose  $A$  and  $B$  are locally finite subsets of  $\mathbb{Q}$  with the same 3-deck. Let  $k$  be some distance that occurs in  $A$ , and again define  $X_A = \{a \in A : a + k \in A\}$  and  $X_B = \{b \in B : b + k \in B\}$  as in the proof of Theorem 1. We may assume  $\min X_A = \min X_B = 0$ . Now suppose  $n$  is an integer such that  $1/n$  divides  $k$  and all differences in  $X_A$  and  $X_B$ . That is,  $nk \in \mathbb{Z}$  and for all  $q, r \in X_A \cup X_B$  we have  $n(q - r) \in \mathbb{Z}$ . In particular  $nq \in \mathbb{Z}$  for all  $q \in X_A \cup X_B$ . We will show that for all  $i$  we have

$$A \cap \frac{1}{in}\mathbb{Z} = B \cap \frac{1}{in}\mathbb{Z}$$

Since  $\mathbb{Q} = \bigcup_{i \geq 1} \frac{1}{in}\mathbb{Z}$  the result will then be proved.

As in the proof of Theorem 1, it is enough to show that the 3-decks of  $A \cap \frac{1}{in}\mathbb{Z}$  and  $B \cap \frac{1}{in}\mathbb{Z}$ , restricted to multisets of form  $\{0, k, \alpha\}$ , are equal. Now if  $a + \{0, k, \alpha\} \subset A$  then  $a \in X_A$ , and so

$$\begin{aligned} a + \{0, k, \alpha\} \subset A \cap \frac{1}{in}\mathbb{Z} &\iff a + \alpha \in \frac{1}{in}\mathbb{Z} \\ &\iff \alpha \in \frac{1}{in}\mathbb{Z}. \end{aligned}$$

Thus the relevant parts of the 3-decks of  $A \cap \frac{1}{in}\mathbb{Z}$  and  $B \cap \frac{1}{in}\mathbb{Z}$  are equal, and hence  $A \cap \frac{1}{in}\mathbb{Z} = B \cap \frac{1}{in}\mathbb{Z}$ . ■

**Theorem 8** *Let  $A \subset \mathbb{Q}^n$  be locally finite. Then  $A$  is reconstructible from its 3-deck.*

**Proof.** Similar to the proof of Theorem 7, with modifications as indicated in the proof of Theorem 6. ■

**Theorem 9** *Let  $A \subset \mathbb{R}$  be locally finite. Then  $A$  is reconstructible from its 3-deck.*

**Proof.** Let  $\{q : q \in I\}$  be a Hamel basis for  $\mathbb{R}$  over  $\mathbb{Q}$ , where the set  $I$  is well-ordered by  $\prec$ . This induces a total ordering on  $\mathbb{R}$  by defining  $x < y$  iff  $y - x = \sum_{i=1}^n a_i q_i$  with  $q_1 \prec q_2 \prec \dots \prec q_n$  and  $a_1 > 0$ . Given a subset  $S \subset \mathbb{R}$  we write  $\langle S \rangle$  for the collection of finite  $\mathbb{Q}$ -linear combinations of elements of  $S$ .

Now suppose that  $A, B \subset \mathbb{R}$  are locally finite, and that the 3-decks of  $A$  and  $B$  are the same. Let  $r$  be a distance that occurs in  $A$  and let  $X_A = \{a \in A : a + r \in A\}$ , and  $X_B = \{b \in B : b + r \in B\}$ . We may assume that  $\min X_A = \min X_B = 0$ . Let  $I_0 \subset I$  be a finite subset of  $I$  such that  $x - y \in \langle I_0 \rangle$  for all  $x, y \in X_A \cup X_B$ , and also  $r \in \langle I_0 \rangle$ . Such a subset exists, since  $X_A \cup X_B$  is finite and every element of  $\mathbb{R}$  can be written as a  $\mathbb{Q}$ -linear combination of a finite set of elements from  $I$ .

We will show that for finite subsets  $J$  with  $I_0 \subset J \subset I$ , the sets  $A \cap \langle J \rangle$  and  $B \cap \langle J \rangle$  are equal, from which it easily follows that  $A = B$ . Consider then such a  $J$ . If  $a + \{0, r, \alpha\} \subset A$  then  $a \in X_A$  and

$$\begin{aligned} a + \{0, r, \alpha\} \subset A \cap \langle J \rangle &\iff a + \alpha \in \langle J \rangle \\ &\iff \alpha \in \langle J \rangle. \end{aligned}$$

Since  $\langle J \rangle$  is isomorphic to  $\mathbb{Q}^N$ , for some  $N$ , and, by the argument above, the 3-decks of  $A \cap \langle J \rangle$  and  $B \cap \langle J \rangle$  restricted to multisets of form  $\{0, r, \alpha\}$  are the same, it follows from Theorem 8 that  $A \cap \langle J \rangle = B \cap \langle J \rangle$ . Since  $\bigcup_{J \supset I_0} \langle J \rangle = \mathbb{R}$ , we have that  $A = B$ . ■

It would be interesting to have a measure-theoretic version of this result. Let  $S$  be a Lebesgue-measurable set of reals, and for every finite set  $X$ , define  $S(X) = \lambda(x : X + x \subset S)$ . Call  $S$  *locally finite* if  $S(X)$  is finite whenever  $|X| > 1$ . We regard sets  $X, Y$  as equivalent if  $\lambda(X \triangle (Y + t)) = 0$  for some



real number  $t$ . Can we reconstruct every set of finite measure from its 3-deck? Can we reconstruct every locally finite set from its 3-deck? Or from the  $k$ -deck for sufficiently large  $k$ ?

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